Medan Vektor: gradient, divergensi dan curl

Tujuan:
Memahami dan menghitung Kalkulus vektor yang penting: gradient, divergensi dan curl

Medan skalar mendefinisikan medan vektor melalui gradien. Sedangkan medan vektor mendefinisikan medan skalar melalui divergensi dan medan vektor melalui curl (rotasi).

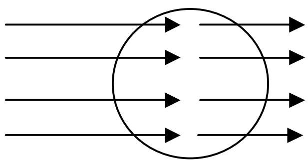
Fungsi skalar / potensial: f(P)

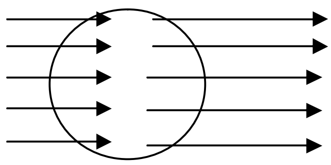
Fungsi vektor v(P) = grad f(P) dimana $grad\ f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

Divergensi dari medan vector

Arti fisis dari divergensi: menghitung berapa perubahan/laju suatu vector dalam arah x, y dan z. (outflow dikurangi inflow)

Divergensi < 0 **Divergensi > 0**





Misal $\vec{v}(v_1, v_2, v_3)$ **adalah fungsi vektor terdiferensialkan, dimana x, y dan z adalah koordinat Kartesius, maka**

$$div\ \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

disebut divergensi dari $\vec{v}$ **atau divergensi dari medan vektor yang didefinisikan oleh** $\vec{v}$**.**

Notasi yang lain dari divergensi: $\nabla \cdot \vec{v}$

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$$div\ \vec{v} = \nabla \cdot \vec{v} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (v_1\hat{i}, v_2\hat{j}, v_3\hat{k}) = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Contoh: Hitunglah divergensi dari $\vec{v} = 3xz\hat{i} + 2xy\hat{j} - yx^2\hat{k}$

Contoh fisis: medan gravitasi

Let a particle A of mass M be fixed at a point P_0 and let a particle B of mass m be free to take up various positions P in space. Then A attracts B. According to **Newton's law of gravitation** the corresponding gravitational force **p** is directed from P to P_0, and its magnitude is proportional to $1/r^2$, where r is the distance between P and P_0, say,

(2) $$|\mathbf{p}| = \frac{c}{r^2}, \qquad c = GMm.$$

Here $G = 6.67 \cdot 10^{-8}\ \text{cm}^3/(\text{gm} \cdot \text{sec}^2)$ is the gravitational constant. Hence **p** defines a vector field in space. If we introduce Cartesian coordinates such that P_0 has the coordinates x_0, y_0, z_0 and P has the coordinates x, y, z, then by the Pythagorean theorem,

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \qquad (\geqq 0).$$

Assuming that $r > 0$ and introducing the vector

$$\mathbf{r} = [x - x_0,\quad y - y_0,\quad z - z_0] = (x - x_0)\mathbf{i} + (y - y_0)\mathbf{j} + (z - z_0)\mathbf{k},$$

we have $|\mathbf{r}| = r$, and $(-1/r)\mathbf{r}$ is a unit vector in the direction of **p**; the minus sign indicates that **p** is directed from P to P_0 (Fig. 196). From this and (2) we obtain

(3) $$\mathbf{p} = |\mathbf{p}|\left(-\frac{1}{r}\mathbf{r}\right) = -\frac{c}{r^3}\mathbf{r} = \left[-c\frac{x - x_0}{r^3},\quad -c\frac{y - y_0}{r^3},\quad -c\frac{z - z_0}{r^3}\right]$$
$$= -c\frac{x - x_0}{r^3}\mathbf{i} - c\frac{y - y_0}{r^3}\mathbf{j} - c\frac{z - z_0}{r^3}\mathbf{k}.$$

Gaya tarik menarik antara 2 partikel:

$$p = -\frac{c}{r^3}\vec{r} = -c\left(\frac{x - x_0}{r^3}\hat{i} + \frac{y - y_0}{r^3}\hat{j} + \frac{z - z_0}{r^3}\hat{k} \right)$$

dengan $\hat{r}$ **jarak antara** $P_0(x_0, y_0)$ **dan** $P(x, y)$**.**

Fungsi p **adalah gradient dari fungsi scalar** $f(x, y, z) = \frac{c}{r}$

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Fungsi f ini memenuhi persamaan paling penting dalam Fisika, yaitu Persamaan Laplace:

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$$

Misal gerakan compressible fluid.

Curl (Rotasi) dari medan vector

Misalkan x, y dan z adalah koordinat Kartesius (arah tangan kanan) dan $\vec{v}(x,y,z)=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ **fungsi yang terdideferensialkan. Fungsi curl atau rotasi adalah fungsi**

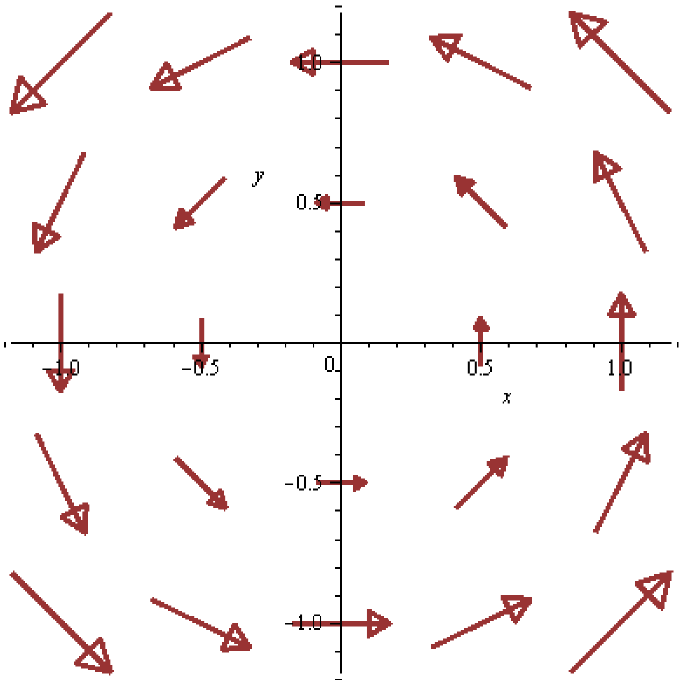
$$curl\ \vec{v}=rot\ \vec{v}=\nabla\times\vec{v}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$$

$$=\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)\hat{i}+\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\hat{j}+\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)\hat{k}$$

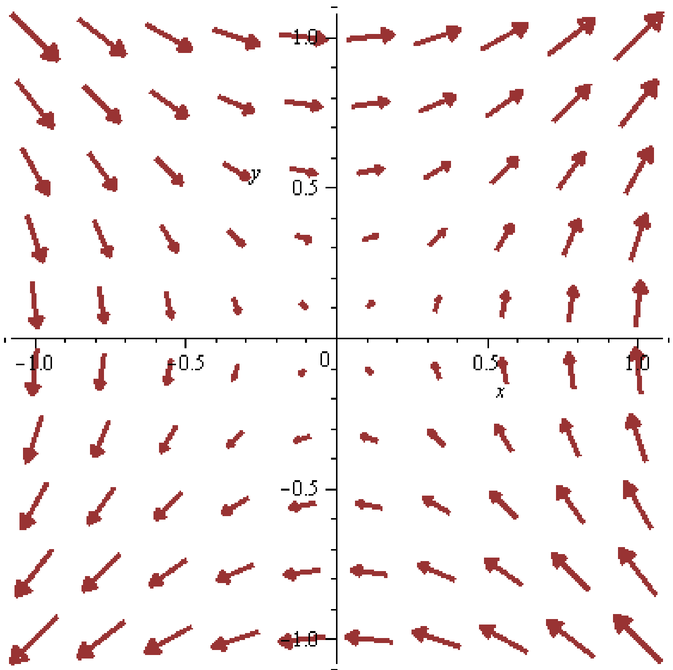
Contoh:

a. **Hitung curl dan divergensi dari** $\vec{f}(x,y)=-y\hat{i}+x\hat{j}$ **.**
b. **Hitung curl dan divergensi dari** $\vec{f}(x,y)=y\hat{i}+x\hat{j}$ **.**

Gambar dari $\vec{f}(x,y)=-y\hat{i}+x\hat{j}$



Gambar dari $\vec{f}(x,y) = y\hat{i} + x\hat{j}$



Contoh:

Rotasi benda pejal.

rot (grad f) = 0

Jadi jika suatu fungsi vektor adalah gradient dari fungsi scalar, rotasinya adalah nol. Medan gradientnya irrotational / konservatif.

$$\textbf{div (rot } \vec{v}\textbf{) = 0}$$

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9.10 no. 6, 10, 16
9 Review: no. 32, 36