Expanding super edge-magic graphs *

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Abstract. For a graph G, with the vertex set V(G) and the edge set E(G) an edge-magic total labelling is a bijection f from V(G) ∪ E(G) to the set of integers \{1, 2, ..., |V(G)| + |E(G)|\} with the property that for each edge uv ∈ E(G) and for a fixed integer k, f(u) + f(v) + f(uv) = k.

An edge-magic total labelling f is called a super edge-magic total labelling if f(V(G)) = \{1, 2, ..., |V(G)|\} and f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, ..., |V(G)| + |E(G)|\}. In this paper we construct the expanded super edge-magic total graphs from cycles C_n, generalized Petersen graphs and generalized prisms.

Key words : Edge-magic, super edge-magic, magic sum.

1 Introduction

All graphs considered here are finite, undirected and simple. As usual, the vertex set and edge set will be denoted V(G) and E(G), respectively. The symbol |A| will denote the cardinality of the set A. Other terminologies or notations not defined here can be found in [2, 7, 15].

Edge-magic total labelling were introduced by Kotzig and Rosa [8] as follow. An edge-magic total labelling on G is a bijection f from V(G) ∪ E(G) onto \{1, 2, ..., |V(G)| + |E(G)|\} with the property that, given any edge uv,

\[ f(u) + f(v) + f(uv) = k \]

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for some constant \( k \). It will be convenient to call \( f(u) + f(v) + f(uv) \) the edge sum of \( uv \) and \( k \) the magic sum of \( f \). A graph is called edge-magic total if it admits any edge-magic total labelling.

Kotzig and Rosa [9] showed that no complete graph \( K_n \) with \( n > 6 \) is edge-magic total and neither is \( K_3 \), and edge-magic total labellings for \( K_3, K_5 \) and for \( K_6 \) for all feasible values of \( k \), are described in [14].

In [8] it is proved that every cycle \( C_n \), every caterpillar (a graph derived from a path by hanging any number of pendant vertices from vertices of the path) and every complete bipartite graph \( K_{m,n} \) (for any \( m \) and \( n \)) are edge-magic total.

Wallis et.al. [14] showed that all paths \( P_n \) and all \( n \)-suns (a cycle \( C_n \) with an additional edge terminating in a vertex of degree 1 attached to each vertex of the cycle) are edge-magic total. It was shown in [16] that the Cartesian product \( C_n \times P_m \) admits an edge-magic total labelling for \( n \) odd.

It is conjectured that all trees are edge-magic total [8] and all wheels \( W_n \) are edge-magic total whenever \( n \not\equiv 3 \pmod 4 \) [4]. Enomoto et.al. [4] showed that the conjectures are true for all trees with less than 16 vertices and wheels \( W_n \) for \( n \leq 30 \). Phillips et.al. [12] solved the conjecture partially by showing that a wheel \( W_n, n \equiv 0 \) or 1 (mod 4), is edge-magic total. Slamin et.al. [13] showed that for \( n \equiv 6 \) (mod 8) every wheel \( W_n \) has an edge-magic total labelling.

An edge-magic total labelling \( f \) is called super edge-magic total if \( f(V(G)) = \{1, 2, \ldots, |V(G)|\} \) and \( f(E(G)) = \{|V(G)| + 1, |V(G)| + 2, \ldots, |V(G)| + |E(G)|\} \). Enomoto et.al. [4] proved that the complete bipartite graphs \( K_{m,n} \) is super edge-magic total if and only if \( m = 1 \) or \( n = 1 \). They also proved the complete graphs \( K_n \) is super edge-magic if and only if \( n = 1, 2 \) or 3.

In this paper we will construct the super edge-magic total graphs by hanging any number of pendant vertices from vertices of the cycles, generalized prisms and generalized Petersen graphs.

2 Results

For \( n \geq 3 \) and \( p \geq 1 \) we denote by \( C_n + A_p \) a graph which is obtained by adding \( p \) vertices and \( p \) edges to one vertex of cycles \( C_n \) (say \( v_1 \)). The vertex set and the edge set of \( C_n + A_p \) are \( V(C_n + A_p) = \{v_i :
Let \((n, p)\)-sun be a graph derived from a cycle \(C_n, n \geq 3\), by hanging \(p\) pendant vertices from all vertices of the cycle. Let us denote the vertex set of \((n, p)\)-sun by \(V((n, p)\text{-sun}) = \{v_i : 1 \leq i \leq n\} \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}\) and the edge set by \(E((n, p)\text{-sun}) = \{v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{v_nv_1\} \cup \{v_1u_j : 1 \leq j \leq p\}\). Observe that \(|V((n, p)\text{-sun})| = |E((n, p)\text{-sun})| = n(p+1)\). The cycle \(C_n, n \geq 3\), is super edge-magic total if and only if \(n\) is odd (see [4]). Now, we shall investigate super edge-magic total labellings for graphs of \(C_n + A_p\) and \((n, p)\)-sun which are expanded from a cycle \(C_n\).

Define a vertex labelling \(f_1\) and an edge labelling \(f_2\) of \(C_n + A_p\) as follows,

\[
\begin{align*}
f_1(v_i) &= \begin{cases}  
\frac{n+i}{2} & \text{if } i \text{ is odd}, \\
\frac{i}{2} & \text{if } i \text{ is even}, 
\end{cases} \\
f_1(u_{i,j}) &= n + j \quad \text{for } 1 \leq j \leq p, \\
f_2(v_iv_{i+1}) &= 2(n+p) + 1 - i \quad \text{for } 1 \leq i \leq n-1, \\
f_2(v_nv_1) &= n + 2p + 1, \\
f_2(v_1u_j) &= n + 2p + 1 - j \quad \text{for } 1 \leq j \leq p.
\end{align*}
\]

**Theorem 1.** If \(n\) is odd, \(n \geq 3\) and \(p \geq 1\), then graph \(C_n + A_p\) is super edge-magic total.

**Proof.** It easy to verify that the values of \(f_1\) are \(1, 2, ..., n+p\) and the values of \(f_2\) are \(n+p+1, n+p+2, ..., 2n+2p\) and furthermore the common edge sum is \(k = 2p + \frac{5n+3}{2}\).

**Theorem 2.** If \(n\) is odd, \(n \geq 3\) and \(p \geq 1\), then \((n, p)\)-sun is super edge-magic total.

**Proof.** Label the vertices and the edges of \((n, p)\)-sun in the following way.

\[
\begin{align*}
f_3(v_i) &= f_1(v_i) \quad \text{for } 1 \leq i \leq n, \\
f_3(u_{1,j}) &= n_j + 1 \quad \text{for } 1 \leq j \leq p, \\
f_3(u_{i,j}) &= n(j + 1) + 2 - i \quad \text{for } 2 \leq i \leq n \text{ and } 1 \leq j \leq p.
\end{align*}
\]
If $\alpha$, consider a bijection, $\beta$ has a super edge-magic total labelling.

**Theorem 3.** If $n$ is odd, $n \geq 5$ and $p \geq 1$ then the graph $P(n, 2) + A_p$ has a super edge-magic total labelling.

**Proof.** Consider a bijection, $f_5 : V(P(n, 2) + A_p) \rightarrow \{1, 2, ..., 2n + p\}$ where,

$$f_5(v_i) = \begin{cases} 
\frac{n + i}{2} & \text{if } i \text{ is even, } 2 \leq i \leq n - 1, \\
\frac{3n + i}{2} & \text{if } i \text{ is odd, } 1 \leq i \leq n,
\end{cases}$$

We can see that the vertices of $(n, p)$-sun are labelled by values $1, 2, ..., n, n+1$ and the edges are labelled by $n(p+1)+1, n(p+1)+2, ..., 2n(p+1)$. Furthermore, all edges have the same magic number $k = 2n(p + 1) + \frac{n+3}{2}$. ■

A generalized Petersen graph $P(n, m)$, $n \geq 3$ and $1 \leq m \leq \lceil \frac{n-1}{2} \rceil$, consists of an outer $n$-cycle $v_1, v_2, ..., v_n$, a set of $n$ spokes $v_i z_i$, $1 \leq i \leq n$, and $n$ inner edges $z_i z_{i+m}$, $1 \leq i \leq n$, with indices taken modulo $n$.

For $n \geq 5$, $m = 2$ and $p \geq 1$, we denote by $P(n, 2) + A_p$, a graph which is obtained by adding $p$ vertices and edges to one vertex of $P(n, 2)$, say $v_1$. Hence, $V(P(n, 2) + A_p) = V(P(n, 2)) \cup \{u_j : 1 \leq j \leq p\}$ and $E(P(n, 2) + A_p) = E(P(n, 2)) \cup \{v_i u_j : 1 \leq j \leq p\}$.

Let $P(n, 2, p)$ be a graph derived from $P(n, 2)$, $n \geq 5$, by hanging $p$ pendant vertices from all vertices $v_i$, $1 \leq i \leq n$ of $P(n, 2)$. Then the vertex set of $P(n, 2, p)$ is $V(P(n, 2, p)) = V(P(n, 2)) \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}$ and the edge set is $E(P(n, 2, p)) = E(P(n, 2)) \cup \{v_i u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}$.

In [11] it is proved that generalized Petersen graph $P(n, 2)$ is edge magic total. Fukuchi [6] showed that $P(n, 2)$ are super edge-magic total.
Define a bijection, \( f_5(z_i) = \begin{cases} 
\frac{n-i+4}{4} & \text{if } i \equiv 1 \pmod{4}, \\
\frac{2n-i+4}{4} & \text{if } i \equiv 2 \pmod{4}, \\
\frac{3n-i+4}{4} & \text{if } i \equiv 3 \pmod{4}, \\
\frac{4n-i+4}{4} & \text{if } i \equiv 0 \pmod{4}, 
\end{cases} \)

and for every edge \( u_j \) we have \( f_5(u_j) = 2n + j \) for \( 1 \leq j \leq p \).

We can observe that under the labelling \( f_5 \), \( \{ f_5(v_i) + f_5(v_{i+1}) : 1 \leq i \leq n \} = \{ \frac{5n+1}{2} + i : 1 \leq i \leq n \} \) and \( \{ f_5(z_i) + f_5(z_{i+2}) : 1 \leq i \leq n \} = \{ \frac{2n+1}{2} + i : 1 \leq i \leq n \} \) with indices taken modulo \( n \).

Moreover, \( \{ f_5(v_i) + f_5(z_i) : 1 \leq i \leq n \} = \{ \frac{2n+1}{2} + i : 1 \leq i \leq n \} \) and \( \{ f_5(v_1) + f_5(u_j) : 1 \leq j \leq p \} = \{ \frac{7n+1}{2} + j : 1 \leq j \leq p \} \). The elements of the set \( \{ f_5(v_i) + f_5(v_{i+1}) : 1 \leq i \leq n \} \cup \{ f_5(z_i) + f_5(z_{i+2}) : 1 \leq i \leq n \} \cup \{ f_5(v_i) + f_5(z_j) : 1 \leq i \leq n \} \cup \{ f_5(v_i) + f_5(u_j) : 1 \leq j \leq p \} \) form an arithmetic sequence \( \frac{n+1}{2} + 1, \frac{n+1}{2} + 2, ..., \frac{7n+1}{2}, \frac{7n+1}{2} + 1, ..., \frac{7n+1}{2} + p \).

We are able to arrange the values \( 2n+p+1, 2n+p+2, ..., 5n+2p \) to the edges of \( P(n, 2) + A_p \) in such way that the resulting labelling is total and for every edge \( xy \in E(P(n, 2) + A_p) \), \( f_5(x) + f_5(y) + f_5(x, y) = \frac{11n+3}{2} + 2p \). Thus we arrive at the desired result.

\[ \text{Theorem 4. If } n \text{ is odd, } n \geq 5 \text{ and } p \geq 1, \text{ then the graph } P(n, 2) \text{ has a super edge-magic total labelling.} \]

\[ \text{Proof. Define a bijection, } f_6 : V(P(n, 2, p)) \rightarrow \{ 1, 2, ..., n(p+2) \} \text{ as follows,} \]

\[ f_6(v_i) = f_5(v_i) \text{ and } f_6(z_i) = f_5(z_i) \text{ for } 1 \leq i \leq n, \]

\[ f_6(u_{1,j}) = n(j+1) + 1 \text{ for } 1 \leq j \leq p, \]

\[ f_6(u_{i,j}) = n(j+2) + 2 - i \text{ for } 2 \leq i \leq n \text{ and } 1 \leq j \leq p. \]

We can see that under the vertex labelling \( f_6 \) the values \( f_6(x) + f_6(y) \) of all edges \( xy \in E(P(n, 2, p)) \) constitute an arithmetic sequence \( \frac{n+1}{2} + 1, \frac{n+1}{2} + 2, ..., \frac{7n+1}{2}, \frac{7n+1}{2} + 1, ..., \frac{7n+1}{2} + np \). If we complete the edge labelling with the consecutive values in the set \( \{ n(p+2) + 1, n(p+2) + 2, n(p+2) + 3, ..., 5n+2np \} \) then we can obtain total labelling where \( f_6(x) + f_6(y) + f_6(xy) = \frac{11n+3}{2} + 2np \) for every edge \( xy \in E(P(n, 2, p)) \).
In the sequel we shall consider a graph of a generalized prism which can be defined as the Cartesian product $C_n \times P_m$ of a cycle on $n$ vertices with a path on $m$ vertices.

Let $V(C_n \times P_m) = \{v_{i,k} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m\}$ be the vertex set and $E(C_n \times P_m) = \{v_{i,k}v_{i,k+1} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m\} \cup \{v_{i,k}v_{i,k+1} : 1 \leq i \leq n \text{ and } 1 \leq k \leq m-1\}$ be the edge set, where $i$ is taken modulo $n$. For $n \geq 3$, $m \geq 2$ and $p \geq 1$, we will consider a graph $(C_n \times P_m) + A_p$ (respectively a graph $(C_n \times P_m) + \sum_{i=1}^{n} A_i^p$) which is obtained by adding $p$ vertices and $p$ edges to one vertex of $C_n \times P_m$, say $v_{1,m}$ (respectively to all vertices $v_{i,m}, 1 \leq i \leq n$ of $C_n \times P_m$). Thus $V((C_n \times P_m) + A_p) = V(C_n \times P_m) \cup \{u_j : 1 \leq j \leq p\}$, $V((C_n \times P_m) + \sum_{i=1}^{n} A_i^p) = V(C_n \times P_m) \cup \{u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}$, $E((C_n \times P_m) + A_p) = E(C_n \times P_m) \cup \{v_{1,m}u_j : 1 \leq j \leq p\}$, and $E((C_n \times P_m) + \sum_{i=1}^{n} A_i^p) = E(C_n \times P_m) \cup \{v_{i,m}u_{i,j} : 1 \leq i \leq n, 1 \leq j \leq p\}$.

Figueroa-Centeno et.al. [5] showed that the generalized prism $C_n \times P_m$ is super edge-magic if $n$ is odd and $m \geq 2$.

The next two theorems show super edge-magic total labellings of graphs $(C_n \times P_m) + A_p$ and $(C_n \times P_m) + \sum_{i=1}^{n} A_i^p$.

Theorem 5. If $n$ is odd, $n \geq 3$, $m \geq 2$ and $p \geq 1$, then the graph $(C_n \times P_m) + A_p$ has a super edge-magic total labelling.

Proof. If $m$ is even, $m \geq 2$, $1 \leq k \leq m$, $1 \leq i \leq n$, then we construct a vertex labelling $f_7$ in the following way,

$$f_7(v_{i,k}) = \begin{cases} 
  n(k-1) + \frac{i+1}{2} & \text{if } i \text{ is odd and } k \text{ is odd}, \\
  nk + \frac{i-n+1}{2} & \text{if } i \text{ is even and } k \text{ is odd}, \\
  nk + \frac{i-n}{2} & \text{if } i \text{ is odd and } k \text{ is even}, \\
  n(k-1) + \frac{i}{2} & \text{if } i \text{ is even and } k \text{ is even},
\end{cases}$$

$$f_7(u_j) = mn + j \text{ for } 1 \leq j \leq p.$$
If $m$ is odd, $m \geq 3$, $1 \leq k \leq m$, $1 \leq i \leq n$, then we define a vertex labelling $f_8$ as follows,

$$f_8(v_{i,k}) = \begin{cases} 
\frac{n+1}{2} + n(k-1) & \text{if } i \text{ is odd and } k \text{ is odd}, \\
\frac{n+1}{2} + n(k-1) & \text{if } i \text{ is even and } k \text{ is odd}, \\
k & \text{if } i = 1 \text{ and } k \text{ is even}, \\
n(k-1) + \frac{i-1}{2} & \text{if } i \text{ is odd and } k \text{ is even}, \\
n(k-1) + \frac{n+1}{2} & \text{if } i \text{ is even and } k \text{ is even},
\end{cases}$$

$$f_8(u_j) = mn + j \text{ for } 1 \leq j \leq p.$$

It is easy to verify that for each edge $xy \in E((C_n \times P_m) + A_p)$ the values $f_7(x) + f_7(y)$ and $f_8(x) + f_8(y)$ form an arithmetic sequence $\frac{n+1}{2} + 1, \frac{n+1}{2} + 2, ..., 2mn - \frac{n+1}{2}, 2mn - \frac{n-3}{2}, ..., 2mn - \frac{n-1}{2} + p$.

Let $f_9$ be a bijection from $E((C_n \times P_m) + A_p)$ onto $\{1, 2, ..., 2mn-n+p\}$. We can combine the vertex labelling $f_7$ or $f_8$ and the edge labelling $f_9 + mn + p$ such that the resulting labelling is total and the edge sum for each edge $xy \in E((C_n \times P_m) + A_p)$ is equal to $3mn + \frac{3-n}{2} + 2p$.

**Theorem 6.** If $n$ is odd, $n \geq 3$, $m \geq 2$ and $p \geq 1$, then the graph $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ has a super edge-magic total labelling.

**Proof.** Define vertex labelling $f_{10}$ and $f_{11}$ such that :

- $f_{10}(v_{i,k}) = f_7(v_{i,k})$ if $m$ is even, $1 \leq k \leq m$, $1 \leq i \leq n$,
- $f_{11}(v_{i,k}) = f_8(v_{i,k})$ if $m$ is odd, $1 \leq k \leq m$, $1 \leq i \leq n$,
- $f_{10}(u_{1,i}) = f_{11}(u_{1,i}) = n(m+j-1) + 1$ for $1 \leq j \leq p$,
- $f_{10}(u_{i,j}) = f_{11}(u_{i,j}) = n(m+j) - i + 2$ for $2 \leq i \leq n$ and $1 \leq j \leq p$.

We can see that vertices of $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ are labelled by values $1, 2, 3, ..., n(m+p)$ and $f_t(x) + f_t(y)$ for all edges $xy \in (C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ and $t \in \{10, 11\}$ constitute an arithmetic sequence $\frac{n+1}{2} + 1, \frac{n+1}{2} + 2, ..., 2mn - \frac{n-1}{2} + np$.

We can complete the consecutive the edge labelling of $(C_n \times P_m) + \sum_{i=1}^{n} A_p^i$ with values in the set $\{n(m+p)+1, n(m+p)+2, ..., n(3m+2p)-$
consecutively such that the common edge sum is \[ k = 3mn + 2pm - \frac{n-3}{2} \]. Thus the total labelling of \((C_n \times P_m) + \sum_{i=1}^{n} A_i^p\) is super edge-magic and the theorem is proved. ■

References