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ORTHOGONALITY IN 2-NORMED SPACES REVISITED

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In this paper we discuss some existing notions of orthogonality in 2-normed spaces and their drawback. We also formulate new definitions of orthogonality that improve the existing ones. In the standard 2-normed space, our notions of orthogonality coincide with the usual one.

1. INTRODUCTION

In a real normed space $(X, \|\cdot\|)$, one can define orthogonality of two vectors x and y in many different ways. For example, the following definitions of Pythagorean, isosceles, and the Birkhoff-James orthogonality are known:

P-orthogonality: x is P-orthogonal to y (denoted by $x \perp_{\rm P} y$) if only if

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

I-orthogonality: x is I-orthogonal to y (denoted by $x \perp_{\rm I} y$) if only if

$$\|x+y\| = \|x-y\|.$$

BJ-orthogonality: x is BJ-orthogonal to y (denoted by $x \perp_{\rm BJ} y$) if only if

$$\|x+\alpha y\| \geq \|x\| \text{ for every } \alpha \in \mathbb{R}.$$

Note that in an inner product space $(X, \langle\cdot,\cdot\rangle)$, $x \perp_{\rm P} y$, $x \perp_{\rm I} y$, and $x \perp_{\rm BJ} y$ are all equivalent to the condition $\langle x, y\rangle = 0$, for which we have the usual orthogonality

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$x \perp y$. In a normed space which is not an inner product space, however, one does not imply another. For further properties of these orthogonalities, see, for example, [**20**]. Related results may be found in [**1**, **2**, **5**, **6**, **14**, **19**, **21**].

Now, suppose that $\dim(X) \geq 2$ and that X is equipped with a 2-*norm* $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ satisfying the following properties:

(N1) $\|x,y\| \geq 0$ for every $x, y \in X$; $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(N2) $\|x,y\| = \|y,x\|$ for every $x, y \in X$;
(N3) $\|x,\alpha y\| = |\alpha|\|y,x\|$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$;
(N4) $\|x, y+z\| \leq \|x,y\| + \|x,z\|$ for every $x, y, z \in X$.

In other words, $(X, \|\cdot,\cdot\|)$ forms a 2-*normed space*. The theory of 2-normed spaces was first introduced by Gähler in 1960's [**12**]. Since then, various notions in normed spaces have been extended to 2-normed spaces by many authors (see, for example, [**3**, **10**, **11**, **15**, **17**, **18**]). In [**16**] Khan and Siddiqui defined the notions of P-, I-, and BJ-orthogonality in 2-normed spaces $(X, \|\cdot,\cdot\|)$ as follows:

P-orthogonality: $x \perp_{\rm P} y$ if only if $\|x+y,z\|^2 = \|x,z\|^2 + \|y,z\|^2$ for every $z \neq 0$;

I-orthogonality: $x \perp_{\rm I} y$ if only if $\|x+y,z\| = \|x-y,z\|$ for every $z \neq 0$;

BJ-orthogonality: $x \perp_{\rm BJ} y$ if only if $\|x+\alpha y,z\| \geq \|x,z\|$ for every $z \neq 0$ and $\alpha \in \mathbb{R}$.

However, in [**13**], Godini remarked that the phrase "for every $z \neq 0$" in the above definitions should be replaced by "for every $z \notin \text{span}\{x,y\}$", for otherwise there do not exist two nonzero vectors x and y such that x is P-, I-, or BJ-orthogonal to y.

In this paper, we shall show that even with the above correction, the three definitions of orthogonality are still "void" in the sense that there do not exist two nonzero vectors x and y such that x is P-, I-, or BJ-orthogonal to y even in the standard case (for which X is an inner product space and the 2-norm $\|x,y\|$ equals the area of the parallelogram spanned by x and y).

Accordingly, we shall offer some alternatives for P-, I-, and BJ-orthogonalities in 2-normed spaces. Just like the three orthogonalities in a normed space X are equivalent to the usual orthogonality when X is actually an inner product space, we shall show here that our definition also coincide with the usual orthogonality when X is the standard 2-normed space.

2. THE DRAWBACK IN THE STANDARD 2-NORMED SPACE

Let us consider the standard case now, where $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, equipped with the *standard* 2-*norm*

$$\|x,y\| := \begin{vmatrix} \langle x,x\rangle & \langle x,y\rangle \\ \langle y,x\rangle & \langle y,y\rangle \end{vmatrix}^{1/2}.$$

Note that geometrically $\|x, y\|$ represents the area of the parallelogram spanned by x and y. The determinant is known as the *Gramian* of x and y.

Along with the 2-norm, we have the *standard* 2-*inner product* $\langle \cdot, \cdot | \cdot \rangle : X \times X \times X \to \mathbb{R}$ given by the formula

$$\langle x, y | z \rangle := \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix}.$$

Observe here that $\|x, y\| = \langle x, x | y \rangle^{1/2}$. Further, one may check that the 2-inner product satisfies

(I1) $\langle x, x | z \rangle \geq 0$ for every $x, z \in X$; $\langle x, x | z \rangle = 0$ if and only if x and z are linearly dependent;
(I2) $\langle x, y | z \rangle = \langle y, x | z \rangle$ for every $x, y, z \in X$;
(I3) $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$ for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$;
(I4) $\langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle$ for every $x_1, x_2, y, z \in X$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$\langle x, y | z \rangle^2 \leq \|x, z\|^2 \, \|y, z\|^2$$

for every $x, y, z \in X$, from which the triangle inequality (N4) follows.

A vector space X equipped with a 2-inner product space $\langle \cdot, \cdot | \cdot \rangle$ satisfying (I1)–(I4) is called a 2-*inner product space* [**7**, **8**, **9**]. In an arbitrary 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$, $x \perp_{\mathrm{P}} y$, $x \perp_{\mathrm{I}} y$, and $x \perp_{\mathrm{BJ}} y$ are equivalent to the condition

$$\langle x, y | z \rangle = 0 \quad \text{for every } z \notin \operatorname{span}\{x, y\}. \tag{1}$$

Cho and Kim [**4**] use the latest condition to define G-orthogonality of two vectors in a 2-inner product space of dimension 3 or higher. (Hereafter, we shall assume that $\dim(X) \geq 3$ unless otherwise stated.)

Unfortunately, in the standard 2-normed space (which is also the standard 2-inner product space), one cannot find two nonzero vectors x and y satisfying (1) Indeed, we have the following fact, which tells us that the definitions of orthogonality introduced by Khan and Siddiqui, as well as the one introduced by Cho and Kim, are "void" in the above sense.

Fact 2.1. *Let $(X, \|\cdot, \cdot\|)$ be the standard 2-normed space. Then, for two nonzero vectors $x, y \in X$, there exists $z \notin \operatorname{span}\{x, y\}$ such that $\langle x, y | z \rangle \neq 0$.*

Proof. Let x and y be two nonzero vectors in X, and $z = \alpha x + \beta y + z'$ where $z' \perp \operatorname{span}\{x, y\}$ (with respect to the usual orthogonality) and $z' \neq 0$. Then, by using properties of an inner product and determinants (with respect to some elementary row and column operations), we have

$$\begin{aligned}
\langle x, y|z\rangle &= \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} \\
&= \begin{vmatrix} \langle x, y\rangle & \langle x, \alpha x + \beta y + z'\rangle \\ \langle \alpha x + \beta y + z', y\rangle & \langle \alpha x + \beta y + z', \alpha x + \beta y + z'\rangle \end{vmatrix} \\
&= \begin{vmatrix} \langle x, y\rangle & \alpha\langle x, x\rangle \\ \beta\langle y, y\rangle & \alpha\beta\langle x, y\rangle + \langle z', z'\rangle \end{vmatrix} \\
&= -\alpha\beta[\|x\|^2\|y\|^2 - \langle x, y\rangle^2] + \langle x, y\rangle\|z'\|^2 \\
&= -\alpha\beta\|x, y\|^2 + \langle x, y\rangle\|z'\|^2.
\end{aligned}$$

Now if $\langle x, y\rangle > 0$, then when $\alpha\beta < 0$ we find that $\langle x, y|z\rangle$ is strictly positive. If $\langle x, y\rangle < 0$, then when $\alpha\beta > 0$ we find that $\langle x, y|z\rangle$ is strictly negative. Finally, if $\langle x, y\rangle = 0$, then when $\alpha, \beta \neq 0$ we find that $\langle x, y|z\rangle = -\alpha\beta\|x, y\|^2 = -\alpha\beta\|x\|^2\|y\|^2 \neq 0$. □

In the next section, we shall offer new definitions of orthogonality of two vectors in a 2-normed space, which can be viewed as an improvement of Khan and Siddiqui's (and of Cho and Kim's in the 2-inner product space case).

3. NEW ORTHOGONALITIES IN 2-NORMED SPACES

Let us learn from the drawback of the existing definitions of orthogonality in 2-normed spaces. First, Khan and Siddiqui require that the orthogonality conditions hold for every $z \neq 0$. This is too strong for we cannot find two nonzero vectors satisfying the orthogonality conditions. Godini, as well as Cho and Kim, weaken the conditions to hold only for $z \notin \text{span}\{x, y\}$. As we have seen in the standard case, one still cannot find two nonzero vectors satisfying the orthogonality condition. This tells us that we should weaken the condition a little further.

Next, we should formulate the definition such that, in the standard case, two vectors are orthogonal with respect to our definition if and only if they are orthogonal in the usual sense (that is, $\langle x, y\rangle = 0$) (just like all definitions of orthogonality in a normed space are equivalent to the usual one when the space is actually an inner product space).

Note that in the standard case, if $\langle x, y\rangle = 0$, then $\langle x, y|z\rangle = 0$ if and only if $\langle x, z\rangle = 0$ or $\langle z, y\rangle = 0$ (that is, if and only if $z \perp x$ or $z \perp y$). This implies that, in general, one can only have a subspace V of X with $\text{codim}(V) = 1$ for which $\langle x, y|z\rangle = 0$ for every $z \in V$.

Based on this observation, we redefine P-, I-, and BJ-orthogonality in a 2-normed space $(X, \|\cdot, \cdot\|)$ as follows:

Definition 3.1. (P-, I-, and BJ-orthogonality)
$x \perp_{\text{P}} y$ if only if there exists a subspace V of X with $\text{codim}(V) = 1$ *such that*

$$\|x + y, z\|^2 = \|x, z\|^2 + \|y, z\|^2 \quad \textit{for every } z \in V;$$

$x \perp_{\rm I} y$ *if only if there exists a subspace* V *of* X *with* $\text{codim}(V) = 1$ *such that*

$$\|x + y, z\| = \|x - y, z\| \quad for\ every\ z \in V;$$

$x \perp_{\rm BJ} y$ *if only if there exists a subspace* V *of* X *with* $\text{codim}(V) = 1$ *such that*

$$\|x + \alpha y, z\| \ge \|x, z\| \quad \text{for every } z \in V \text{and } \alpha \in \mathbb{R}.$$

At the same time, we also redefine G-orthogonality in a 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$ as follows:

Definition 3.2. (G-orthogonality) $x \perp_{\rm G} y$ *if and only if there exists a subspace* V *of* X *with* $\text{codim}(V) = 1$ *such that* $\langle x, y|z\rangle = 0$ *for every* $z \in V$*.*

In a 2-inner product space, it is clear that P-, I-, and BJ-orthogonality are equivalent to G-orthogonality. Furthermore, in the standard 2-normed space, if $x \perp y$ (that is, $\langle x, y\rangle = 0$), then $x \perp_{\rm G} y$ (because we can choose $V^{\perp} = \text{span}\{x\}$ or $\text{span}\{y\}$, so that $\langle x, y|z\rangle = 0$ for every $z \in V$). The following proposition states that the converse is also true.

Proposition 3.3. *Let* $(X, \|\cdot, \cdot\|)$ *be the standard* 2*-normed space of dimension* 3 *or higher. If* $x \perp_{\rm G} y$*, then* $x \perp y$*.*

Proof. Suppose that, for nonzero x and y, we have $x \not\perp y$ (that is, $\langle x, y\rangle \neq 0$). We shall show that $x \not\perp_{\rm G} y$. So let V be a subspace of X with $\text{codim}(V) = 1$. We claim that there must exist $z \in V$ such that $\langle x, y|z\rangle \neq 0$.

To prove the claim, we consider several cases.

Case 1. If x and y are linearly dependent, that is, $y = kx$ for $k \neq 0$, then

$$\langle x, y|z\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix} = k\|x, z\|^2.$$

In this case we can choose $z \in V \setminus \text{span}\{x\}$ (we can do so because $\text{codim}(V) = 1$ and $\dim(X) \geq 3$), so that $\|x, z\| > 0$ and hence $\langle x, y|z\rangle \neq 0$.

Case 2. If x and y are linearly independent (so that $\|x, y\| > 0$), then we consider several subcases:
Case 2*a*. If $x \in V$ and $y \in V$, then we may choose $z = x + y \in V$, so that $\langle x, y|z\rangle = -\|x, y\|^2 < 0$.
Case 2*b*. If $x \in V$ and $y \notin V$, then we can choose a nonzero vector $z \in V$ perpendicular to x, so that $\langle x, y|z\rangle = \langle x, y\rangle \|z\|^2 \neq 0$.
Case 2*c*. If $x \notin V$ and $y \in V$, then we can choose a nonzero vector $z \in V$ perpendicular to y, so that $\langle x, y|z\rangle = \langle x, y\rangle \|z\|^2 \neq 0$.
Case 2*d*. If $x \notin V$ and $y \notin V$, then we can write $x = au + v_x$ and $y = bu + v_y$ where u is a nonzero vector perpendicular to V, $v_x, v_y \in V$, and $a, b \in \mathbb{R} \setminus \{0\}$. Hence, choosing $z = bx - ay = bv_x - av_y \in V$, we have $\langle x, y|z\rangle = ab\|x, y\|^2 \neq 0$. □

4. REMARKS ON THE 2-DIMENSIONAL CASE

Let us consider the standard 2-normed space $(X, \|\cdot,\cdot\|)$ where $\dim(X) = 2$. (Clearly CHO and KIM's G-orthogonality does not apply here.) Take two nonzero vectors x and y in X. If x and y are linearly dependent, that is, $y = kx$ for $k \neq 0$, then for every $z \in X$ we have

$$\langle x, y|z\rangle = k\|x, z\|^2.$$

If x and y are linearly independent, then for $z = \alpha x + \beta y$ we have

$$\langle x, y|z\rangle = -\alpha\beta\|x, y\|^2.$$

Hence, for nonzero vectors x and y, we see that $\langle x, y|z\rangle = 0$ if only if (a) $y = kx$ and $z = lx$ (that is, x, y, and z are colinear) or (b) x and y are linearly independent, and $z = \alpha x$ or $z = \beta y$.

With respect to our G-orthogonality defined in §3, we find that two arbitrary vectors x and y are G-orthogonal to each other, because we can always find a 1-dimensional subspace V of X such that $\langle x, y|z\rangle = 0$ for every $z \in V$, even though x is parallel to y. Thus we have moved from one extreme situation (where there do not exist two nonzero vectors that are orthogonal) to another (where every pair of vectors are orthogonal).

To get around this, we notice that in the case where x and y are linearly dependent, there is only one subspace V of X such that $\langle x, y|z\rangle = 0$ for every $z \in V$; while in the other case where x and y are linearly independent, there are two such subspaces. Hence if we define G-orthogonality in the standard 2-normed space X of dimension 2 as follows:

> "$x \perp_G y$ if and only if there exist two distinct 1-dimensional subspaces V_1 and V_2 of X such that $\langle x, y|z\rangle = 0$ for every $z \in V_1 \cup V_2$"

then we have excluded two nonzero parallel vectors x and y from being G-orthogonal. All other pairs of vectors, however, will still be G-orthogonal.

If, in the latest definition, we require the two subspaces V_1 and V_2 to be perpendicular (with respect to the usual orthogonality), then clearly $x \perp_G y$ if and only if $x \perp y$ in the usual sense. This last definition, however, cannot be applied to arbitrary 2-normed spaces (for we do not always have the notion of perpendicular subspaces in the first place).

To define orthogonality in 2-normed spaces in general, it seems that we have to use a different way. In light of [**15**], one can actually define a norm in an arbitrary 2-normed space $(X, \|\cdot,\cdot\|)$ and use this norm to define P-, I-, and BJ-orthogonality. If the 2-normed space is standard and 2-dimensional, then the norm can be derived from the 2-norm in such a way that it coincides with the existing norm (induced from the inner product). In this case, the P-, I-, and BJ-orthogonality will be equivalent to the usual orthogonality, as desired.

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