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IDEAL CONVERGENCE IN 2-NORMED SPACES

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Abstract. In this paper we introduce and investigate $\mathcal{I}-$convergence in $2-$normed spaces, and also define and examine some new sequence spaces using $2-$norm.

1. Introduction

The notion of ideal convergence was introduced first by P. Kostyrko et al. [6] as an interesting generalization of statistical convergence $[1, 11]$.

The concept of $2-$normed spaces was initially introduced by Gähler [2] in the 1960's. Since then, this concept has been studied by many authors, see for instance $[3, 10]$.

In a natural way, one may unite these two concepts, and study $\mathcal{I}-$convergence in $2-$normed spaces. This is actually what we offer in this article. Furthermore we define and investigate some sequence spaces by using $2-$norm.

Throughout this paper $\mathbb{N}$ will denote the set of positive integers. Let $(X, \|.\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

A family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (**iii**) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ $[7, 8]$.

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ in X is said to be $\mathcal{I}-$convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ $[6, 7]$.

Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A $2-$norm on X is a function $\|., .\| : X \times X \to \mathbb{R}$ which satisfies (**i**) $\|x, y\| = 0$ if and only

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if x and y are linearly dependent; (**ii**) $\|x,y\| = \|y,x\|$; (**iii**) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$, $\alpha \in \mathbb{R}$; (**iv**) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$. The pair $(X, \|.,.\|)$ is then called a $2-$normed space [2]. As an example of a $2-$normed space we may take $X = \mathbb{R}^2$ being equipped with the $2-$norm $\|x,y\| :=$ the area of the parallelogram spanned by the vectors x and y, which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|\,, \quad x = (x_1,x_2)\,, \quad y = (y_1,y_2)\,.$$

Recall that $(X, \|.,.\|)$ is a $2-$Banach space if every Cauchy sequence in X is convergent to some x in X.

2. Ideal Convergence of 2-Normed Spaces

Throughout the paper we assume X to be a $2-$normed space having dimension d, where $2 \leq d < \infty$.

Definition 2.1. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_n) of X is said to be $\mathcal{I}-$convergent to x, if for each $\varepsilon > 0$ and z in X the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x, z\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.

If (x_n) is $\mathcal{I}-$convergent to x then we write $\mathcal{I} - \lim\limits_{n\to\infty} \|x_n - x, z\| = 0$ or $\mathcal{I} - \lim\limits_{n\to\infty} \|x_n, z\| = \|x, z\|$. The number x is $\mathcal{I}-$limit of the sequence (x_n).

Further we will give some examples of ideals and corresponding $\mathcal{I}-$ convergences.

(i) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence coincides with usual convergence in [2] .

(ii) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$ convergence coincides with the statistical convergence in [4].

Now we give an example of $\mathcal{I}-$convergence in $2-$normed spaces.

Example 2.1. Let $\mathcal{I} = \mathcal{I}_\delta$. Define the (x_n) in $2-$normed space $(X, \|.,.\|)$ by

$$x_n = \begin{cases} (0,n) & , n = k^2, k \in \mathbb{N} \\ (0,0) & , \text{otherwise.} \end{cases}$$

and let $L = (0,0)$ and $z = (z_1, z_2)$. Then for every $\varepsilon > 0$ and $z \in X$

$$\{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\} \subset \left\{1, 4, 9, 16, ..., n^2, ...\right\}.$$

We have that $\delta(\{n \in \mathbb{N} : \|x_n - L, z\| \geq \varepsilon\}) = 0$, for every $\varepsilon > 0$ and $z \in X$. This implies that $st - \lim\limits_{n\to\infty} \|x_n, z\| = \|L, z\|$. But, the sequence (x_n) is not convergent to L.

We note that the stated claims given in Proposition 3.1 and Remark 3.1 of [6] are also hold in $2-$normed spaces.

We next provide a proof of the fact that $\mathcal{I}-$limit operation for sequences in $2-$normed space $(X, \|.,.\|)$ is linear with respect to summation and scalar multiplication.

Theorem 2.1. *Let $\mathcal{I}$ be an admissible ideal. For each $z \in X$,*

(i) *If* $\mathcal{I}-\lim_{n\to\infty} \|x_n, z\| = \|x, z\|$, $\mathcal{I}-\lim_{n\to\infty} \|y_n, z\| = \|y, z\|$ *then* $\mathcal{I}-\lim_{n\to\infty} \|x_n + y_n, z\| = \|x + y, z\|$;

(ii) $\mathcal{I}-\lim_{n\to\infty} \|ax_n, z\| = \|ax, z\|$, $a \in \mathbb{R}$;

Proof. (**i**) Let $\varepsilon > 0$. Then $K_1, K_2 \in \mathcal{I}$ where

$$K_1 = K_1(\varepsilon) := \left\{ n \in \mathbb{N} : \|x_n - x, z\| \geq \frac{\varepsilon}{2} \right\}$$

and

$$K_2 = K_2(\varepsilon) := \left\{ n \in \mathbb{N} : \|y_n - y, z\| \geq \frac{\varepsilon}{2} \right\}$$

for each $z \in X$. Let

$$K = K(\varepsilon) := \{ n \in \mathbb{N} : \|(x_n + y_n) - (x + y), z\| \geq \varepsilon \}.$$

Then the inclusion $K \subset K_1 \cup K_2$ holds and the statement follows.

(**ii**) Let $\mathcal{I}-\lim_{n\to\infty} \|x_n, z\| = \|L, z\|$, $a \in \mathbb{R}$ and $a \neq 0$. Then

$$\left\{ n \in \mathbb{N} : \|x_n - L, z\| \geq \frac{\varepsilon}{|a|} \right\} \in \mathcal{I}.$$

Then by definition 2.1, we have

$$\{ n \in \mathbb{N} : \|ax_n - aL, z\| \geq \varepsilon \} = \left\{ n \in \mathbb{N} : \|x_n - L, z\| \geq \frac{\varepsilon}{|a|} \right\}.$$

Hence, the right hand side of above equality belongs to $\mathcal{I}$. Hence, $\mathcal{I}-\lim_{n\to\infty} \|ax_n, z\| = \|aL, z\|$ for every $z \in X$. ■

Fix $u = \{u_1, ..., u_d\}$ to be a basis for X. Then we have the following:

Lemma 2.2. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_n) in X is $\mathcal{I}-$convergent to x in X if and only if* $\mathcal{I}-\lim_{n\to\infty} \|x_n - x, u_i\| = 0$ *for every* $i = 1, ..., d$.

Using Lemma 2.2 and the norm $\|.\|_\infty$, we have:

Lemma 2.3. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_n) in X is $\mathcal{I}-$convergent to x in X if and only if $\mathcal{I}-\lim_{n\to\infty}\|x_n - x\|_\infty = 0$.*

Using open balls $B_u(x,\varepsilon)$, Lemma 2.3 becomes:

Lemma 2.4. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_n) in X is $\mathcal{I}-$convergent to x in X if and only if $A(\varepsilon) = \{n \in \mathbb{N} : x_n \notin B_u(x,\varepsilon)\}$ belongs to ideal.*

Now we introduce the concept $\mathcal{I}-$Cauchy sequence in $2-$normed spaces X.

Definition 2.2. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_n) of X is said to be $\mathcal{I}-$Cauchy sequence in X, if for each $\varepsilon > 0$ and $z \in X$ there exists a number $N = N(\varepsilon, z)$ such that

$$\left\{k \in \mathbb{N} : \left\|x_k - x_{N(\varepsilon,z)}, z\right\| \geq \varepsilon\right\} \in \mathcal{I}.$$

We give a similar result as in [3, Lemma 1.2].

Theorem 2.5. *Let $\mathcal{I}$ be an admissible ideal. For a given $\mathcal{I}-$Cauchy sequence (x_n) in X with any of the norms $\|.,.\|$ or $\|.\|_\infty$, the following are equivalent.*

(i) (x_n) is $\mathcal{I}-$convergent in $(X, \|.,.\|)$.

(ii) (x_n) is $\mathcal{I}-$convergent in $(X, \|.\|_\infty)$.

Proof. From Lemma 2.3, $\mathcal{I}-$convergence in the $2-$norm is equivalent to that in the $\|.\|_\infty$ norm. That is,

$$\mathcal{I}-\lim_{n\to\infty}\|x_n - x, z\| = 0, \forall z \in X \Leftrightarrow \mathcal{I}-\lim_{n\to\infty}\|x_n - x\|_\infty .$$

It is sufficient to show that (x_n) is $\mathcal{I}-$Cauchy sequence with respect to the $2-$norm iff it is $\mathcal{I}-$Cauchy sequence with respect to the norm $\|.\|_\infty$. However the proof of the latter can be obtained in a very similar way as in [3, Lemma 2.6] by using ideals. ∎

Note that all of these results imply the similar theorems for convergence of sequences in $2-$normed space X which are investigated in [3].

3. New Sequence Spaces

In this section we introduce some new sequence spaces and verify some of their properties.

Let $(X, \|.,.\|)$ be any $2-$normed spaces and $S(2-X)$ denotes $X-$ valued sequences spaces. Clearly $S(2-X)$ is a linear space under addition and scalar multiplication.

Recall that a map $g : X \to \mathbb{R}$ is called a paranorm (on X) if it satisfies the following conditions : (**i**) $g(\theta) = 0$ (Here θ is zero of the space); (**ii**) $g(x) = g(-x)$; (**iii**) $g(x+y) \leq g(x) + g(y)$; (**iv**) $\lambda^n \to \lambda$ $(n \to \infty)$ and $g(x^n - x) \to 0$ $(n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0$ $(n \to \infty)$ for all $x, y \in X$ [9].

Now we define the following sequence space.

Definition 3.1.

$$l(2-p) = \left\{ x \in S(2-X) : \sum_k \|x_k, z\|^{p_k} < \infty, \ \forall z \in S(2-X) \right\}.$$

Lemma 3.1. *The sequence space $l(2-p)$ is a linear space.*

Proof. Let $p_k > 0$, $(\forall k)$, $H = \sup p_k$ and $a_k, b_k \in \mathbb{C}$ (complex numbers). Then

$$|a_k + b_k|^{p_k} \leq C\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\}, \quad C = \max\left\{1, 2^{H-1}\right\},$$

[9]. Hence, if $|\lambda| \leq L$ and $|\mu| \leq M$; L, M integers, $x, y \in l(2-p)$ (omitting subscript k) then we get

$$\|\lambda x + \mu y, z\|^{p_k} \leq CL^H (\|x, z\|)^{p_k} + CM^H (\|y, z\|)^{p_k}.$$

The desired result is obtained by taking sum over k. ■

Definition 3.2. Let $t_k = \sum_{i=1}^{k} \|x_i, z\|^{p_i}$ and $\mathcal{I}$ be an admissible ideal. Then we define the new sequences space as follows:

$$l^{\mathcal{I}}(2-p) = \{x \in S(2-X) : \{k \in \mathbb{N} : \|t_k - t, z\| \geq \varepsilon \ \forall z \in S(2-X)\} \in \mathcal{I}\}.$$

Theorem 3.2. *Let $\mathcal{I}$ an admissible ideal. $l^{\mathcal{I}}(2-p)$ sequences space is a linear space.*

Proof. This can be easily verified by using properties of ideal and partial sums of sequences as in the above Lemma 4.1. ■

Theorem 3.3. *$l(2-p)$ space is a paranormed space with the paranorm defined by $g : l(2-p) \to \mathbb{R}$,*

$$g(x) = \left(\sum_k \|x_k, z\|^{p_k} \right)^{\frac{1}{M}},$$

where $0 < p_k \leq \sup p_k = H,\ M = \max(1, H)$.

Proof.

(i) $g(\theta) = \left(\sum_k \|\theta_k, z\|^{p_k}\right)^{\frac{1}{M}} = 0$

(ii) $g(-x) = \left(\sum_k \|-x_k, z\|^{p_k}\right)^{\frac{1}{M}} = \left(\sum_k |-1|\,\|x_k, z\|^{p_k}\right)^{\frac{1}{M}} = g(x)$

(iii) Using well known inequalities

$$\begin{aligned} g(x+y) &= \left(\sum_k \|x_k + y_k, z\|^{p_k}\right)^{\frac{1}{M}} \\ &\leq \left(\sum_k \left(\|x_k, z\|^{\frac{p_k}{M}}\right)^M\right)^{\frac{1}{M}} + \left(\sum_i \left(\|y_k, z\|^{\frac{p_k}{M}}\right)^M\right)^{\frac{1}{M}} \\ &= g(x) + g(y). \end{aligned}$$

(iv) Now let $\lambda^n \to \lambda$ and $g(x^n - x) \to 0\ (n \to \infty)$. We have

$$\begin{aligned} g(\lambda^n x^n - \lambda x) &= \left(\sum_k \|\lambda^n x_k^n - \lambda x_k, z\|^{p_k}\right)^{\frac{1}{M}} \\ &\leq |\lambda|^{\frac{H}{M}} \left(\sum_k \|x_k^n - x_k, z\|^{pk}\right)^{\frac{1}{M}} + \left(\sum_k |\lambda^n - \lambda|\,\|x_k, z\|^{p_k}\right)^{\frac{1}{M}}. \end{aligned}$$

In this inequality, the first term of the right hand side tends to zero because $g(x^n - x) \to 0\ (n \to \infty)$. On the other hand, since $\lambda^n \to \lambda\ (n \to \infty)$, the second term also tends to zero by Lemma 4.1. ■

Theorem 3.4. *If* $(X, \|.,.\|)$ *is finite dimensional* $2-$*Banach space then* $(l(2-p), g)$ *is complete.*

Proof. Let (x^n) be a Cauchy sequence in $(l(2-p), g)$. Then for each $\varepsilon > 0$ there exists some $N_0 \in \mathbb{N}$ such that for each $m, n > N_0$ we have

$$g(x^n - x^m) = \left(\sum_k \|x_k^n - x_k^m, z\|^{p_k}\right)^{\frac{1}{M}} < \varepsilon,$$

which implies $\left(\|x_k^n - x_k^m, z\|^{p_n}\right)^{\frac{1}{M}} < \varepsilon$. So, (x^n) is a Cauchy sequence in $(X, \|.,.\|)$ and since $(X, \|.,.\|)$ is a $2-$Banach space, there exists an x in X such that $\|x_k^n - x_k, z\| \to 0 \ (n \to \infty) \ (\forall z \in X)$. ∎

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References

1. H. Fast, Sur la convergence statistique, *Colloq. Math.,* **2** (1951), 241-244.
2. S. Gähler, $2-$metrische Räume und ihre topologische Struktur, *Math. Nachr.*, **26** (1963), 115-148.
3. H. Gunawan and Mashadi, On Finite Dimensional $2-$normed spaces, *Soochow J. Math.*, **27(3)** (2001), 321-329.
4. M. Gürdal and S. Pehlivan, The Statistical Convergence in $2-$Banach Spaces, *Thai J. Math.,* **2(1)** (2004), 107-113.
5. J. L. Kelley, General Topology, Springer-Verlag, New York (1955).
6. P. Kostyrko, M. Macaj and T. Salat, $I-$Convergence, *Real Anal. Exchange*, **26(2)** (2000), 669-686.
7. P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, $I-$Convergence and Extremal $I-$Limit Points, *Math. Slovaca.*, **55** (2005), 443-464.
8. C. Kuratowski, Topology I., PWN, Warszawa, (1958).
9. I. J. Maddox, Elements of Functional Analysis, Cambridge at the Uni. Press, (1970).
10. W. Raymond, Y. Freese and J. Cho, Geometry of linear $2-$normed spaces, N.Y. Nova Science Publishers, Huntington, 2001.
11. H. Steinhaus, Sur la convergence ordinarie et la convergence asymptotique, *Colloq. Math.,* **2** (1951), 73-74.

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