Hampiran Turunan

Secara analitik, turunan suatu fungsi di suatu titik ditentukan oleh nilai limit dari

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

Komputer tidak dapat melakukan perhitungan dengan limit sehingga digunakan hampiran nilai turunan yang diperoleh dari Deret Taylor

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

Untuk turunan pertama gunakan hampiran sebagai berikut:

$$f'(a) = \frac{f(a+h) - f(a)}{h} - \frac{h}{2!} f''(a) + \cdots$$

di mana komponen sisa mengandung $O(h)$ sehingga

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

adalah hampiran turunan Beda Hingga dengan galat orde 1.

Beberapa jenis hampiran dengan Beda Hingga:

Misal ditanyakan $f'(x_0) =?$

a. Beda Maju
   Gunakan $(x_0, y_0), (x_1, y_1)$,

$$f'(x_0) \approx \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

   Yang diperoleh dari uraian sebelumnya.

b. Beda Mundur
   Gunakan $(x_0, y_0), (x_{-1}, y_{-1})$, dimana

$$f(x_{-1}) = f(x_0) - hf'(x_0)$$

   Sehingga : $f'(x_0) \approx \frac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}} = \frac{y_0 - y_{-1}}{x_0 - x_{-1}}$

c. Beda Pusat
   Gunakan $(x_1, y_1), (x_{-1}, y_{-1})$, dimana

$$f'(x_0) \approx \frac{f(x_0) - f(x_{-1})}{x_0 - x_{-1}} = \frac{y_0 - y_{-1}}{x_0 - x_{-1}}$$

Aproksimasilah f_0'' dengan data dari (x_0, f_0), (x_1, f_1) dan (x_2, f_2).
Jawab:
Di sini kita gunakan persamaan (4) dan (5) dari halaman (2).
Kita eliminasi faktor f_0' dengan cara: pers$(5) - 2$ pers(4)

$$f_2 - 2f_1 = -f_0 + \frac{2h^2}{2!} f_0'' + \frac{6h^3}{3!} f_0''' + \cdots$$

$$f_0'' = \frac{f_0 - 2f_1 + f_2}{h^2} - \frac{6h}{3!} f_0''' + \cdots$$

$$f_0'' = \frac{f_0 - 2f_1 + f_2}{h^2} \quad \text{dengan galat} \quad O(h)$$

Penerapan persamaan beda pada persamaan diferensial :

Misal diketahui persamaan diferensial sebagai berikut:

$y'' - y' + 2y = 0.$ (*)

Misal digunakan persamaan beda maju untuk diskritisasi y' dan y'' seperti pada kasus 1 dan 3 di atas, di titik ke-n (x_n, y_n), akan diperoleh

$$\frac{y_n - 2y_{n+1} + y_{n+2}}{h^2} - \frac{y_{n+1} - y_n}{h} + 2y_n = 0$$

sehingga menjadi

$$y_{n+2} + (-2 - h)y_{n+1} + (1 - h + 2h^2)y_n = 0$$

yang merupakan persamaan beda untuk persamaan diferensial (*).

Pada bab berikutnya akan diterangkan mengenai penyelesaian persamaan beda yang ditulis dalam rumus rekursif (nilai sekarang bergantung pada nilai pada titik/waktu sebelumnya) menjadi persamaan dalam indeks n.

1.3 Difference equations

Whereas a differential equation is an equation in an unknown function, a *difference equation* is an equation in an unknown *sequence*. For example, suppose we know that a certain sequence of numbers $y_0, y_1, y_2, \ldots$ satisfies the following conditions:

$$y_{n+2} + 5y_{n+1} + 6y_n = 0 \qquad n = 0, 1, 2, \ldots \tag{1.3.1}$$

and furthermore that $y_0 = 1$ and $y_1 = 3$.

Evidently, we can compute as many of the y_n's as we need from (1.3.1), thus we would get $y_2 = -21$, $y_3 = 87$, $y_4 = -309$ so forth. The entire sequence of y_n's is determined by the difference equation (1.3.1) together with the two starting values.

$y = \alpha^n$, where α is a constant.

Can we somehow "solve" a difference equation by obtaining a formula for the values of the solution sequence? The answer is that we can, as long as the difference equation is linear and has constant coefficients, as in (1.3.1).

$$\alpha^{n+2} + 5\alpha^{n+1} + 6\alpha^n = \alpha^n(\alpha^2 + 5\alpha + 6) = 0. \tag{1.3.2}$$

Just as we were able to cancel the common factor $e^{\alpha x}$ in the differential equation case, so here we can cancel the α^n, and we're left with the quadratic equation

$$\alpha^2 + 5\alpha + 6 = 0. \tag{1.3.3}$$

The two roots of this *characteristic equation* are $\alpha = -2$ and $\alpha = -3$. Therefore the sequence $(-2)^n$ satisfies (1.3.1) and so does $(-3)^n$. Since the difference equation is linear, it follows that

$$y_n = c_1(-2)^n + c_2(-3)^n \tag{1.3.4}$$

is also a solution, whatever the values of the constants c_1 and c_2.

When we take account of the given data $y_0 = 1$ and $y_1 = 3$, we get the two equations

$$\left\{ \begin{array}{lcl} 1 & = & c_1 + c_2 \\ 3 & = & (-2)c_1 + (-3)c_2 \end{array} \right. \tag{1.3.5}$$

from which $c_1 = 6$ and $c_2 = -5$. Finally, we use these values of c_1 and c_2 in (1.3.4) to get

$$y_n = 6(-2)^n - 5(-3)^n \qquad n = 0, 1, 2, \ldots. \tag{1.3.6}$$

Equation (1.3.6) is the desired formula that represents the unique solution of the given difference equation together with the prescribed starting values.

So much for the equation (1.3.1). Now let's look at the general case, in the form of a linear difference equation of order p:

$$y_{n+p} + a_1 y_{n+p-1} + a_2 y_{n+p-2} + \cdots + a_p y_n = 0. \tag{1.3.7}$$

We try a solution of the form $y_n = \alpha^n$, and after substituting and canceling, we get the characteristic equation

$$\alpha^p + a_1\alpha^{p-1} + a_2\alpha^{p-2} + \cdots + a_p = 0. \tag{1.3.8}$$

Let α^* be one of these p roots. If α^* is simple (*i.e.*, has multiplicity 1) then the part of the general solution that corresponds to α^* is $c(\alpha^*)^n$. If, however, α^* is a root of multiplicity $k > 1$ then we must multiply the solution $c(\alpha^*)^n$ by an arbitrary polynomial in n, of degree $k-1$, just as in the corresponding case for differential equations we used an arbitrary polynomial in x of degree $k-1$.

We illustrate this, as well as the case of complex roots, by considering the following difference equation of order five:

$$y_{n+5} - 5y_{n+4} + 9y_{n+3} - 9y_{n+2} + 8y_{n+1} - 4y_n = 0. \tag{1.3.9}$$

This example is rigged so that the characteristic equation can be factored as

$$(\alpha^2 + 1)(\alpha - 2)^2(\alpha - 1) = 0 \tag{1.3.10}$$

from which the roots are obviously i, $-i$, 2 (multiplicity 2), 1.

Corresponding to the roots i, $-i$, the portion of the general solution is $c_1 i^n + c_2(-i)^n$. Since

$$i^n = e^{in\pi/2} = \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right) \tag{1.3.11}$$

and similarly for $(-i)^n$, we can also take this part of the general solution in the form

$$c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right). \tag{1.3.12}$$

The double root $\alpha = 2$ contributes $(c_3 + c_4 n)2^n$, and the simple root $\alpha = 1$ adds c_5 to the general solution, which in its full glory is

$$y_n = c_1 \cos\left(\frac{n\pi}{2}\right) + c_2 \sin\left(\frac{n\pi}{2}\right) + (c_3 + c_4 n)2^n + c_5. \tag{1.3.13}$$

The five constants would be determined by prescribing five initial values, say y_0, y_1, y_2, y_3 and y_4, as we would expect for the equation (1.3.9).

Stability theory of difference equations

In the previous section we discussed the stability of differential equations. The key ideas were that such an equation is stable if every one of its solutions remains bounded as t approaches infinity, and strongly stable if the solutions actually approach zero.

Similar considerations apply to difference equations, and for similar reasons. As an example, take the equation

$$y_{n+1} = \frac{5}{2}y_n - y_{n-1} \qquad (n \geq 1) \tag{1.6.1}$$

along with the initial equations

$$y_0 = 1; \quad y_1 = 0.5\,. \tag{1.6.2}$$

It's easy to see that the solution is $y_n = 2^{-n}$, and of course, this is a function that rapidly approaches zero with increasing n.

Now let's change the initial data (1.6.2), say to

$$y_0 = 1; \quad y_1 = 0.50000001 \tag{1.6.3}$$

instead of (1.6.2).

The solution of the difference equation with these new data is

$$y = (0.0000000066\ldots)2^n + (0.9999999933\ldots)2^{-n}. \tag{1.6.4}$$

The point is that the coefficient of the growing term 2^n is small, but 2^n grows so fast that after a while the first term in (1.6.4) will be dominant. For example, when $n = 30$, the solution is $y_{30} = 7.16$, compared to the value $y_{30} = 0.0000000009$ of the solution with the original initial data (1.6.2). A change of one part in fifty million in the initial condition produced, thirty steps later, an answer one billion times as large.

The fault lies with the difference equation, because it has both rising and falling components to its general solution. It should be clear that it is hopeless to do extended computation with an unstable difference equation, since a small roundoff error may alter the solution beyond recognition several steps later.

As in the case of differential equations, we'll say that a difference equation is *stable* if every solution remains bounded as n grows large, and that it is *strongly stable* if every solution approaches zero as n grows large. Again, we emphasize that *every* solution must be well behaved, not just the solution that is picked out by a certain set of initial data. In other words, the stability, or lack of it, is a property of the equation and not of the starting values.

Now consider the case where the difference equation is linear with constant coefficients. The we know that the general solution is a sum of terms of the form

$$(\text{polynomial in } n)\alpha^n. \tag{1.6.5}$$

Under what circumstances will such a term remain bounded or approach zero?

Suppose $|\alpha| < 1$. Then the powers of α approach zero, and multiplication by a polynomial in n does not alter that conclusion. Suppose $|\alpha| > 1$. Then the sequence of powers grows unboundedly, and multiplication by a nonzero polynomial only speeds the parting guest.

Theorem 1.6.1 *A linear difference equation with constant coefficients is stable if and only if all of the roots of its characteristic equation have absolute value at most 1, and those of absolute value 1 are simple. The equation is strongly stable if and only if all of the roots have absolute value strictly less than 1.*

Exercises

Determine, for each of the following difference equations whether it is strongly stable, stable, or unstable.

(a) $y_{n+2} - 5y_{n+1} + 6y_n = 0$

(b) $8y_{n+2} + 2y_{n+1} - 3y_n = 0$

(c) $3y_{n+2} + y_n = 0$

(d) $3y_{n+3} + 9y_{n+2} - y_{n+1} - 3y_n = 0$

(e) $4y_{n+4} + 5y_{n+2} + y_n = 0$