## Numerical integration

Many of the integrals that are required in practical calculation turn out to be either very hard or cannot be done using well known functions. An approximate solution is all that can be easily obtained. In this section we shall look at some simple ways to find approximations to integrals. We shall consider estimating

$$I = \int_a^b f(x)\,dx$$

which, of course, is the area under the curve $f(x)$ between $x = a$ and $x = b$.

### Method 1 The mid-point rule

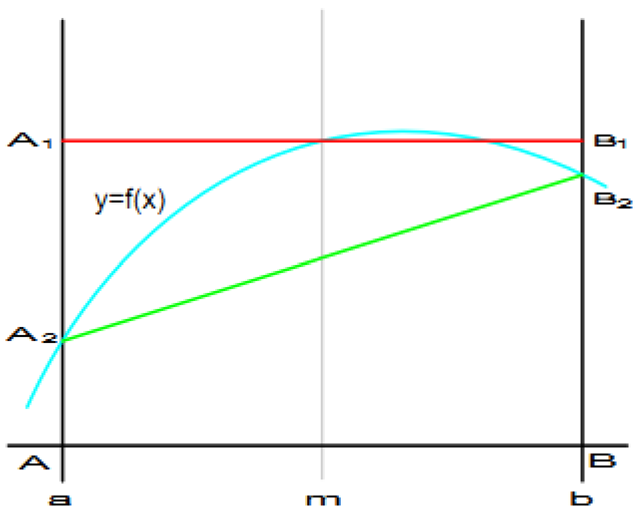
Let $m$ be the mid-point between $a$ and $b$, that is $m = (a+b)/2$, then the mid-point rule approximates $I$ by

$$I \approx M = (b-a)\,f(m)$$

and the error in this approximation can be shown to be

$$\mathrm{E}_M = I - M = \frac{(b-a)^3}{24} f^{(2)}(t_m)$$

where $f^{(p)} = \dfrac{d^p f}{dx^p}$ and $t_m$ is some point between $a$ and $b$.
We can derive this formula by approximating the function $f(x)$ by a constant that is exact at the midpoint and then integrating this approximation from $a$ to $b$, see figure 1. Thus we approximate $I$ by the area of the rectangle $ABB_1A_1$ namely $AB \times BB_1$ which is $(b-a)\,f(m)$.




### Method 2 The trapezium rule

The trapezium rule approximates $I$ by the area of the trapezium $ABB_2A_2$ indicated in figure 1. Thus

$$I \approx T = \frac{(b-a)}{2}\left(f(a) + f(b)\right).$$

The error in this approximation can be shown to be

$$E_T = I - T = -\frac{(b-a)^3}{12} f^{(2)}(t_T).$$

This rule can also be derived by approximating $f(x)$ by a linear function $L_1(x)$ which is exact at the points $x = a$ and $x = b$ and then integrating this approximation from $a$ to $b$.

To show this derivation we note that

$$L_1(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

and therefore

$$\begin{aligned} T &= \int_a^b L_1(x)\,dx = \frac{f(a)}{(a-b)}\int_a^b (x-b)\,dx + \frac{f(b)}{(b-a)}\int_a^b (x-a)\,dx \\ &= \frac{f(a)}{(a-b)}\left[\frac{x^2}{2} - bx\right]_a^b + \frac{f(b)}{(b-a)}\left[\frac{x^2}{2} - ax\right]_a^b = \frac{1}{2}(b-a)(f(a)+f(b)). \end{aligned}$$

**Method 3 Simpson's rule**

Simpson's rule provides an approximation $S$ to $I$ by finding a quadratic function $L_2(x)$ that approximates $f(x)$ at $a$, $b$ and $m$ and integrating this approximation from $a$ to $b$. The quadratic function is

$$L_2(x) = f(a)\frac{(x-m)(x-b)}{(a-m)(a-b)} + f(m)\frac{(x-a)(x-b)}{(m-a)(m-b)} + f(b)\frac{(x-a)(x-m)}{(b-a)(b-m)}.$$

Thus, after some algebraic manipulation,

$$S = \int_a^b L_2(x)\,dx = \frac{(b-a)}{6}(f(a) + 4f(m) + f(b))$$

and it can be shown that

$$E_s = I - S = -\frac{(b-a)^5}{2880}f^{(4)}(t_s).$$

**Example**

We shall approximate the integral

$$I = \int_0^1 \frac{1}{1+x^2}\,dx$$

which has exact value 0.78540 by each of the above formulae. The three rules give the approximations $M$, $T$ and $S$ with errors denoted by $E_M$, $E_T$ and $E_S$ respectively.

$$\begin{aligned} M &= \tfrac{1}{1+\left(\frac{1}{2}\right)^2} = \tfrac{4}{5} = 0.8 & E_M &= -0.0146 \\ T &= \tfrac{1}{2}\left(\tfrac{1}{1+0^2} + \tfrac{1}{1+1^2}\right) = 0.75 & E_T &= +0.0354 \\ S &= \tfrac{1}{6}\left(\tfrac{1}{1+0^2} + 4\tfrac{1}{1+\left(\frac{1}{2}\right)^2} + \tfrac{1}{1+1^2}\right) = 0.7833 & E_S &= +0.0021 \end{aligned}$$

We can make the following comments about these results. We would expect that the more work we do, the better the result. Moreover, the formula based on the most accurate interpolation would be expected to give the best answers. Thus it is no surprise that $S$ is the most accurate.

## 5.3 Improving the accuracy in numerical integration

To get a more accurate approximation we could integrate a more accurate polynomial approximation to $f(x)$ but this is seldom done. A common approach is to use the so-called composite formulae. The basic idea is to split up the range of integration into a number of smaller ranges (usually of equal length) and use either the **Trapezium rule** or **Simpson's rule** in each of the smaller intervals. For example, we get the formula $T_1 = T$ if we have just one interval, the formula $T_2$ in two intervals, the formula $T_n$ in $n$ intervals.

Thus, letting $x_0 = a$, $x_i = a + ih$, $x_n = b$ ($\therefore\ h = (b-a)/n$) and using the **trapezium rule** in each interval we obtain:

$$\begin{aligned}\int_a^b f dx &= \int_{x_0}^{x_1} f dx + \int_{x_1}^{x_2} f dx + \cdots + \int_{x_{n-1}}^{x_n} f dx \\ &\approx \frac{h}{2}\left(f(x_0) + f(x_1)\right) + \frac{h}{2}\left(f(x_1) + f(x_2)\right) + \cdots + \frac{h}{2}\left(f(x_{n-1}) + f(x_n)\right) \\ &= \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \cdots + 2f_{n-1} + f_n\right) = T_n.\end{aligned}$$

We can show that

$$E_{T_n} = I - T_n = -\frac{h^2}{12}(b-a) f^{(2)}(\theta_n).$$

Thus, if we compute $T_n$ and $T_{2n}$ we would expect the error to decrease by approximately a factor of 4.
**Example** Returning to the example above and using more trapeziums yields,

$$\begin{array}{lclcl} T &=& 0.75000 & E_{T_1} &=& 0.03540 \\ T_2 &=& \frac{\frac{1}{2}}{2}\left(\frac{1}{1+0^2} + 2\frac{1}{1+\left(\frac{1}{2}\right)^2} + \frac{1}{1+1^2}\right) = 0.7750 & E_{T_2} &=& 0.01040 \\ T_4 &=& \frac{\frac{1}{4}}{2}\left(\frac{1}{1+0^2} + 2\frac{1}{1+\left(\frac{1}{4}\right)^2} + 2\frac{1}{1+\left(\frac{2}{4}\right)^2} + 2\frac{1}{1+\left(\frac{3}{4}\right)^2} + \frac{1}{1+1^2}\right) & E_{T_4} &=& 0.00261 \\ &=& 0.78729 & & & \\ T_8 &=& 0.78475 & E_{T_8} &=& 0.00065 \\ T_{16} &=& 0.78524 & E_{T_{16}} &=& 0.00016 \end{array}$$

We observe that $E_{T_{2n}} \approx \frac{1}{4} E_{T_n}$.

We can do the same sort of thing with **Simpson's rule** except we define $2n+1$ points in $(a, b)$ by $x_0 = a$, $x_i = a + ih$, $x_n = a + 2nh$, ($\therefore\ h = (b-a)/(2n)$) and apply Simpson's rule in each of the intervals $(x_{2i-2}, x_{2i})$ giving the composite Simpson's rule $S_{2n}$ defined by

$$S_{2n} = \frac{h}{3}\left(f_0 + 4\sum_{odd} f_i + 2\sum_{even<2n} f_i + f_{2n}\right).$$

We can show that the error term has the form

$$E_{S_{2n}} = I - S_{2n} = -\frac{h^4}{180}(b-a) f^{(4)}(\sigma_{2n})$$

and we expect the error to decrease by about a factor of 16 when we halve $h$.

## Example

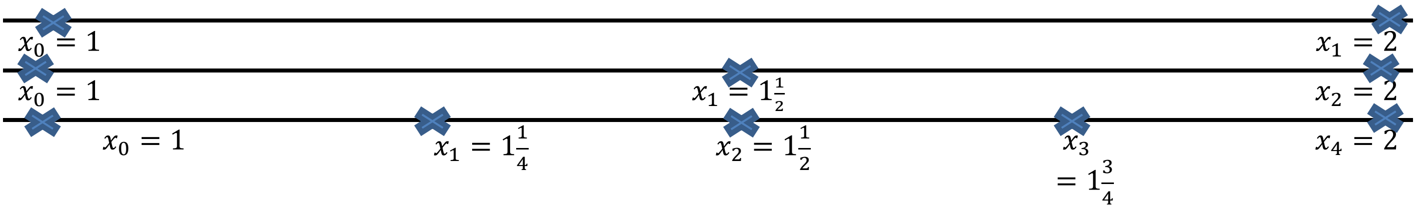
$$\begin{array}{lclcl} S_2 &=& S = 0.78333333 & E_{S_2} &=& 0.00206483 \\ S_4 &=& 0.78539216 & E_{S_4} &=& 0.00000601 \\ S_8 &=& 0.78539813 & E_{S_8} &=& 0.00000004 \\ S_{16} &=& 0.78539816 & & & \text{exact to 8dp} \end{array}$$

## Integrasi Romberg

Ini merupakan proses untuk mendapatkan hasil integrasi yang lebih akurat menggunakan teknik Ekstrapolasi Richardson. Misal hitung $\int_1^2 \frac{dx}{x}$ menggunakan metode Trapesium komposit.

Untuk h=1, $\int_1^2 \frac{dx}{x} \approx T(h) = \frac{1}{2}\big(f(1) + f(2)\big) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$.

Untuk $h = \frac{1}{2}$, $\int_1^2 \frac{dx}{x} \approx S\left(\frac{1}{2}h\right) = \frac{1/2}{3}\left(f(1) + 4f\left(\frac{3}{2}\right) + f(2)\right) = \frac{17}{24}$




Dengan menggunakanTeknik Ekstrapolasi Richardson atau bisa disebut Integrasi Romberg:

$$A = \frac{1}{3}\left(4S\left(\frac{1}{2}h\right) - T(h)\right) = 0{,}6944$$

Demikian juga untuk meningkatkan akurasi dapat menggunakan matriks berikut:

| | | | |
|---|---|---|---|
| $R_{0,0}$ | | | |
| $R_{1,0}$ | $R_{1,1}$ | | |
| $R_{2,0}$ | $R_{2,1}$ | $R_{2,2}$ | |
| $R_{3,0}$ | $R_{3,1}$ | $R_{3,2}$ | $R_{3,3}$ |

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$$R_{0,0} = \frac{b-a}{2}(f(a) + f(b))$$

$$R_{j,0} = \frac{R_{j-1,0}}{2} + h_j \sum_{i=1}^{n/2} f(x_{2i-1})$$

Penambahan nilai fungsi pada titik-titik berindeks ganjil, sedangkan nilai fungsi pada titik-titik berdindeks genap sudah termasuk dalam $R_{j-1,0}$.

$$R_{j,k} = \frac{4^k R_{j,k-1} - R_{j-1,k-1}}{4^k - 1}, k = 1,2,\cdots,j.$$

Kriteria penghentian iterasi : $\left|\frac{R_{j,j} - R_{j-1,j-1}}{R_{j-1,j-1}}\right| < \varepsilon$.