AN INNER PRODUCT THAT MAKES A SET OF VECTORS ORTHONORMAL

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Given a linearly independent set $\{a_1, \ldots, a_n\}$ in a real inner product space $(X, \langle \cdot, \cdot \rangle)$ of dimension $d \geq n$ (d may be infinite), an average undergraduate student can do Gram-Schmidt process to obtain an orthonormal set $\{a_1^*, \ldots, a_n^*\}$ from $\{a_1, \ldots, a_n\}$.

Now, leaving the set $\{a_1, \ldots, a_n\}$ as it is, how can we explicitly derive a new inner product $\langle \cdot, \cdot \rangle^*$ from the given inner product $\langle \cdot, \cdot \rangle$ such that, with respect to the new inner product, $\{a_1, \ldots, a_n\}$ becomes an orthonormal set in X?

Basically, this can be done in the following way. Let S be the subspace spanned by $\{a_1, \ldots, a_n\}$, P be the projection on S and $Q = I - P$ be its complementary projection. Next let T be a linear transformation that maps $\{a_1, \ldots, a_n\}$ to an orthonormal basis for S and define a new inner product $\langle \cdot, \cdot \rangle^*$ on X by

$$\langle x, y \rangle^* := \langle TPx, TPy \rangle + \langle Qx, Qy \rangle. \tag{1}$$

Then clearly we have $\langle a_i, a_j \rangle^* = \delta_{ij}$ for all $i, j \in \{1, \ldots, n\}$, that is, $\{a_1, \ldots, a_n\}$ is orthonormal with respect to $\langle \cdot, \cdot \rangle^*$.

To get an explicit formula for $\langle \cdot, \cdot \rangle^*$ from (1), however, one needs to work out the expressions for P and T and then plug them to (1). Here P will be of the form $P(x) = \sum_{i=1}^{n} \alpha_i a_i$ with $\alpha_i = \alpha_i(x, a_1, \ldots, a_n)$, $i = 1, \ldots, n$. Meanwhile T can be represented by an $n \times n$ matrix, but one has to choose or, most likely, construct an orthonormal basis for S first (this can be done, e.g., by applying the Gram-Schmidt

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process to $\{a_1, \ldots, a_n\}$ — oh, no!). Although possible, this will be demanding and might not be the best way to do it.

But is there an alternative way? Well, here is one that invokes the notion of n-inner products, which will be explained below.

Suppose that $n \geq 2$ and let V be a real vector space of dimension $d \geq n$. A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on V^{n+1} satisfying the following five properties:

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$; $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of (1, $\ldots, n$);

(I3) $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle, \ \alpha \in \mathbf{R}$;

(I5) $\langle x + x', y | x_2, \ldots, x_n \rangle = \langle x, y | x_2, \ldots, x_n \rangle + \langle x', y | x_2, \ldots, x_n \rangle$,

is called an *n-inner product* on V and the pair $(V, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an *n-inner product space.*

On an n-inner product space $(V, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$, the following function

$$\|x_1, x_2, \ldots, x_n\| := \langle x_1, x_1 | x_2, \ldots, x_n \rangle^{1/2}$$

defines an n-norm, which enjoys the following four properties:

(N1) $\|x_1, \ldots, x_n\| \geq 0$; $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;

(N2) $\|x_1, \ldots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|, \ \alpha \in \mathbf{R}$;

(N4) $\|x + y, x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|y, x_2, \ldots, x_n\|$.

Historically, the notion of n-norms were introduced earlier by Gähler in order to generalize the idea of length, area and volume in a real vector space (see [3], [4] and [5]). The concept of n-inner products were developed later in [1] and [2] (for $n = 2$) and [8] (for general $n \geq 2$).

Just like an inner product space, an n-inner product space $(V, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ has many nice properties. For instance, we have the Cauchy-Schwarz inequality

$$|\langle x, y|x_2,\dots,x_n\rangle| \le \|x, x_2,\dots,x_n\| \, \|y, x_2,\dots,x_n\|,$$

and the equality holds if and only if $x, y, x_2, \dots, x_n$ are linearly dependent (see [6]).

We also have the polarization identity

$$\|x+y, x_2,\dots,x_n\|^2 - \|x-y, x_2,\dots,x_n\|^2 = 4\langle x, y|x_2,\dots,x_n\rangle$$

and the parallelogram law

$$\|x+y, x_2,\dots,x_n\|^2 + \|x-y, x_2,\dots,x_n\|^2 = 2(\|x, x_2,\dots,x_n\|^2 + \|y, x_2,\dots,x_n\|^2),$$

which characterizes an n-norm that comes from an n-inner product.

Further, by the polarization identity and the property (I2), we have

$$\langle x, y|x_2,\dots,x_n\rangle = \langle x, y|x_{i_2},\dots,x_{i_n}\rangle$$

for every permutation $(i_2,\dots,i_n)$ of $(2,\dots,n)$. One can also verify that

$$\langle x, y|x_2,\dots,x_n\rangle = 0$$

whenever x or y is a linear combination of $x_2,\dots,x_n$. These two facts will be useful later.

On an inner product space $(V, \langle\cdot,\cdot\rangle)$, we can define an n-inner product $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle$ by

$$\langle x, y|x_2,\dots,x_n\rangle := \begin{vmatrix} \langle x,y\rangle & \langle x,x_2\rangle & \dots & \langle x,x_n\rangle \\ \langle x_2,y\rangle & \langle x_2,x_2\rangle & \dots & \langle x_2,x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n,y\rangle & \langle x_n,x_2\rangle & \dots & \langle x_n,x_n\rangle \end{vmatrix}. \tag{2}$$

Conversely, on an n-inner product space $(V, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$, we can define an inner product $\langle\cdot,\cdot\rangle^*$ with respect to a fixed linearly independent set $\{a_1,\dots,a_n\}$ in V by

$$\langle x, y\rangle^* := \kappa \sum_{\{i_2,\dots,i_n\}\subseteq\{1,\dots,n\}} \langle x, y|a_{i_2},\dots,a_{i_n}\rangle \tag{3}$$

for any $\kappa > 0$ (see [7]).

Given an inner product space $(V, \langle \cdot, \cdot \rangle)$ and a linearly independent set $\{a_1, \ldots, a_n\}$ in V, we can in general derive a new inner product $\langle \cdot, \cdot \rangle^*$ from the given inner product $\langle \cdot, \cdot \rangle$ by first defining an n-inner product $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on V as in (2), and then defining the new inner product $\langle \cdot, \cdot \rangle^*$ on V with respect to $\{a_1, \ldots, a_n\}$ as in (3).

So, to accomplish our task, let us do those two steps to our inner product space $(X, \langle \cdot, \cdot \rangle)$ and linearly independent set $\{a_1, \ldots, a_n\}$. We claim that, with respect to the new inner product $\langle \cdot, \cdot \rangle^*$ obtained in this way, the set $\{a_1, \ldots, a_n\}$ is orthogonal and $\|a_i\|^* = \sqrt{\kappa}\, \|a_1, \ldots, a_n\|$ for every $i = 1, \ldots, n$. (Here $\|x\|^*$ denotes the induced norm from $\langle \cdot, \cdot \rangle^*$, that is, $\|x\|^* := \sqrt{\langle x, x \rangle^*}$.)

To verify our claim, observe that if $i \neq j$, then $\langle a_i, a_j | a_{i_2}, \ldots, a_{i_n} \rangle = 0$ for every subset $\{i_2, \ldots, i_n\}$ of $\{1, \ldots, n\}$ (because a_i or a_j will always equal one of $a_{i_2}, \ldots, a_{i_n}$), and hence $\langle a_i, a_j \rangle^* = 0$. This tells us that $\{a_1, \ldots, a_n\}$ is orthogonal.

Next, for each $i \in \{1, \ldots, n\}$, we have $\|a_i, a_{i_2}, \ldots, a_{i_n}\| = 0$ for every subset $\{i_2, \ldots, i_n\}$ of $\{1, \ldots, n\}$ except for the case where $i \notin \{i_2, \ldots, i_n\}$, for which we have $\|a_i, a_{i_2}, \ldots, a_{i_n}\| = \sqrt{\kappa}\, \|a_1, a_2, \ldots, a_n\|$. Hence $\|a_i\|^* = \sqrt{\kappa}\, \|a_1, \ldots, a_n\|$ for every $i = 1, \ldots, n$, as claimed.

Now, if we take $\kappa = \|a_1, \ldots, a_n\|^{-2}$, then the new inner product $\langle \cdot, \cdot \rangle^*$ on X, given by

$$(4) \qquad \langle x, y \rangle^* := \|a_1, \ldots, a_n\|^{-2} \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle,$$

$$= \det(\langle a_i, a_j \rangle)^{-1} \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}} \begin{vmatrix} \langle x, y \rangle & \langle x, a_{i_2} \rangle & \ldots & \langle x, a_{i_n} \rangle \\ \langle a_{i_2}, y \rangle & \langle a_{i_2}, a_{i_2} \rangle & \ldots & \langle a_{i_2}, a_{i_n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle a_{i_n}, y \rangle & \langle a_{i_n}, a_{i_2} \rangle & \ldots & \langle a_{i_n}, a_{i_n} \rangle \end{vmatrix},$$

will make the set $\{a_1, \ldots, a_n\}$ orthonormal in X. We also see that it preserves the orthogonal complement of the linear span of $\{a_1, \ldots, a_n\}$: if $\langle x, a_i \rangle = 0$ for every $i = 1, \ldots, n$, then $\langle x, a_i \rangle^* = 0$ for every $i = 1, \ldots, n$.

Note that in n-dimensional case, (4) agrees with (1). In general, the inner product that makes the set $\{a_1, \ldots, a_n\}$ orthonormal in X is unique up to restriction on the linear span of $\{a_1, \ldots, a_n\}$.

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