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Stummel Class and Morrey Spaces

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Abstract. In this paper we present some generalization of Ragusa and Zamboni's results on Stummel class and Morrey spaces.

Keywords: Morrey spaces; Stummel class.

1. Introduction and Main Results

For a given function $\psi : (0, \infty) \longrightarrow (0, \infty)$, define the Stummel modulus of a locally integrable function f on $\mathbf{R}^n$ by

$$\psi_f(r) = \sup_{x \in \mathbf{R}^n} \int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{|x-y|^n} \, dy, \quad 0 < r < \infty,$$

and the Stummel class $S_\psi = S_\psi(\mathbf{R}^n)$ by

$$S_\psi = \{f \in L^1_{\mathrm{loc}}(\mathbf{R}^n) : \lim_{r \to 0} \psi_f(r) = 0\}.$$

If $\psi(t) = t^p, \ 1 < p < n$, then S_ψ is the Stummel class S_p introduced by Ragusa and Zamboni [2]. Also note that for each $1 < p < n$ we have $S_p \subseteq S_\psi$ provided that $\psi(t) \leq Ct^p$ for some positive constant C.

Now, for each $1 \leq p < \infty$, one may define the generalized Morrey spaces $\mathcal{M}_{p,\psi} = \mathcal{M}_{p,\psi}(\mathbf{R}^n)$ by

$$\mathcal{M}_{p,\psi} = \{f \in L^p_{\mathrm{loc}}(\mathbf{R}^n) : \|f\|_{\mathcal{M}_{p,\psi}} < \infty\},$$

where $\|f\|_{\mathcal{M}_{p,\psi}} = \sup\limits_B \frac{1}{\psi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{\frac{1}{p}}$. Here the supremum is taken over all open balls $B = B(a, r)$ in $\mathbf{R}^n$, $|B|$ denotes the Lebesgue measure of B

in $\mathbf{R}^n$, and $\psi(B) = \psi(r)$. Observe that if $\psi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda \le n$, then $\mathcal{M}_{p,\psi} = M_{p,\lambda}$, the classical Morrey spaces (see [1]).

In this paper, we study the relation between the Stumnel class S_ψ and the Morrey spaces $\mathcal{M}_{1,\phi}$, particularly for ψ and ϕ that satisfy *the doubling condition*: a function ψ is said to satisfy the doubling condition if

$$\frac{1}{2} \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{A} \le \frac{\psi(t)}{\psi(r)} \le A,$$

where $A \ge 1$ is independent of $t, r > 0$. Our main results are the following:

Theorem 1. *Suppose that $\phi, \psi : (0,\infty) \longrightarrow (0,\infty)$ satisfy the doubling condition and $\lim\limits_{r\to 0} \int_0^r \psi(t)\phi(t)t^{-1}dt = 0$. Then $\mathcal{M}_{1,\phi} \subseteq S_\psi$.*

Theorem 2. *Suppose $\psi : (0,\infty) \longrightarrow (0,\infty)$ satisfies the doubling condition and $f \in S_\psi$. If $\phi : (0,\infty) \longrightarrow (0,\infty)$ satisfies $\int_0^r \frac{\psi_f(t)}{\psi(t)} t^{n-1}dt \le Cr^n\phi(r)$ for all $r > 0$, then $f \in \mathcal{M}_{1,\phi}$.*

Note that if one puts $\psi(t) = t^p$, with $1 < p < n$, and $\phi(t) = t^{\lambda-n}$, with $n - p < \lambda < n$, in the above theorems, one will get the result of Ragusa and Zamboni [2], p. 56.

2. Proof of the Theorems

In the following, the letter C's denote positive constants, which may vary from line to line. In general these constants depend on n.

Proof of Theorem 1. Let $f \in \mathcal{M}_{1,\phi}$. For a given ball $B = B(x,r)$, where $x \in \mathbf{R}^n$ and $r > 0$ sufficiently small, we have

$$\begin{aligned}
\int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{|x-y|^n}\, dy &\le \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy \\
&\le C \sum_{k=-\infty}^{-1} \frac{\psi(2^k r)}{(2^k r)^n} \int_{|x-y|<2^{k+1}r} |f(y)|\, dy \\
&\le C \sum_{k=-\infty}^{-1} \psi(2^k r)\phi(2^k r)\|f\|_{\mathcal{M}_{1,\phi}}.
\end{aligned}$$

But, for each negative integer k, we have $\psi(2^k r)\phi(2^k r) \sim \int_{2^k r}^{2^{k+1} r} \psi(t)\phi(t)t^{-1}dt$, because ψ and ϕ satisfy the doubling condition. Hence

$$\int_{|x-y|<r} |f(y)| \frac{\psi(|x-y|)}{|x-y|^n}\, dy \le C\|f\|_{\mathcal{M}_{1,\phi}} \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \psi(t)\phi(t)t^{-1} dt$$

$$= C\|f\|_{\mathcal{M}_{1,\phi}} \int_0^r \psi(t)\phi(t)t^{-1} dt.$$

Now the hypothesis that $\lim_{r\to 0} \int_0^r \psi(t)\phi(t)t^{-1}dt = 0$ brings us to the conclusion that $f \in S_\psi$. This proves the theorem. ■

To prove Theorem 2, we shall use the following Lemma, which may be viewed as a generalization of Lemma 1.2 of [2].

Lemma 3. *If $\psi : (0,\infty) \longrightarrow (0,\infty)$ satisfies the doubling condition, then so does ψ_f for every $f \in S_\psi$.*

Proof. First note that if $\frac{1}{2} \le \frac{t}{r} \le 2$, then $\frac{1}{2}r \le t \le 2r$, and so we have

$$\psi_f\left(\frac{1}{2}r\right) \le \psi_f(t) \le \psi_f(2r),$$

because ψ_f in a nondecreasing function. To prove the Lemma, it thus suffices to find a constant $C > 0$ such that $\psi_f(r) \le C\psi_f\left(\frac{1}{2}r\right)$ for all $r > 0$.

For a given ball $B = B(x,r)$, choose $x_1, x_2, \ldots, x_m \in B$ such that

$$B \subseteq \bigcup_{i=1}^{m} B\left(x_i, \frac{1}{2}r\right).$$

Hence, we have the following estimate

$$\int_{|x-y|<r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy \le \sum_{i=1}^{m} \int_{|x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy = \sum_{i=1}^{m} I_i.$$

Now, for each $i \in \{1,\ldots,m\}$, we have $B(x_i, \frac{1}{2}r) \subseteq B(x, \frac{3}{2}r)$, and so we can write I_i as

$$I_i = A_{1i} + A_{2i} + A_{3i}, \quad i \in \{1,2,\ldots,m\},$$

where

$$A_{1i} = \int_{|x-y|\le|x_i-y|,\, |x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy,$$

$$A_{2i} = \int_{|x_i-y|<|x-y|\le 2|x_i-y|,\, |x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy,$$

$$A_{3i} = \int_{2|x_i-y|<|x-y|\le 3|x_i-y|,\, |x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy.$$

We estimate each of these three terms as follows:

$$A_{1i} \le \int_{|x-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy \le \psi_f(\tfrac{1}{2}r),$$

$$\begin{aligned} A_{2i} &= \int_{|x_i-y|<|x-y|\le 2|x_i-y|,\, |x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy, \\ &\le C \int_{|x_i-y|<\frac{1}{2}r} \frac{|f(y)|\psi(|x_i-y|)}{|x_i-y|^n}\, dy \le C\psi_f(\tfrac{1}{2}r), \end{aligned}$$

and with the same method as before, we also have

$$A_{3i} \le C\psi_f\left(\frac{1}{2}r\right).$$

This completes the proof. ■

Proof of Theorem 2. For a given ball $B = B(x,r)$, where $x \in \mathbf{R}^n$ and $0 < r < \infty$, we have

$$\begin{aligned} \int_{|x-y|<r} |f(y)|\, dy &= C \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} (2^k r)^n \frac{|f(y)|}{|x-y|^n}\, dy \\ &\le C \sum_{k=-\infty}^{-1} \frac{(2^k r)^n}{\psi(2^k r)} \int_{|x-y|<2^{k+1} r} \frac{|f(y)|\psi(|x-y|)}{|x-y|^n}\, dy \\ &\le C \sum_{k=-\infty}^{-1} \frac{(2^k r)^n \psi_f(2^k r)}{\psi(2^k r)}. \end{aligned}$$

By the above fact, we know that ψ_f, as well as ψ, satisfies the doubling condition, so that $\frac{\psi_f(2^k r)}{\psi(2^k r)} \sim \int_{2^k r}^{2^{k+1} r} \frac{\psi_f(t)}{\psi(t)}\, t^{-1} dt$ for each negative integer k. Hence

$$\begin{aligned} \int_{|x-y|<r} |f(y)|\, dy &\le C \sum_{k=-\infty}^{-1} \int_{2^k r}^{2^{k+1} r} \frac{\psi_f(t)}{\psi(t)}\, t^{n-1} dt \\ &= C \int_0^r \frac{\psi_f(t)}{\psi(t)}\, t^{n-1} dt \le C r^n \phi(r), \end{aligned}$$

by our hypothesis on ϕ. This implies that $f \in \mathcal{M}_{1,\phi}$, as desired. ■

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