The Ramsey numbers of large cycles versus wheels

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Abstract
In this paper we show that the Ramsey number $R(C_n, W_m) = 2n - 1$ for even $m$ and $n \geq \frac{5m}{2} - 1$.

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1. Introduction

Throughout the paper, all graphs are finite and simple. Let $G$ be such a graph. We write $V(G)$ or $V$ for the vertex set of $G$ and $E(G)$ or $E$ for the edge set of $G$. The graph $\overline{G}$ is the complement of the graph $G$, i.e., the graph obtained from the complete graph $K_{|V(G)|}$ on $|V(G)|$ vertices by deleting the edges of $G$.

The graph $H = (V', E')$ is a subgraph of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. For any non-empty subset $S \subseteq V$, the subgraph induced by $S$ is the maximal subgraph of $G$ with the vertex set $S$; it is denoted by $G[S]$. If graphs $G$ and $H$ are isomorphic we shall denote this by $G \cong H$.

If $e = \{u, v\} \in E$ (in short, $e = uv$), then $u$ is called adjacent to $v$, and $u$ and $v$ are called neighbors. For $x \in V$ and a subgraph $B$ of $G$, define $N_B(x) = \{y \in V(B) : xy \in E\}$ and $N_B[x] = N_B(x) \cup \{x\}$. The degree $d_G(x)$ of a vertex $x$ is $|N_G(x)|$; $\delta(G)$ denotes the minimum degree in $G$.

The connectivity $\kappa(G)$ of a graph $G$ is defined as the minimum value of $|U|$ among all subsets $U \subseteq V(G)$ such that $G - U$ is either disconnected or trivial.

A cycle $C_n$ of length $n \geq 3$ is a connected graph on $n$ vertices in which every vertex has degree two. A wheel $W_n$ is a graph on $n + 1$ vertices obtained from a $C_n$ by adding one vertex $x$, called the hub of the wheel, and making $x$ adjacent to all vertices of $C_n$, called the rim of the wheel. If $G$ contains cycles, let $c(G)$ be the circumference of $G$, that is, the length of a longest cycle, and $g(G)$ be the girth, that is, the length of a shortest cycle. A graph on $n$ vertices is pancyclic if it contains cycles of every length $l$, $3 \leq l \leq n$. A graph is weakly pancyclic if it contains cycles of length from the girth to the circumference.

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Theorem 1. \( R(C_n, W_m) = 2n - 1 \) for even \( m \) and \( n \geq 5m/2 - 1 \).

For given graphs \( G \) and \( H \), the *Ramsey number* \( R(G, H) \) is the smallest positive integer \( N \) such that for every graph \( F \) of order \( N \) the following holds: either \( F \) contains \( G \) as a subgraph or the complement of \( F \) contains \( H \) as a subgraph.

Several results have been obtained for cycles versus wheels. For instance, Burr and Erdős [3] showed that \( R(C_3, W_m) = 2m + 1 \) for each \( m \geq 5 \). Later Hendry [8] showed \( R(C_5, W_4) = 9 \). Radziszowski and Xia [12] gave a simple and unified method to establish the Ramsey number \( R(C_3, G) \), where \( G \) is either a path, a cycle or a wheel. Jayawardane and Rousseau [9] showed \( R(C_5, W_5) = 11 \), and Surahmat et al. [15] showed \( R(C_4, W_m) = 9, 10 \) and 9 for \( m = 4, 5 \) and 6, respectively. Independently, Tse [16] showed \( R(C_4, W_m) = 9, 10, 9, 11, 12, 13, 14, 15 \) and 17 for \( m = 4, 5, 6, 7, 8, 9, 10, 11 \) and 12, respectively. In general case, Zhou [17] proved \( R(C_n, W_m) = 2m + 1 \) if \( n \) is odd and \( m \geq 5n - 7 \). Recently, in [14] the Ramsey numbers of cycles versus small wheels were obtained, e.g., \( R(C_n, W_4) = 2n - 1 \) for \( n \geq 5 \) and \( R(C_n, W_5) = 3n - 2 \) for \( n \geq 5 \).

More information about the Ramsey numbers of other graph combinations can be found in the survey [11].

The aim of this paper is to determine the Ramsey number of large cycles \( C_n \) versus wheels \( W_m \) for even \( m \). The Zhou results naturally complements the theorem in the present paper by handling a large number of cases where \( m \) is much bigger than \( n \), whereas this paper handles cases where \( n \) is much bigger than \( m \). The main result of this paper is the following.

**Theorem 1.** \( R(C_n, W_m) = 2n - 1 \) for even \( m \) and \( n \geq 5m/2 - 1 \).

For given graphs \( G \) and \( H \), Chvátal and Harary [5] established the lower bound \( R(G, H) \geq (\zeta(G) - 1)(\chi(H) - 1) + 1 \), where \( \zeta(G) \) is the number of vertices of the largest component of \( G \) and \( \chi(H) \) is the chromatic number of \( H \). In particular, if \( G = C_n \) and \( H = W_m \) for even \( m \), then we have \( R(C_n, W_m) \geq 2n - 1 \). In order to prove this theorem, we need the following known results and lemmas.

**2. Some lemmas**

Some lemmas in what follows will be used to prove the main result of this paper.

**Proposition 1** (Faudree and Schelp [7], Rosta [13]).

\[
R(C_n, C_m) = \begin{cases} 
2n - 1 & \text{for } 3 \leq m \leq n, \text{ } m \text{ odd, } (n, m) \neq (3, 3); \\
n + m/2 - 1 & \text{for } 4 \leq m \leq n, \text{ } m \text{ even and } n \text{ even, } (n, m) \neq (4, 4); \\
\max\{n + m/2 - 1, 2m - 1\} & \text{for } 4 \leq m < n, \text{ } m \text{ even and } n \text{ odd.}
\end{cases}
\]

**Lemma 1** (Bondy [1]). Let \( G \) be a graph of order \( n \). If \( \delta(G) \geq n/2 \) then either \( G \) is pancyclic or \( n \) is even and \( G \cong K_{n/2,n/2} \).

**Lemma 2** (Brandt et al. [2]). Every non-bipartite graph \( G \) of order \( n \) with \( \delta(G) \geq (n + 2)/3 \) is weakly pancyclic and has girth \( 3 \) or \( 4 \).

**Lemma 3** (Dirac [6]). Let \( G \) be a 2-connected graph of order \( n \geq 3 \) with \( \delta(G) = \delta \). Then \( c(G) \geq \min\{2\delta, n\} \).

**Lemma 4** (Chvátal and Erdős [4], Zhou [17]). If \( H = C_s \subseteq F \) for a graph \( F \), while \( F \not\supseteq C_{s+1} \) and \( \overline{F} \not\supseteq K_r \), then \( |N_H(x)| \leq r - 2 \) for each \( x \in V(F) \setminus V(H) \).

**Lemma 5.** Let \( F \) be a graph with \( |V(F)| \geq R(C_n, C_m) + 1 \). If there is a vertex \( x \in V(F) \) such that \( |N_F[x]| \leq |V(F)| - R(C_n, C_m) \) and \( F \not\supseteq C_n \), then \( \overline{F} \supseteq W_m \).

**Proof.** Let \( A = V(F) \setminus N_F[x] \) and so \( |A| \geq R(C_n, C_m) \). If the subgraph \( F[A] \) of \( F \) induced by \( A \) contains no \( C_n \), then by the definition of \( R(C_n, C_m) \) we get that \( \overline{F}[A] \) contains a \( C_m \) and hence \( \overline{F} \) contains a \( W_m \) (with hub \( x \)). \( \square \)

**Lemma 6.** Let \( F \) be a graph with \( 2n - 1 \) vertices without a \( C_n \). If \( \overline{F} \) contains no \( W_m \), then \( \delta(F) \geq n - m/2 \) for even \( m \geq 4 \) and \( n \geq 3m/2 \).
Proof. Suppose \( \delta(F) < n - m/2 \) for some even \( m \geq 4 \) and \( n \geq 3m/2 \). Then, there exists a vertex \( x \in V(F) \) such that \( |N_F[x]| = d_F(x) + 1 = \delta(F) + 1 < n - m/2 = (2n - 1) - (n + (m/2) - 1) \). Using Proposition 1 we get that \( |N_F[x]| \leq |V(F)| - R(C_n, C_m) \). By Lemma 5, we conclude that \( \overline{F} \) contains a \( W_m \) with hub \( x \). \( \square \)

3. Proof of Theorem

Proof of Theorem 1. Let \( G \) be a graph of order \( 2n - 1 \) for even \( m \) and \( n \geq 5m/2 - 1 \) containing no \( C_n \). We shall show that \( \overline{G} \) contains \( W_m \).

By contradiction, suppose \( \overline{G} \) contains no \( W_m \). By Lemma 6, we have \( \delta(G) \geq n - m/2 \). Now we shall distinguish two cases below.

Case 1: \( G \) is non-bipartite. Since \( \delta(G) \geq n - m/2 \geq (2n - 1)/3 \), by Lemma 2, we get that \( G \) is weakly pancyclic with girth 3 or 4. If \( \kappa(G) \geq 2 \) then \( G \) is a 2-connected graph. By Lemma 3, we have \( c(G) \geq \min\{2n - m, 2n - 1\} \). This implies that \( G \) contains \( C_n \), a contradiction.

Let \( \kappa(G) = 1 \). There exists a cut vertex \( v \in V(G) \) such that \( G - v \) is disconnected. Let \( G_1, \ldots, G_r \) be the components of \( G - v \). Since \( \delta(G) \geq n - m/2 \) we deduce \( \delta(G_i) \geq n - (m/2) - 1 \). This implies \( |V(G_i)| \geq n - m/2 \). We can write \( 2n - 1 = |V(G)| = 1 + \sum_{i=1}^{r}|V(G_i)| \geq 1 + 3(n - m/2) \) for \( r \geq 3 \). But this is equivalent to \( n \leq 3m/2 - 2 \), a contradiction, since \( n \geq 5m/2 - 1 \). It follows that \( r = 2 \). Suppose that \( |V(G_1)| = p \). Because \( 2n - 1 = 1 + |V(G_1)| + |V(G_2)| \) one deduces \( p = 2n - 2 - |V(G_2)| \leq n + (m/2) - 2 \). Since \( n \geq 5m/2 - 1 \) we obtain \( \delta(G_1) \geq n - (m/2) - 1 \), \( p/2 \), hence Lemma 1 applies again for both \( G_1 \) and \( G_2 \). In a similar way we deduce that \( G \) contains \( W_m \), a contradiction, or both \( G_1 \) and \( G_2 \) are pancyclic, hence \( G_1 \supseteq C_n \) for each \( i \in \{1, 2\} \). Let \( \varepsilon = \max\{|N_{G_1}(v)|, |N_{G_2}(v)|\} \). By Lemma 4, to avoid a \( K_{m+1} \) (and then \( W_m \)) in \( \overline{G}[V(G_1) \cup \{v\}] \) we have that \( \varepsilon \leq m - 1 \). So \( 2m - 2 \geq 2\varepsilon \geq |N_{G_1}(v)| + |N_{G_2}(v)| = |N_{G}(v)| \geq n - m/2 \geq (5m/2 - 1) - m/2 = 2m - 1 \), a contradiction.

Let \( \kappa(G) = 0 \). Then \( G \) is disconnected and we deduce as above that \( G \) has exactly two components, \( G_1 \) and \( G_2 \). Since \( \delta(G_i) \geq n - m/2 \) we get \( n - (m/2) + 1 \geq |V(G_i)| \geq n + (m/2) - 2 \) for each \( i \in \{1, 2\} \). If \( |V(G_1)| \geq |V(G_2)| \) we find that \( |V(G_1)| \geq n \) and \( \delta(G_1) \geq n - m/2 > |V(G_1)|/2 \). By Lemma 1 we get that \( G_1 \) is either pancyclic and so \( G_1 \supseteq C_n \), a contradiction, or \( G_1 \supseteq K_{p/2, p/2} \), where \( p = |V(G_1)| \). Since \( p/2 \geq n/2 > m \) we deduce as above, that \( \overline{G} \supseteq W_m \), a contradiction.

Case 2: \( G \) is bipartite. Since \( G \) is bipartite and \( \delta(G) \geq n - m/2 \), we deduce that \( G \) is isomorphic to a spanning subgraph of \( K_{j,t} \) for \( j \geq n - m/2 \) and \( t \geq n - m/2 \). This implies \( \overline{G} \supseteq W_m \), a contradiction, since \( E(\overline{G}) \supseteq E(K_j) \cup E(K_t) \) and \( n - m/2 > m + 1 \). This completes the proof. \( \square \)

4. Conjecture

Finally, we propose the following conjecture: \( R(C_n, W_m) = 3n - 2 \) for odd \( m \) and \( n \geq m \geq 3 \), \( (m, n) \neq (3, 3) \). This is true at least for \( m = 5 \) [14].

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