

PROPER INCLUSIONS OF MORREY SPACES

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ABSTRACT. In this paper, we prove that the inclusions between Morrey spaces, between weak Morrey spaces, and between a Morrey space and a weak Morrey space are all proper. The proper inclusion between a Morrey space and a weak Morrey space is established via the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. In addition, we also give a necessary condition for each inclusion. Our results refine previous inclusion properties studied in [4].

1. INTRODUCTION

Morrey spaces were first introduced by C. B. Morrey in [7] in relation to the study of the solution of certain elliptic partial differential equations. For $1 \leq p \leq q < \infty$, the *Morrey space* $\mathcal{M}_q^p = \mathcal{M}_q^p(\mathbb{R}^d)$ is defined to be the set of all $f \in L_{loc}^p(\mathbb{R}^d)$ such that

$$\|f\|_{\mathcal{M}_q^p} := \sup_{a \in \mathbb{R}^d, r > 0} |B(a, r)|^{\frac{1}{q}} \left(\frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty.$$

Here, $B(a, r)$ is an open ball centered at a with radius r , and $|B(a, r)|$ denotes its Lebesgue measure. Notice that, when $p = q$, one can recover the Lebesgue space $L^p = L^p(\mathbb{R}^d)$ as the special case of \mathcal{M}_q^p . See [9] for various spaces related to Morrey spaces. Many researchers have proved the boundedness of classical integral operators on Morrey spaces and their generalizations. See, for instance, [1, 2] and the references therein.

Concerning the Hardy-Littlewood maximal operator (defined in Section 3), one may prove its boundedness on Morrey spaces using the inclusion $\mathcal{M}_q^p \subseteq$

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\mathcal{M}_q^1 . In general, we have the following inclusions of Morrey spaces

$$L^q = \mathcal{M}_q^q \subseteq \mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1} \subseteq \mathcal{M}_q^1$$

provided that $1 \leq p_1 \leq p_2 \leq q < \infty$. These inclusions may be obtained by applying Hölder's inequality. Note that, for $1 \leq p_2 < q < \infty$, we have $f(x) := |x|^{-\frac{d}{q}} \in \mathcal{M}_q^{p_2} \setminus \mathcal{M}_q^q$. This tells us that the inclusion $\mathcal{M}_q^q \subseteq \mathcal{M}_q^{p_2}$ is proper for $1 \leq p_2 < q < \infty$.

Besides the 'strong' Morrey spaces, we also have weak Morrey spaces whose definitions are given as follows.

DEFINITION 1.1. *Let $1 \leq p \leq q < \infty$. A measurable functions f on \mathbb{R}^d is said to belong to the weak Morrey space $w\mathcal{M}_q^p = w\mathcal{M}_q^p(\mathbb{R}^d)$ if the quasi-norm*

$$\|f\|_{w\mathcal{M}_q^p} := \sup_{\gamma > 0} \|\gamma \chi_{\{|f| > \gamma\}}\|_{\mathcal{M}_q^p}$$

is finite.

Note that, by using the inequality $\gamma \chi_{\{|f| > \gamma\}} \leq |f|$ for every $\gamma > 0$, we have $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$. The inclusion properties of weak Morrey spaces, generalized Morrey spaces, generalized weak Morrey spaces, and their necessary conditions were discussed in [4]. In particular, for the case of Morrey spaces and weak Morrey spaces, the results can be stated as follows.

THEOREM 1.2 ([4]). *For $1 \leq p_1 \leq p_2 \leq q < \infty$, the following inclusion holds*

$$w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}.$$

Further, if $p_1 < p_2$, then

$$w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}.$$

In addition to the above inclusion relations of Morrey spaces, we have the following theorems.

THEOREM 1.3. *Let $1 \leq p_1 < p_2 < q < \infty$. Then each of the following inclusions is proper:*

- (i) $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$;
- (ii) $w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$;
- (iii) $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$.

THEOREM 1.4. *Let $1 \leq p \leq q < \infty$. Then the inclusion $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$ is proper.*

REMARK 1.5. The claim about the proper inclusion $\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$ is stated in [4, p. 2] without proof. We shall see the detailed explanation of this claim in the proof of Theorem 1.3 (i). In [4, Remark 2.4], the authors refer to [3] for the proper inclusion between the generalized Morrey space $L^{1,\phi}$ and the corresponding weak type space $wL^{1,\phi}$, where $\phi(t) = t^{-1} \log(3+t)$. Since

$L^{1,\phi} \neq \mathcal{M}_q^1$ for this choice of ϕ , Theorem 1.4 can be seen as a complement of the result in [3].

We also obtain the following necessary conditions for inclusion of Morrey spaces and weak Morrey spaces which can be seen as a refinement of some necessary conditions given in [4].

THEOREM 1.6. *Let $1 \leq p_i \leq q_i < \infty$ for $i = 1, 2$. Then the following implications hold:*

- (i) $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$ implies $q_1 = q_2$ and $p_1 \leq p_2$;
- (ii) $w\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_1}^{p_1}$ implies $q_1 = q_2$ and $p_1 \leq p_2$;
- (iii) $w\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$ implies $q_1 = q_2$ and $p_1 < p_2$.

REMARK 1.7. A necessary and sufficient condition for inclusion of Morrey spaces on a bounded domain can be found in [10, Theorem 2.1] and [11]. The case of Morrey spaces on \mathbb{R}^d is mentioned in [6, Eq. (3.9)] and the authors refer to [12, Satz 1.6]. However, we do not have the access to the paper, so that we do not know how the proof goes. See also [6, Corollary 3.14] for weighted version of Theorem 1.6. Here we present a proof of the necessary and sufficient condition for the inclusion property, which is different from and simpler than that in [10].

The organization of this paper is as follows. In the next section, we prove that for $1 \leq p_1 < p_2 < q < \infty$ the set $\mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$ is not empty. By the same example, we also show that for $1 \leq p_1 < p_2 < q$ the inclusion $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$ is proper. In Section 3, we give the proof of Theorem 1.4 using the unboundedness of the Hardy-Littlewood maximal operator on Morrey spaces of exponent 1. The proof of Theorem 1.6 is given in the last section. Throughout this paper, we denote by C a positive constant which is independent of the function f and its value may be different from line to line.

2. THE PROOF OF THEOREM 1.3

We shall first prove Theorem 1.3 (i) by constructing a function which belongs to $\mathcal{M}_q^{p_1}$ but not to $\mathcal{M}_q^{p_2}$, for $1 \leq p_1 < p_2 < q < \infty$.

PROOF OF THEOREM 1.3 (i). Let $1 \leq p_1 < p_2 < q < \infty$ and $\beta := \frac{d(p_1+p_2)}{2q}$. Then we have

$$(2.1) \quad \frac{dp_1}{q} < \beta < \frac{dp_2}{q}$$

and

$$d - \beta = \frac{d(q - p_1) + d(q - p_2)}{2q} > 0.$$

Define $g(x) := \chi_{B(0,1)}(x) + \chi_{\mathbb{R}^n \setminus B(0,1)}(x)|x|^{-\beta}$. Then, for each $k \in \mathbb{N}$, we choose $r_k \in (k, k+1)$ such that

$$\int_{B(0,k+1) \setminus B(0,k)} g(x) dx = |B(0, r_k) \setminus B(0, k)|.$$

Next define

$$(2.2) \quad f(x) := \chi_{B(0,1)}(x) + \sum_{k=1}^{\infty} \chi_{B(0,r_k) \setminus B(0,k)}(x).$$

We shall show that $f \in \mathcal{M}_q^{p_1} \setminus \mathcal{M}_q^{p_2}$. First observe that

$$\int_{B(a,r)} |f(x)|^p dx \leq \int_{B(0,r)} |f(x)|^p dx$$

for every $1 \leq p < \infty$, $a \in \mathbb{R}^d$, $r > 0$. Now, for $1 \leq p < \infty$ and $r > 2$, we have

$$\int_{B(0,r)} |f(x)|^p dx = \int_{B(0,r)} |f(x)| dx \leq \int_{B(0,2r)} g(x) dx,$$

so

$$(2.3) \quad \int_{B(0,r)} |f(x)|^p dx \leq \int_{B(0,2r)} |x|^{-\beta} dx = Cr^{d-\beta}$$

and

$$(2.4) \quad \begin{aligned} \int_{B(0,r)} |f(x)|^p dx &\geq \int_{B(0,r) \setminus B(0,1)} |x|^{-\beta} dx \\ &= C(r^{d-\beta} - 1) \geq C \left(1 - \frac{1}{2^{d-\beta}}\right) r^{d-\beta}. \end{aligned}$$

Therefore, by substituting $p = p_1$ into (2.3) and recalling (2.1), we have

$$(2.5) \quad |B(0,r)|^{\frac{1}{q} - \frac{1}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \leq Cr^{\frac{d}{q} - \frac{d}{p_1}} r^{\frac{d}{p_1} - \frac{\beta}{p_1}} = Cr^{\frac{d}{q} - \frac{\beta}{p_1}} \leq C.$$

On the other hand, for each $r \leq 2$, we have

$$(2.6) \quad \begin{aligned} |B(0,r)|^{\frac{1}{q} - \frac{1}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} &\leq Cr^{\frac{d}{q} - \frac{d}{p_1}} \left(\int_{B(0,r)} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ &\leq Cr^{\frac{d}{q}} \leq C. \end{aligned}$$

By combining (2.5) and (2.6) we conclude that $f \in \mathcal{M}_q^{p_1}$.

Meanwhile, by substituting $p = p_2$ into (2.4), we have

$$|B(0,r)|^{\frac{1}{q} - \frac{1}{p_2}} \left(\int_{B(0,r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \geq Cr^{\frac{d}{q} - \frac{d}{p_2}} r^{\frac{d-\beta}{p_2}} = Cr^{\frac{d}{q} - \frac{\beta}{p_2}}.$$

Since $\frac{d}{q} - \frac{\beta}{p_2} > 0$, we have

$$\begin{aligned} \sup_{a \in \mathbb{R}^d, r > 0} |B(a, r)|^{\frac{1}{q} - \frac{1}{p_2}} \left(\int_{B(a, r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \\ \geq C \sup_{r > 2} |B(0, r)|^{\frac{1}{q} - \frac{1}{p_2}} \left(\int_{B(0, r)} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \\ \geq C \sup_{r > 2} r^{\frac{d}{q} - \frac{\beta}{p_2}} = \infty. \end{aligned}$$

Thus $f \notin \mathcal{M}_q^{p_2}$, and we are done. \square

Theorem 1.3 (ii) and (iii) are proved by using the function f from the proof of Theorem 1.3 (i) and its relation with the characteristic function of its level set. The detailed proof goes as follows.

PROOF OF THEOREM 1.3 (ii)-(iii). For $1 \leq p_1 < p_2 < q < \infty$, let f be defined by (2.2). Observe that

$$\chi_{\{|f| > \gamma\}} = \begin{cases} 0, & \gamma \geq 1, \\ f, & \gamma \in (0, 1). \end{cases}$$

This together with the fact that $f \notin \mathcal{M}_q^{p_2}$ gives

$$\|f\|_{w\mathcal{M}_q^{p_2}} = \sup_{\gamma \in (0, 1)} \gamma \|\chi_{\{|f| > \gamma\}}\|_{\mathcal{M}_q^{p_2}} = \sup_{\gamma \in (0, 1)} \gamma \|f\|_{\mathcal{M}_q^{p_2}} = \|f\|_{\mathcal{M}_q^{p_2}} = \infty,$$

and hence $f \in \mathcal{M}_q^{p_1} \setminus w\mathcal{M}_q^{p_2}$. Thus we have shown that $w\mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}$ is a proper inclusion. Since $\mathcal{M}_q^{p_1} \subseteq w\mathcal{M}_q^{p_1}$, we also have $f \in w\mathcal{M}_q^{p_1} \setminus w\mathcal{M}_q^{p_2}$, so the inclusion (iii) is proper. \square

3. THE PROOF OF THEOREM 1.4

In order to prove Theorem 1.4, we need the following lemma.

LEMMA 3.1. *Let $1 \leq p \leq q < \infty$. Then*

$$\|f\|_{\mathcal{M}_q^p} = \| |f|^p \|_{\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}$$

for every $f \in \mathcal{M}_q^p$ and

$$\|f\|_{w\mathcal{M}_q^p} = \| |f|^p \|_{w\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}$$

for every $f \in w\mathcal{M}_q^p$.

PROOF. We calculate

$$\|f\|_{\mathcal{M}_q^p} = \sup_B \left(|B|^{\frac{p}{q}-1} \int_B |f(x)|^p dx \right)^{\frac{1}{p}} = \| |f|^p \|_{\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}}.$$

By applying the first identity for $\chi_{\{|f|>\gamma^{\frac{1}{p}}\}}$, we have

$$\| |f|^p \|_{w\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}} = \sup_{\gamma>0} \gamma^{\frac{1}{p}} \| \chi_{\{|f|^p>\gamma\}} \|_{\mathcal{M}_{\frac{q}{p}}^1}^{\frac{1}{p}} = \sup_{\gamma>0} \gamma^{\frac{1}{p}} \| \chi_{\{|f|>\gamma^{\frac{1}{p}}\}} \|_{\mathcal{M}_q^p} = \|f\|_{w\mathcal{M}_q^p},$$

as desired. □

We also use the following fact about the unboundedness of the Hardy-Littlewood maximal operator M on Morrey spaces of exponent 1. The operator M maps a locally integrable function f to Mf which is given by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^d.$$

LEMMA 3.2. *The Hardy-Littlewood maximal operator M is not bounded on the Morrey space \mathcal{M}_q^1 for $1 < q < \infty$.*

REMARK 3.3. Lemma 3.2 is a consequence of a necessary condition of the boundedness of M on generalized Orlicz-Morrey spaces given in [8, Corollary 5.3]. The Morrey space \mathcal{M}_q^p in this paper is recognized as the Orlicz-Morrey space $L^{(\Phi,\phi)}$ with $\Phi(t) = t^p$ and $\phi(t) = t^{-\frac{1}{q}}$. Based on [8, Corollary 5.3], the maximal operator M is bounded on $L^{(\Phi,\phi)}$ if and only if $\Phi \in \nabla_2$ (that is, $\Phi(r) \leq \frac{1}{2k} \Phi(kr)$ for some $k \geq 1$). Clearly $\Phi(t) = t \notin \nabla_2$.

Now, we are ready to prove Theorem 1.4.

PROOF OF THEOREM 1.4. Let $1 \leq p \leq q$. If $p = q$, then $f(x) := |x|^{-\frac{d}{q}} \in w\mathcal{M}_q^p \setminus \mathcal{M}_q^p$. So assume that $p < q$ and write $r = \frac{q}{p}$. In view of Lemma 3.1, it suffices for us to prove that $\mathcal{M}_r^1 \subset w\mathcal{M}_r^1$ properly. Suppose to the contrary that $\mathcal{M}_r^1 = w\mathcal{M}_r^1$. Since the Hardy-Littlewood maximal operator M is bounded from \mathcal{M}_r^1 to $w\mathcal{M}_r^1$, we obtain

$$\|Mg\|_{w\mathcal{M}_r^1} \leq C \|g\|_{\mathcal{M}_r^1},$$

for every $g \in \mathcal{M}_r^1$. Meanwhile, by the Closed Graph Theorem, there must exist a constant $C' > 0$ such that

$$\|Mg\|_{\mathcal{M}_r^1} \leq C' \|Mg\|_{w\mathcal{M}_r^1}$$

for every $g \in \mathcal{M}_r^1$. Combining the two inequalities, we obtain

$$\|Mg\|_{\mathcal{M}_r^1} \leq C \|g\|_{\mathcal{M}_r^1}$$

for every $g \in \mathcal{M}_r^1$. This tells us that M is bounded on \mathcal{M}_r^1 , which contradicts Lemma 3.2. Therefore, $w\mathcal{M}_r^1 \setminus \mathcal{M}_r^1 \neq \emptyset$, as desired. □

To conclude this section, we write a proposition which gives us a subset of weak Morrey spaces with norm equivalence between the Morrey norm $\|\cdot\|_{\mathcal{M}_q^p}$ and the weak Morrey quasi-norm $\|\cdot\|_{w\mathcal{M}_q^p}$.

PROPOSITION 3.4. *Let $1 \leq p < q < \infty$. Suppose that f is a positive radial decreasing function in $w\mathcal{M}_q^p(\mathbb{R}^d)$. Then $f \in \mathcal{M}_q^p(\mathbb{R}^d)$ with*

$$\|f\|_{w\mathcal{M}_q^p} \leq \|f\|_{\mathcal{M}_q^p} \leq \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|} \right)^{\frac{1}{p}} \|f\|_{w\mathcal{M}_q^p},$$

that is, $\|f\|_{w\mathcal{M}_q^p} \sim \|f\|_{\mathcal{M}_q^p}$.

PROOF. Recall that, since $\gamma\chi_{\{|f|>\gamma\}} \leq |f|$ for every $\gamma > 0$, we have $\|f\|_{w\mathcal{M}_q^p} \leq \|f\|_{\mathcal{M}_q^p}$. Next, let $x \in \mathbb{R}^d$. Since $\{y \in B(0,|x|) : f(y) > f(x)\} = B(0,|x|)$, we have

$$\begin{aligned} f(x) &= \frac{f(x)|\{y \in B(0,|x|) : f(y) > f(x)\}|^{\frac{1}{p}}}{|B(0,|x|)|^{\frac{1}{p}}} \\ &\leq \frac{|B(0,|x|)|^{\frac{1}{p}-\frac{1}{q}}\|f\|_{w\mathcal{M}_q^p}}{|B(0,|x|)|^{\frac{1}{p}}} = |B(0,1)|^{-\frac{1}{q}}\|f\|_{w\mathcal{M}_q^p}|x|^{-\frac{d}{q}}. \end{aligned}$$

By combining the last estimate and

$$\| |x|^{-\frac{d}{q}} \|_{\mathcal{M}_q^p} = |B(0,1)|^{\frac{1}{q}} \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|} \right)^{\frac{1}{p}},$$

where ω_{d-1} is the surface area of the unit sphere \mathbb{S}^{d-1} , we get

$$\|f\|_{\mathcal{M}_q^p} \leq (|B(0,1)|^{-\frac{1}{q}}\| |x|^{-\frac{d}{q}} \|_{\mathcal{M}_q^p})\|f\|_{w\mathcal{M}_q^p} = \left(\frac{q\omega_{d-1}}{d(q-p)|B(0,1)|} \right)^{\frac{1}{p}} \|f\|_{w\mathcal{M}_q^p}.$$

Hence $\|f\|_{w\mathcal{M}_q^p} \sim \|f\|_{\mathcal{M}_q^p}$. □

4. THE PROOF OF THEOREM 1.6

PROOF OF THEOREM 1.6 (i). It follows from the inclusion $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$ that

$$\|\chi_{B(0,r)}\|_{\mathcal{M}_{q_1}^{p_1}} \leq C\|\chi_{B(0,r)}\|_{\mathcal{M}_{q_2}^{p_2}},$$

for every $r > 0$. Therefore

$$r^{\frac{d}{q_1}-\frac{d}{q_2}} \leq C$$

for every $r > 0$, which implies that $q_1 = q_2$. Now choose $\epsilon \in (0, \min\{\frac{dp_1}{q_1}, \frac{dp_2}{q_2}\})$. For $j \in \mathbb{N}$, define $h_j(x) := \chi_{\{j \leq |x| \leq j+\epsilon\}}(x)$, and for $K \in \mathbb{N}$ write

$f(x) := \chi_{\{0 \leq |x| < 1\}}(x) + \sum_{j=1}^K h_j(x)$. Then

$$(4.1) \quad \begin{aligned} \|f\|_{\mathcal{M}_{q_1}^{p_1}} &\geq |B(0, K + K^{-\epsilon})|^{\frac{1}{q_1} - \frac{1}{p_1}} \left(\int_{B(0, K + K^{-\epsilon})} |f(x)|^{p_1} dx \right)^{\frac{1}{p_1}} \\ &\geq C(K + K^{-\epsilon})^{\frac{d}{q_1} - \frac{d}{p_1}} (K + K^{-\epsilon})^{\frac{d}{p_1} - \frac{\epsilon}{p_1}} = C(K + K^{-\epsilon})^{\frac{d}{q_1} - \frac{\epsilon}{p_1}}. \end{aligned}$$

Meanwhile, for each $L \in \mathbb{N}$, $L \leq K$, we observe that

$$|B(0, L + L^{-\epsilon})|^{\frac{1}{q_2} - \frac{1}{p_2}} \left(\int_{B(0, L + L^{-\epsilon})} |f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \leq C(L + L^{-\epsilon})^{\frac{d}{q_2} - \frac{\epsilon}{p_2}}.$$

Hence,

$$(4.2) \quad \|f\|_{\mathcal{M}_{q_2}^{p_2}} \leq C(K + K^{-\epsilon})^{\frac{d}{q_2} - \frac{\epsilon}{p_2}}.$$

By combining (4.1), (4.2), $q_1 = q_2$, and $\|f\|_{\mathcal{M}_{q_1}^{p_1}} \leq C\|f\|_{\mathcal{M}_{q_2}^{p_2}}$, we get

$$(K + K^{-\epsilon})^{\frac{\epsilon}{p_2} - \frac{\epsilon}{p_1}} \leq C.$$

As this holds for every $K \in \mathbb{N}$, we conclude that $p_1 \leq p_2$. \square

REMARK 4.1. Note that the difference between the proof of Theorem 1.6 (i) and [4, Remark 3.4] is that we do not assume $p_1 \leq p_2$.

PROOF OF THEOREM 1.6 (ii). By arguing as in the proof of Theorem 1.6 (i) and using the identities

$$\|\chi_{B(0,r)}\|_{w\mathcal{M}_{q_1}^{p_1}} = |B(0,r)|^{\frac{1}{q_1}}$$

and

$$\|\chi_{B(0,r)}\|_{w\mathcal{M}_{q_2}^{p_2}} = |B(0,r)|^{\frac{1}{q_2}},$$

we have $q_1 = q_2$. Assume to the contrary that $p_1 > p_2$. Define f by (2.2). By a similar argument as in the proof of Theorem 1.3 (ii)-(iii), we have $f \in w\mathcal{M}_{q_2}^{p_2}$ but $f \notin w\mathcal{M}_{q_1}^{p_1}$, which contradicts $w\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_1}^{p_1}$. Hence $p_1 \leq p_2$. \square

REMARK 4.2. Observe that unlike [4, Theorem 4.4 and Remark 4.5], the condition $p_1 \leq p_2$ is not assumed in Theorem 1.6 (ii).

PROOF OF THEOREM 1.6 (iii). Since $\mathcal{M}_{q_2}^{p_2} \subseteq w\mathcal{M}_{q_2}^{p_2}$, we have $\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$. Therefore, by virtue of Theorem 1.6 (ii), we have $q_1 = q_2$ and $p_1 \leq p_2$. Now, assume to the contrary that $p_1 = p_2$. According to Theorem 1.4, there exists $f_0 \in w\mathcal{M}_{q_2}^{p_2}$ such that $f_0 \notin \mathcal{M}_{q_1}^{p_1}$. This contradicts $w\mathcal{M}_{q_2}^{p_2} \subseteq \mathcal{M}_{q_1}^{p_1}$. Thus $p_1 < p_2$, as desired. \square

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