Invited Review

Fractional Integral Operators and Their Boundedness on Various Spaces

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Abstract

This is a survey paper which presents and discusses some results on fractional integral operators and their generalized version on both homogeneous and non-homogeneous spaces.

Keywords: Fractional integral operators, Homogeneous spaces, Non-homogeneous spaces

Abstrak

Makalah ini merupakan makalah survey yang membahas hasil-hasil yang berkenaan dengan operator integral fraksional dan perumumannya, baik di ruang homogen maupun di ruang tak homogen.

Kata kunci: Operator integral fraksional, Ruang homogen, Ruang tak-homogen

1. Introduction

One family of objects studied in (modern) Fourier Analysis is integral operators, from the classical Fourier integral operator to various operators, such as maximal operators, singular integral operators, and fractional integral operators. Most of these operators are related to classical differential equations such as wave equations, heat equations, and Poisson equation.

One of the main aspects we are concerned with the operators is how they act from one space of functions to another. In particular, we often ask whether the operators are *bounded* on certain spaces. Recall that an operator T is bounded from a normed space U to another normed space V if there exists a constant C such that

$$\|Tu\|_V \le C\|u\|_U$$

for all u in U. Here $\|\cdot\|\,\|\cdot\|_U$ denotes the *norm* on U.

If $U = V$, we simply say that T is bounded on U. A bounded linear operator on U is *continuous* on U, which intuitively means that a small change in u will yield a small change in Tu. A continuous operator is thus *regular*.

The regularity, and accordingly the boundedness, is an important feature that one would like to see in an operator, especially when the operator is involved in an integral or differential equation related to a certain physical phenomenon. From numerical point of view, things will often be easier when the operator is regular or bounded.

In this paper, we will pay our attention to the study of fractional integral operators. These operators take functions to other functions on $\mathbf{R}^d$. For $0 < \alpha < d$, the fractional integral operator I_α is defined by the formula

$$I_\alpha f(x) = \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,dy,$$

where f is a real-valued function on $\mathbf{R}^d$. Here $dy = d\mu(y)$ denotes the usual Lebesgue measure on $\mathbf{R}^d$.

The operator I_α, also known as the Riesz potential (of degree α), was first studied by Hardy and Littlewood in the 1920s (1927; 1930; 1932) and also by Sobolev (1938). For $\alpha = 2$, it is known as the Newtonian potential.

Note that I_α can be expressed as a multiple of the Laplacian $-\Delta$ raised to the power of $-\alpha/2$, that is

$$I_\alpha f = C(-\Delta)^{-\alpha/2} f,$$

for some constant C. Here the Laplacian $-\Delta$ is given by the formula

$$\Delta f = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$$

If $d \ge 3$, we see that $C^{-1} I_2 f(x) = (-\Delta)^{-1} f$ is a solution of the Poisson equation

$$-\Delta u = f.$$

We know that, in general, the function expressed as an integral - such as in the definition of $I_\alpha f$ - is smoother than the integrand. Hardy and Littlewood, as well as Sobolev, proved that I_α takes a function in L^p boundedly to a function in L^q, for some $q > p$. Here, $L^p = L^p(\mathbf{R}^d)$, for $1 \le p \le \infty$, is the space of p-th power integrable functions on $\mathbf{R}^d$.

To be precise, they proved the following result.

Theorem 1.1 (Hardy-Littlewood-Sobolev). *For 1 < p < $\frac{d}{\alpha}$, we have the inequality*

$$\|I_\alpha f\|_{L^q} \le C\|f\|_{L^p},$$

that is, I_α *is bounded from* L^p *to* L^q, *provided that* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.

The inequality tells us that $I_\alpha f$ is better worth of degree α than *f* with respect to local behaviour. Sobolev's embedding theorems then follows from the above theorem (Stein, 1970). From Theorem 1.1, we also know that if *f* is in L^p for $1 < p < d/2$, then the solution to the Poisson equation, namely $u = (-\Delta)^{-1} f$, is in $L^{\frac{dp}{d-2}}$; see Evans (1998) for further discussion about the application in partial differential equations.

The old proof of the theorem uses the idea of splitting the integral into two parts, one is near 0 and the other is away from 0, and then estimating each integral separately. Now there are two other ways of proving it, one is due to (Stein, 1970) and the other is based on the work of Hedberg (1972). Stein uses an interpolation technique, while Hedberg employs the pointwise estimate

$$|I_\alpha f(x)| \le CMf(x)^{1-\alpha p/d} \|f\|_{L^p}^{\alpha p/d},$$

which involves another operator *M*, known as the Hardy-Littlewood maximal operator.

The operator *M* is defined by the formula:

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, dy,$$

where $\mu(B(x,r)) = Cr^d$ is the Lebesgue measure of the *d*-dimensional (open) ball with centre $x \in \mathbf{R}^d$ and radius *r*. One basic property of the maximal operator is that it is bounded on L^p for every *p* > 1 (Hardy and Littlewood, 1930).

In the next sections, we shall present and discuss some recent results on the fractional integral operators and their generalization, in the context of homogeneous spaces as well as non-homogeneous spaces. We organize this paper as follows. In Section 2, we summarize known results for the fractional integral operators in homogeneous spaces. In Section 3, we present some recent results in non-homogeneous spaces. Next, in Section 4, we discuss generalized fractional integral operators in both homogeneous and non-homogeneous spaces. Some of the results in the non-homogeneous case are new and have been submitted for publication elsewhere. Finally, concluding remarks and references will be given at the end of the paper.

2. The Fractional Integral Operator on Homogeneous Spaces

In proving the boundedness of the fractional integral operators on various spaces, some researchers find that the translation invariance and the *doubling* properties of the Lebesgue measure play an important role. This is also true in studying other operators such as maximal operators and various types of singular integral operators. Thus, inspired by this fact, they studied the operators in the homogeneous setting.

By a homogeneous space we mean a metric space equipped with a (positive) Borel measure μ satisfying the *doubling condition*, that is, there exists a constant C such that

$$\mu(B(x,2r)) \le C\mu(B(x,r))$$

for every *x* in $\mathbf{R}^d$ and *r* > 0; see (Coifman and de Gusmán, 1970; 1971) or (Coifman and Weiss, 1971). An example of a homogeneous space is of course the metric space $\mathbf{R}^d$ equipped with the Lebesgue measure (and the usual Euclidean distance). More examples of homogeneous spaces can be found in (Christ, 1990) and (Coifman and Weiss, 1977).

In his book (Stein, 1993), Stein also used the homogeneous setting in studying various operators, including the fractional integral operators. More recent results on homogeneous spaces can be found in (Martell, 2004), and Nakai (2001b and 2006).

2.1 Let us now review some results on the fractional integral operators in specific homogeneous spaces, which were published by many authors since the 1960s.

For $p \ge 1$ and $0 \le \lambda \le d$, we define the (classical) Morrey spaces $L^{p,\lambda} = L^{p,\lambda}(\mathbf{R}^d)$ to be the space of all functions $f \in L^p_{loc}(\mathbf{R}^d)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(x,r)} \left(\frac{1}{r^\lambda} \int_{B(x,r)} |f(y)|^p\, dy \right)^{\frac{1}{p}} < \infty.$$

Note that *f* is in $L^{p,\lambda}$ if and only if there exists a constant *C* such that

$$\int_{B(x,r)} |f(y)|^p\, dy \le Cr^\lambda$$

for every *x* in $\mathbf{R}^d$ and $r > 0$. Clearly $L^{p,0} = L^p$ and $L^{p,d} = L^\infty$. For a further discussion on the structure of Morrey spaces and their generalization, we refer the reader to (Campanato, 1964), (Morrey, 1940), (Peetre, 1969), and (Zorko, 1986).

As stated in Peetre (1969), Spanne proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{d}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ and $0 \le \lambda < d$. Spanne's results can now be seen as a consequence of the following theorem.

Theorem 2.1 (Adams, 1975). *For* $1 < p < \frac{d}{\alpha}$ *and* $0 \le \lambda < d - \alpha p$, *we have the inequality*

$$\|I_\alpha f\|_{q,\lambda} \le C\|f\|_{p,\lambda},$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d-\lambda}$.

This theorem was reproved in (Chiarenza and Frasca, 1987) by using an inequality similar to Hedberg's. A generalization of this result in homogeneous spaces may be found in (Nakai, 2001b and 2006).

The inequality in Theorem 2.1 can be used to understand the behaviour of the solution to a Schrödinger equation with a small perturbed potential *W*, as discussed in (Olsen, 1995). The key is the following result:

Theorem 2.2 (Olsen, 1995). *For* $1 < p < \frac{d}{\alpha}$ *and* $0 \le \lambda < d - \alpha p$, *we have*

$$\|WI_\alpha f\|_{p,\lambda} \le C\|W\|_{(d-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

that is WI_α *is bounded on* $L^{p,\lambda}$ *provided that* $W \in L^{\frac{d-\lambda}{\alpha},\lambda}$.

2.2 In Nakai (1994), a generalization of Morrey spaces was introduced and an extension of Spanne's result was obtained. Suppose that $1 \le p < \infty$ and $\phi : (0,\infty) \to (0,\infty)$ satisfies the doubling condition with a doubling constant *C*, that is, ϕ satisfies

$$\frac{1}{2} \le \frac{r}{s} \le C \Rightarrow \frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C.$$

The generalized Morrey space $\mathrm{M}^{p,\phi} = \mathrm{M}^{p,\phi}(\mathbf{R}^d)$ is defined to be the space of all functions $f \in L^p_{loc}(\mathbf{R}^d)$ for which

$$\|f\|_{p,\phi} := \sup_B \frac{1}{\phi(r)} \left(\frac{1}{\mu(B)} \int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty$$

where $B = B(x,r)$. Note that for $\phi(t) = t^{\frac{\lambda - d}{p}}$ ($0 \le \lambda \le d$), we have $\mathbf{M}^{p,\varphi} = L^{p,\lambda}$.

Theorem 2.3 (Nakai, 1994). *Suppose* $1 < p < q < \infty$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$. *Suppose also* $\psi : (0,\infty) \to (0,\infty)$ *satisfies*

$$r^\alpha \phi(r) \le C\psi(r),$$

where $C > 0$ *is independent of* $r > 0$. *If*

$$\int_r^\infty t^{\alpha-1}\phi(t)\, dt \le Cr^\alpha \phi(r),$$

then there exists $C > 0$ *such that*

$$\|I_\alpha f\|_{q,\psi} \le C\|f\|_{p,\phi}$$

for every $f \in \mathrm{M}^{p,\phi}$.

In companion to Nakai's result we have the following result, which can be considered as an extension of Adams'.

Theorem 2.4 (Gunawan and Eridani, 2008). *Suppose that* ϕ *satisfies the doubling condition. Suppose also that*

$$\int_r^\infty \frac{\phi^p(t)}{t} dt \le C\phi^p(r)$$

for $1 < p < \infty$ *and* $\phi(t) < Ct^\beta$ *for* $-\frac{d}{p} \le \beta < -\alpha$, $1 < p < \frac{d}{\alpha}$. *Then, we have*

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C\|f\|_{p,\phi}$$

where $q = \frac{\beta p}{\alpha + \beta}$.

We refer the reader to (Kurata *et al*., 2002) for an application of the boundedness of I_α on generalized Morrey spaces.

3. The Fractional Integral Operator on Non-homogeneous Spaces

The success of the development of many theories in Fourier Analysis in homogeneous spaces for almost three decades is due to the fact that most of the central results in the Euclidean setting can be generalized without too much difficulties to the homogeneous setting (Verdera, 2002). In recent years, however, researchers found that many results still hold without the assumption of doubling condition on the measure (see the works of (Nazarov *et al.,* 1998), (Tolsa, 1998; 1999), and (García-Cuerva and Gatto, 2004).

When the measure is non-doubling, we are working on the so called *non-homogeneous spaces*. Let μ be a (positive) Radon measure on $\mathbf{R}^d$. Then $(\mathbf{R}^d,\mu)$ is a non-homogeneous space if μ satisfies the *growth condition* of order n ($0 < n \le d$), that is,

$$\mu(Q) \le C(l(Q))^n ,$$

for any cube $Q \subset \mathbf{R}^d$ with the sides parallel to the coordinate axis. Here $l(Q)$ stands for the side length of Q. Note that instead of dealing with balls, we choose to deal with cubes for convenience.

In non-homogeneous context, we define the fractional integral operator I_α^n (for $0<\alpha<n\le d$) by

$$I_\alpha^n f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y)$$

(García-Cuerva and Martell, 2001). As in the homogeneous case, the $L^p(\mu)-L^q(\mu)$ boundedness of I_α^n relies on the boundedness of the Hardy-Littlewood maximal operator on $L^p(\mu)$.

One of the problems we have to solve first is how to define the maximal operator in non-homogeneous spaces. As shown in (Nazarov *et al.*, 1998), the usual Hardy-Littlewood maximal operator M is not bounded on $L^p(\mu)$. Thus we have to modify its definition somehow. As we shall discuss below, there are two alternatives to do so.

3.1 The first alternative is to define the Hardy and Littlewood maximal operator M^n by the formula

$$M^n f(x) := \sup_{r>0} \frac{1}{r^n} \int_{Q(x,r)} |f(y)|\, d\mu(y) .$$

Here $Q(x,r)$ denotes the cube centered at $x\in\mathbf{R}^d$ with side length r. In Garcia-Cuerva and Martell (2001), it is shown that M^n is bounded on $L^p(\mu)$ and using this fact they proved the following theorem.

Theorem 3.1 (Garcia-Cuerva and Martell, 2001). *The operator I_α^n is bounded from $L^p(\mu)$ to $L^q(\mu)$ for $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.*

This result can be easily generalized as follows. For $1\le p<\infty$, let us define the generalized non-homogeneous Morrey space $\mathrm{M}^{p,\phi}(\mu) = \mathrm{M}^{p,\phi}(\mathbf{R}^d,\mu)$ to be the set of all functions $f\in L^p_{loc}(\mu)$ such that

$$\|f\|_{p,\phi,\mu} := \sup_{r>0}\frac{1}{\phi(r)}\left(\frac{1}{r^n}\int_{Q(x,r)}|f(y)|^p\, d\mu\right)^{\frac{1}{p}} < \infty.$$

Then we have:

Theorem 3.2 (Sihwaningrum and Suryawan, 2008). *Suppose that we have the inequality*

$$\int_r^\infty t^{\alpha-1}\phi(t)\, dt \le Cr^\alpha\phi(r)$$

and that $\psi:(0,\infty)\to(0,\infty)$ satisfies

$$r^\alpha\phi(r)\le C\psi(r)$$

where C is a constant which is independent of r > 0. Then I_α^n is bounded from $\mathrm{M}^{p,\phi}(\mu)$ to $\mathrm{M}^{q,\psi}(\mu)$ for $1<p<q<\infty$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$.

As a consequence of the above results, Olsen-type inequalities involving I_α^n on Lebesgue and generalized Morrey spaces in non-homogeneous setting are obtained, but we shall not present them here to avoid too much repetition.

3.2 The second alternative is proposed by (Nazarov *et al.*, 1998) and is elaborated further by Sawano (2005). For $k > 1$, one can define the k-dilated Hardy-Littlewood maximal operator M_k by the formula

$$M_k f(x) := \sup_{Q\ni x}\frac{1}{\mu(kQ)}\int_Q f(y)\, d\mu(y)$$

for all locally integrable functions f on $\mathbf{R}^d$. Here kQ denotes the cube with the same centre as Q and with $l(kQ)=kl(Q)$. In the case where $k=3$ and x is the centre of Q, the operator M_k is the modified Hardy-Littlewood maximal operator studied in (Nazarov *et al.*, 1998). Sawano proved that M_k is bounded on $L^p(\mu)$ for $p > 1$, for every $k>1$.

In line with the choice of the parameter $k>1$ in the definition of M_k, we define the Morrey spaces $L^{p,s}(k,\mu)$ for $1\le s\le p<\infty$ to be the set of all $f\in L^s_{loc}$ such that

$$\|f\|_{L^{p,s}(k,\mu)} := \sup_{Q\in\mathcal{Q}(\mu)}\mu(kQ)^{\frac{1}{p}-\frac{1}{s}}\left(\int_Q|f|^s\, d\mu\right)^{\frac{1}{s}}$$

is finite (Sawano and Tanaka, 2006).

Note that $L^{p,p}(k,\mu) = L^p(\mu)$ and for $1\le s_2\le s_1\le p<\infty$, we have

$$L^{p,p}(k,\mu)\subset L^{p,s_1}(k,\mu)\subset L^{p,s_2}(k,\mu).$$

Also, for $k_1, k_2>1$, we have

$$L^{p,s}(k_1,\mu) \approx L^{p,s}(k_2,\mu),$$

that is, $L^{p,s}(k_1,\mu)$ and $L^{p,s}(k_2,\mu)$ coincide as a set and their norm are mutually equivalent.

Moreover, we have the following theorem on the fractional integral operator I_α^n.

Theorem 3.3 (Sawano and Tanaka, 2006). *Let* $1 < s \le p < \infty$, $1 < t \le q$, $\frac{t}{q} = \frac{s}{p}$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then we have*

$$\left\| I_\alpha^n f \right\|_{L^{q,t}(k,\mu)} \le C \left\| f \right\|_{L^{p,s}(k,\mu)}.$$

In the above theorem, the boundedness of I_α^n from $L^p(\mu)$ to $L^q(\mu)$ is obtained by taking $s = p$ and $t = q$. More results on the boundedness of I_α^n on non-homogeneous Morrey spaces can be found in (Sawano *et al.,* 2006).

3.3 In Sawano (2007), the boundedness of I_α^n on generalized non-homogeneous Morrey spaces is studied and a result analogous to (Nakai, 1994) is obtained. We shall here present our result which is analogous to Gunawan (2003). In the classical case, Sawano's result reduces to Spanne's while ours reduces to Adams'.

We define the generalized non-homogeneous Morrey spaces $\mathrm{M}^{p,\phi}(k,\mu)$ to be the set of all μ-locally integrable functions *f* on $\mathbf{R}^d$ for which

$$\|f\|_{\mathrm{M}^{p,\phi}(k,\mu)} := \sup_Q \frac{1}{\phi(\mu(kQ))} \left(\frac{1}{\mu(kQ)} \int_Q |f|^p \, d\mu \right)^{\frac{1}{p}}$$

is finite. Here $1 < p < \infty$, and we always assume that the function ϕ is *almost decreasing*, that is, there exists $C > 0$ such that $\phi(s) \ge C\phi(t)$ for $s < t$. For $k_1, k_2 > 1$, the generalized Morrey spaces $\mathrm{M}^{p,\phi}(k_1,\mu)$ and $\mathrm{M}^{p,\phi}(k_2,\mu)$ coincide as a set and their norms are equivalent (Sawano, 2007).

For the shake of simplicity let us now set $\mathrm{M}^{p,\phi}(\mu) = \mathrm{M}^{p,\phi}(2,\mu)$. To study the boundedness of I_α^n, we define the *fractional maximal operator* $I_{a,\kappa}^*$ defined by the formula

$$I_{a,\kappa}^* f(x) := \int_{\mathbf{R}^d} K_{a,\kappa}^*(x,y) f(y) \, d\mu(y)$$

where

$$K_{a,\kappa}^*(x,y) := \sup_{Q \ni x,y} \mu(\kappa Q)^{a-1}.$$

Here $0 < a < 1$ and $\kappa > 1$.

Theorem 3.4 (Gunawan *et al.,* 2007). *Let* $a = \frac{\alpha}{n}$. *If the function* ϕ *is surjective and satisfies the inequality* $\phi(t) \le Ct^b$ *with* $-\frac{1}{p} \le b \le -a < 0$, *then* $I_{a,\kappa}^*$ *is bounded from* $\mathrm{M}^{p,\phi}(\mu)$ *to* $\mathrm{M}^{q,\phi^{p/q}}(\mu)$, *where* $p > 1$ *and* $q = \frac{bp}{a+b}$.

The proof of the theorem employs the boundedness of M_k on $\mathrm{M}^{p,\phi}(\mu)$ for $p > 1$ and a covering lemma.

Since $I_\alpha^n f \le C I_{\alpha/n,\kappa}^*$ for all positive μ-measurable functions (Sawano *et al.*, 2006), we have the following corollary.

Corollary 3.5 (Gunawan *et al.,* 2007). *Let a, b and* ϕ *be as in Theorem 3.2. Then* I_α^n *is bounded from* $\mathrm{M}^{p,\phi}(\mu)$ *to* $\mathrm{M}^{q,\phi^{p/q}}(\mu)$, *where* $p > 1$ *and* $q = \frac{bp}{a+b}$.

Furthermore, as in the case of homogeneous spaces, we also have an Olsen-type inequality.

Corollary 3.6 (Gunawan *et al.,* 2007). *Let a, b and* ϕ *be as in Theorem 3.2. Then* WI_α^n *is bounded on* $\mathrm{M}^{p,\phi}(\mu)$ *provided that* $W \in \mathrm{M}^{s,\phi^{p/s}}(\mu)$ *where* $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

4. The Generalized Fractional Integral Operator T_ρ

We shall now discuss the generalized fractional integral operators, which were introduced by Nakai (2001a) and Nakai and Sumitomo (2001).

4.1 As a generalization of I_α (in homogeneous setting), we may define the fractional integral operator T_ρ - for a given function $\rho : (0,\infty) \to (0,\infty)$ - by

$$T_\rho f(x) := \int_{\mathbf{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y) \, dy.$$

Note that for $\rho(t) = t^\alpha$, $0 < \alpha < d$, we have $T_\rho = I_\alpha$.

In general, we assume that the function $\rho : (0,\infty) \to (0,\infty)$ satisfies the doubling condition. Examples other than $\rho(t) = t^\alpha$ can be found in (Eridani *et al.*, 2004).

Under appropriate conditions on ρ, ϕ and ψ, Nakai (2001a) showed the $\mathrm{M}^{1,\phi} - \mathrm{M}^{1,\psi}$ boundedness of

T_ρ. Under similar assumptions on ρ, ϕ and ψ, (Eridani, 2002) extended Nakai's result by showing the $\mathrm{M}^{p,\phi} - \mathrm{M}^{p,\psi}$ boundedness of T_ρ for $1 < p < \infty$.

The above result, however, cannot be viewed as the generalization of the known result for I_α. Later on, Eridani and Gunawan (2002) proved the $\mathrm{M}^{p,\phi} - \mathrm{M}^{q,\phi^{p/q}}$ boundedness (for $1 < p < q < \infty$) of T_ρ, which generalizes the result for I_α. Nevertheless their assumptions on ρ and ϕ are not compatible with Nakai's. The following theorem provides a link between the result of Nakai (2001a) and Eridani and Gunawan (2002).

Theorem 4.1 (Gunawan, 2003). *Suppose that ρ satisfies the doubling condition. Suppose also ϕ is surjective and satisfies*

$$\int_r^\infty \frac{\phi(t)}{t}\,dt \le C\phi(r)$$

and for $1 < p < q < \infty$

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\phi(r)^{p/q}$$

for every $r > 0$. Then we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C\|f\|_{p,\phi}.$$

Observe that for $\rho(t) = t^\alpha$ and $\phi(t) = t^{\frac{\lambda-d}{p}}$, the above result reduces to Adams'. For more results on T_ρ, see Nakai (2002; 2004) and (Eridani *et al.,* 2004).

4.2 Moving to the non-homogeneous case, we put

$$D(x,y) := \inf_{x,y\in Q\in\mathcal{Q}(\mu)} \mu(\kappa Q).$$

for some $\kappa > 1$. Here $D(x,y)$ replaces the expression $|x-y|^n$ in I_α^n. Now put

$$l_\rho(x,y) := \frac{\rho(D(x,y))}{D(x,y)}.$$

and define the operator T_ρ^n by

$$T_\rho^n f(x) := \int_{\mathbf{R}^d} l_\rho(x,y) f(y)\,d\mu(y).$$

Observe that $\rho(t) = t^a$, the operator T_ρ^n is of the form $I_{a,\kappa}^*$ which we have discussed earlier.

Theorem 4.2 (Gunawan *et al.,* 2007). *Let $1 < p < q < \infty$. Assume that ϕ is surjective and ρ satisfies the doubling condition. Suppose further that there exists $C > 0$ such that*

$$\phi(r)\int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \le C\phi(r)^{\frac{p}{q}}$$

for every $r > 0$. Then, there exists $C > 0$ such that

$$\|T_\rho f\|_{q,\phi^{p/q},\mu} \le C\|f\|_{p,\phi,\mu}$$

for all positive $f \in \mathrm{M}^{p,\phi}(\mu)$.

Corollary 4.3 (Gunawan *et al.,* 2007). Let ρ and ϕ satisfy the hypotheses of Theorem 4.2. Then we have

$$\|W.T_\rho f\|_{p,\phi,\mu} \le C.\|W\|_{s,\phi^{p/s},\mu}\|f\|_{p,\phi,\mu}$$

for $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$, $1 < p < q < \infty$.

5. Concluding Remarks

The results that we presented and discussed here are not the whole known results for fractional integral operators. If one searches through *MathSciNet* database http://www.ams.org/mathscinet, one would find more than five hundreds papers on "fractional integrals" and almost two hundreds papers on "Riesz potentials". This shows how intensive the research on the subject is.

Besides the "strong" boundedness property, many researchers also studied the "weak-type" inequality for fractional integral operators, especially for end points values of p (and q). Many other researchers are interested in studying the behaviour of the operators on other function spaces, and some other researchers investigate its applications in differential equations or physics.

What we shall do in the future is to investigate whether we can soften the assumptions on ρ, ϕ and ψ but yet the $\mathrm{M}^{p,\phi} - \mathrm{M}^{q,\psi}$ boundedness of the operator T_ρ still holds in both homogeneous and non-homogeneous contexts. In particular, we would like to know if the assumption that ϕ and ψ satisfy the doubling condition is really necessary or can be replaced by a weaker condition.

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