FIXED POINT THEOREMS ON BOUNDED SETS IN AN n-NORMED SPACE

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Abstract

In this paper we prove fixed point theorems for contraction mappings and φ-contraction mappings on a bounded and closed set with respect to n linearly independent vectors in an n-normed space. Our results rectify those obtained recently by Kir and Kiziltunc [11].

1 Introduction

In the 1960's, Gähler introduced the notion of 2-normed spaces [3] and extended it to the concept of n-normed spaces [4, 5, 6]. For a fixed $n \in \mathbb{N}$, an n-norm on a real vector space E ($\dim(E) \geq n$) is a mapping $\|\cdot, \dots, \cdot\| : E^n \to [0, \infty)$ which satisfies the following conditions: For each $x_0, x_1, \dots, x_n \in E$

1. $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, ..., x_n$ linearly dependent;
2. $\|x_1, \dots, x_n\|$ is invariant under permutation;
3. $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$, for $\alpha \in \mathbb{R}$;
4. $\|x_0 + x_1, x_2 \dots, x_n\| \leq \|x_0, x_2, \dots, x_n\| + \|x_1, x_2, \dots, x_n\|$.

The pair $(E, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Several authors such as Iseki [10] and Gunawan and Mashadi [8] studied some aspects of the fixed point theory and proved fixed point theorems in 2-Banach spaces. Recently, Harikrishnan and Ravindran [9] showed a fixed point theorem for a closed and bounded set in 2-normed spaces. Their result is extended by Kir and Kiziltunc [11] to n-normed spaces. In particular, Kir and Kiziltunc defined the set K to be bounded in $(E, \|\cdot, \dots, \cdot\|)$ if there is $M \in \mathbb{R}$ such that $\|x, a_2, \dots, a_n\| \leq M$ for all $x, a_2, \dots, a_n \in K$. They, then, proved a fixed point theorem for a contraction mapping on such a bounded and closed set.

Although the fixed point theorem was proved, we found that there are some issues with Kir and Kiziltunc's definition on bounded sets, which in turn limits the validity of the theorem. To be precise, let $E = \mathbb{R}^d$ $(d \geq n)$ be equipped with the standard n-norm [7], which is given by

$$\|x_1, x_2, \dots, x_n\|_S = \sqrt{\det(\langle x_i, x_j \rangle)}$$

for $x_1, \dots, x_n \in E$ and $i, j = 1, \dots, n$. (Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^d$.) Choose any line $L_x = \{z | z = tx, t \in \mathbb{R}\}$ passing through the origin and a nonzero vector $x \in \mathbb{R}^d$. Obviously, L_x is unbounded according to Euclidean norm. However, L_x satisfies the boundedness condition of Kir and Kiziltunc. If $z, x_2, \dots, x_n \in L_x$, then

$$\|z, x_2, \dots, x_n\|_S = 0$$

because $z, x_2, \dots, x_n$ are linearly dependent. Thus L_x is bounded according to Kir and Kiziltunc's definition. Now, consider the translation $T : L_x \to L_x$ given by $T(tx) = (t+1)x$. Clearly T has no fixed points. However the following inequality

$$\|Tx - Tx', x_2, \dots, x_n\| \leq k\|x - x', x_2, \dots, x_n\|$$

holds since both sides are zero. Consequently, T is a contraction mapping on L_x, and thus, according to Kir and Kiziltunc, T should have a fixed point — which is impossible.

In general, any subset K of E with $\operatorname{rank}(K) < n$ is bounded according to Kir and Kiziltunc's definition, and this leads to unintended results as above. The boundedness definition of Kir and Kiziltunc makes sense only for $K \subset E$ with $n \leq \operatorname{rank}(K) \leq \operatorname{rank}(E)$. Accordingly, the theorem applies to such subsets only. Even so, there are rooms for improvement in the formulation of bounded sets and the criteria for contractive mappings in an n-normed space.

In this paper, the notion boundedness is corrected by using the method of Ekariani *et al.* [2]. Let ℓ^p be the space of p-summable sequences $(1 \leq p \leq \infty)$, equipped with an n-norm denoted by $\|\cdot, \dots, \cdot\|_p$. Gunawan [7] then proved that if $T : \ell^p \to \ell^p$ is a contraction mapping, that is, there exists $k \in (0, 1)$ such that

$$\|Tx - Tx', x_2, \dots, x_n\|_p \leq k\|x - x', x_2, \dots, x_n\|_p$$

for all $x, x', x_2, \dots, x_n$ in ℓ^p, then T has a unique fixed point in ℓ^p. Ekariani *et al.* improved the result by weakening the contraction condition, as follows. Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a set of n linearly independent vectors in ℓ^p, and let $T : \ell^p \to \ell^p$ satisfy a contraction mapping with respect to $\mathcal{A}$, that is, there exists $k \in (0,1)$ such that for all $x, x' \in \ell^p$

$$\|Tx - Tx', a_{i_2}, \ldots, a_{i_n}\|_p \leq k\|x - x', a_{i_2}, \ldots, a_{i_n}\|_p,$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. Ekariani *et al.* then showed that such an operator T has a unique fixed point.

We shall define bounded and closed sets in an n-normed space by using similar ideas to those in [2]. Using the new definition, we reprove the fixed point theorems for contraction mappings and φ-contraction mappings on bounded and closed sets in an n-normed space. Overall, we do not only address the issues in Kir and Kiziltunc's paper, but also simplify the criteria for bounded sets and contractive mappings in an n-normed space.

2 Fixed Point Theorem for Contractive Mappings

Hereafter, let $(E, \|\cdot, \ldots, \cdot\|)$ denote an n-normed space and $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a set of n linearly independent vectors in E. Before we state our fixed point theorem, we introduce the following definitions:

Definition 1. *A sequence $\{x_m\}_{m=0}^{\infty}$ in E* **converges** *to an $x \in E$ with respect to $\mathcal{A}$, denoted by $x_m \xrightarrow{\mathcal{A}} x$, if*

$$\lim_{m\to\infty} \|x_m - x, a_{i_2}, \ldots, a_{i_n}\| = 0$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. The element x is called the **limit** *of the sequence $\{x_m\}$.*

Definition 2. *A sequence $\{x_m\}_{m=0}^{\infty}$ in E is called a* **Cauchy** *sequence with respect to $\mathcal{A}$ if*

$$\lim_{l,m\to\infty} \|x_m - x_l, a_{i_2}, \ldots, a_{i_n}\| = 0$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. If every Cauchy sequence in E converges to an $x \in E$, then $(E, \|\cdot, \ldots, \cdot\|)$ is said to be **complete** *with respect to $\mathcal{A}$.*

Definition 3. *Let $B \subseteq E$ be nonempty set. Then we say that B is* **closed** *if for every sequence $\{x_n\}_{n=0}^{\infty}$ in B which converges in E, its limit is in B.*

Definition 4. *Let E be an n-normed space. A mapping $T : E \to E$ is called a* **contraction mapping with respect to** $\mathcal{A}$ *if there is $k \in (0,1)$ such that*

$$\|Tx - Tx', a_{i_2}, \ldots, a_{i_n}\| \leq k\|x - x', a_{i_2}, \ldots, a_{i_n}\| \tag{1}$$

for every $x, x' \in E$ and $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$.

The following is a revision of Kir and Kiziltunc's definition of bounded sets in an n-normed space.

Definition 5. *Let B be a nonempty subset of E. Then B is called* **bounded with respect to $\mathcal{A}$** *if there is $M > 0$ such that*

$$\|b, a_{i_2}, \ldots, a_{i_n}\| \leq M$$

for every $b \in B$ and $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$.

The reader should now note that the set L_x which is bounded according to Kir and Kiziltunc's definition, is unbounded according to ours. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ be the set of the first n basis vectors in $E = \mathbb{R}^d$. Then one may observe that for any $z \in E$

$$\|z, e_{i_2}, \ldots, e_{i_n}\|_S = |z_{i_1}|$$

for every $\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, n\}$. Since $z = t_z x$ for some $t_z \in \mathbb{R}$, we have $|z_{i_1}| \to \infty$ as $t_z \to \infty$. Consequently, L_x is not a bounded set.

The following lemma is used to show when a vector is zero, which is essential in proving our theorems later.

Lemma 1. *Let $b \in E$. If*

$$\|b, a_{i_2}, \ldots, a_{i_n}\| = 0$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$, then $b = 0$.

Proof. If $\|b, a_{i_2}, \ldots, a_{i_n}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$, then b is in the span of $\{a_{i_2}, \ldots, a_{i_n}\}$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. This can only happen if $b = 0$. □

The following fixed point theorem holds for mappings which are contractive on a closed and bounded subset with respect to $\mathcal{A}$.

Theorem 1. *Let $(E, \|\cdot, \ldots, \cdot\|)$ be a complete n-normed space and $B \subset E$ be a nonempty, closed, and bounded with respect to $\mathcal{A}$. If $T : B \to B$ is a contraction mapping with respect to $\mathcal{A}$, then T has a unique fixed point in B.*

Proof. Let x_0 be any element in B. We first construct an iterative sequence $\{x_m\}_{m=0}^{\infty}$ where

$$x_m = T^m x_0 \text{ for } m = 0, 1, 2, \ldots.$$

Second, we show that $\{x_m\}_{m=0}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{A}$. Since T is contractive, there is $k \in (0,1)$ such that for any two consecutive terms in $\{x_m\}_{m=0}^{\infty}$, we have

$$\begin{aligned}
\|x_m - x_{m+1}, a_{i_2}, \ldots, a_{i_n}\| &= \|Tx_{m-1} - Tx_m, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq k\|x_{m-1} - x_m, a_{i_2}, \ldots, a_{i_n}\| \\
&= k\|Tx_{m-2} - Tx_{m-1}, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq k^2\|x_{m-2} - x_{m-1}, a_{i_2}, \ldots, a_{i_n}\| \\
&\vdots \\
&\leq k^m\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|
\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. Hence by using the triangle inequality and the formula for the sum of a geometric progression, we obtain for $m > l$

$$\begin{aligned}
\|x_m - x_l, a_{i_2}, &\ \ldots\ , a_{i_n}\| \\
&= \|x_m - x_{m-1} + x_{m-1} - \ldots + x_{l+1} - x_l, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq \|x_m - x_{m-1}, a_{i_2}, \ldots, a_{i_n}\| + \|x_{m-1} - x_{m-2}, a_{i_2}, \ldots, a_{i_n}\| + \\
&\quad \ldots + \|x_{l+1} - x_l, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq k^{m-1}\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\| + k^{m-2}\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\| + \\
&\quad \ldots + k^l\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\| \\
&= \left(k^{m-1} + k^{m-2} + \ldots + k^l\right)\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\| \\
&< \frac{k^l}{1-k}\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|
\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. Because $k \in (0,1)$ and $\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|$ is bounded, we can make the right hand side of above inequality as small as we like, by taking l sufficiently large. Since this holds for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$, the sequence $\{x_m\}_{m=0}^{\infty}$ is Cauchy with respect to $\mathcal{A}$. As E is a complete n-normed space and B is closed, there exists $x \in B$ such that $x_m \overset{\mathcal{A}}{\to} x$.

Third, we prove that x is a fixed point of T, that is, $Tx = x$. By using the triangle inequality and contraction mapping (1) we have

$$\begin{aligned}
\|Tx - x, a_{i_2}, \ldots, a_{i_n}\| &\leq \|Tx - x_m, a_{i_2}, \ldots, a_{i_n}\| + \|x_m - x, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq k\|x - x_{m-1}, a_{i_2}, \ldots, a_{i_n}\| + \|x_m - x, a_{i_2}, \ldots, a_{i_n}\|
\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. By taking a sufficiently large m, the sum in the second line can be made smaller than any preassigned $\epsilon > 0$, because $x_m \overset{\mathcal{A}}{\to} x$. We conclude that $\|Tx - x, a_{i_2}, \ldots, a_{i_n}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. By Lemma 1, we have $Tx = x$.

Fourth, we prove that T has no other fixed points. Let $x' \in X$ be another fixed point of T, so that $Tx' = x'$. We obtain

$$\begin{aligned} \|x - x', a_{i_2}, \ldots, a_{i_n}\| &= \|Tx - Tx', a_{i_2}, \ldots, a_{i_n}\| \\ &\leq k\|x - x', a_{i_2}, \ldots, a_{i_n}\| \end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. Since $k \in (0, 1)$, we have $\|x - x', a_{i_2}, \ldots, a_{i_n}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. By Lemma 1, we conclude that $x = x'$. □

3 Fixed Point Theorem for φ-Contraction Mapping

Herewith, we define what is called a comparison function, φ, which will be used to generalize the notion of contraction mappings. We also have a version of the fixed point theorem for φ-contraction mappings which generalizes Theorem 1.

Definition 6. *[1]*
A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies following conditions:

1. *φ is monotone increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$, and*
2. *$\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$,*

is called a **comparison function***.*

The followings are some example of comparison functions:

1. $\varphi(t) = kt, t \in \mathbb{R}^+, k \in [0, 1)$,
2. $\varphi(t) = \frac{t}{1+t}, t \in \mathbb{R}^+$,
3. $\varphi(t) = \frac{1}{2}t$ for $0 \leq t \leq 1$ and $\varphi(t) = t - \frac{1}{2}$ for $t > 1$.

The following definition is more general than the ordinary contraction condition.

Definition 7. *Let E be an n-normed space and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a comparison function. A mapping $T : E \to E$ is called φ-***contraction mapping with respect to $\mathcal{A}$** *if*

$$\|Tx - x', a_{i_2}, \ldots, a_{i_n}\| \leq \varphi(\|x - x', a_{i_2}, \ldots, a_{i_n}\|) \tag{2}$$

for every $x, x' \in E$ and $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$.

For any comparison function, we have the following lemma.

Lemma 2. *[1] If φ is a comparison function, then $\varphi(t) < t$ for all $t > 0$.*

Proof. Suppose the contrary holds, $\varphi(t) \geq t$, or $\varphi(t) = t + \psi_1$ for some $\psi_1 > 0$. Applying monotonic increasing property of φ to the pair t and $\varphi(t) = t + \psi_1$, we get

$$\varphi^2(t) = \varphi(\varphi(t)) \geq \varphi(t) = t + \psi_1.$$

Hence, for some $\psi_2 > 0$,

$$\varphi^2(t) = t + \psi_1 + \psi_2.$$

Using induction argument, it can be shown that for any $k \in \mathbb{N}$, we have the monotonic condition $\varphi^{k+1}(t) \geq \varphi^k(t)$, and thus $\varphi^n(t) \geq t$ for all n. This contradicts the fact that $\varphi^n(t)$ converges to 0 as $n \to \infty$. □

The following theorem is a generalization of the fixed point theorem in n-normed space where contractive nature of map is generalized by φ-contraction mapping in closed and bounded set with respect to $\mathcal{A}$.

Theorem 2. *Let $(E, \|\cdot, \ldots, \cdot\|)$ be a complete n-normed space and $B \subset E$ be a nonempty, closed, and bounded with respect to $\mathcal{A}$. If $T : B \to B$ is φ-contraction mapping with respect to $\mathcal{A}$, then T has a unique fixed point in B.*

Proof. As in the proof of Theorem 1, starting with an element x_0 in B, we construct an iterative sequence $\{x_m\}_{m=0}^{\infty}$, where

$$x_m = T^m x_0 \text{ for } m = 0, 1, 2, \ldots.$$

Then we show that $\{x_m\}_{m=0}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{A}$. Since T is contractive, there is $k \in (0,1)$ such that for any two consecutive terms in $\{x_m\}_{m=0}^{\infty}$, we have

$$\begin{aligned}
\|x_m - x_{m+1}, a_{i_2}, \ldots, a_{i_n}\| &= \|Tx_{m-1} - Tx_m, a_{i_2}, \ldots, a_{i_n}\| \\
&\leq \varphi(\|x_{m-1} - x_m, a_{i_2}, \ldots, a_{i_n}\|) \\
&= \varphi(\|Tx_{m-2} - Tx_{m-1}, a_{i_2}, \ldots, a_{i_n}\|) \\
&\leq \varphi^2(\|x_{m-2} - x_{m-1}, a_{i_2}, \ldots, a_{i_n})\| \\
&\vdots \\
&\leq \varphi^m(\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|)
\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, b\}$. Observe that $\varphi(t) < t$ for each $t > 0$. By monotonicity of Definition 7, we have

$$\|T^{m+1}(x) - T^m(x), a_{i_2}, \ldots, a_{i_n}\| \leq \varphi^m(\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|).$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. For $m > l$ and by applying the triangle inequality on $\|x_m - x_l, a_{i_2}, \ldots, a_{i_n}\|$, we have

$$\begin{aligned}\|x_m - x_l, a_{i_2}, \quad \ldots \quad , a_{i_n}\| & \\ = \quad & \|x_m - x_{m-1} + x_{m-1} - \ldots + x_{l+1} - x_l, a_{i_2}, \ldots, a_{i_n}\| \\ \leq \quad & \|x_m - x_{m-1}, a_{i_2}, \ldots, a_{i_n}\| + \|x_{m-1} - x_{m-2}, a_{i_2}, \ldots, a_{i_n}\| + \\ & \ldots + \|x_{l+1} - x_l, a_{i_2}, \ldots, a_{i_n}\| \\ \leq \quad & \varphi^{m-1}(\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|) + \varphi^{m-2}(\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|) + \\ & \ldots + \varphi^{l}(\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\|) \qquad (3)\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. Because B is bounded with respect to $\mathcal{A}$, there is a constant $M > 0$ such that $\|x_0 - x_1, a_{i_2}, \ldots, a_{i_n}\| \leq M$ for every $x_0 - x_1 \in B$ and $\{i_2, \ldots, i_n\} \subset \{1, 2, \ldots, n\}$. Hence, by (3), we have

$$\|x_m - x_l, a_{i_2}, \ldots, a_{i_n}\| \leq \varphi^{m-1}(M) + \varphi^{m-2}(M) + \ldots + \varphi^{l}(M)$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. By the condition (2) of Definition 6, we have

$$\begin{aligned}\lim_{l \to \infty} \|x_m - x_l, a_{i_2}, \ldots, a_{i_n}\| & \leq \lim_{l \to \infty} \left(\varphi^{m-1}(M) + \varphi^{m-2}(M) + \ldots + \varphi^{l}(M)\right) \\ & = 0\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. This proves that $\{x_m\}_{m=0}^{\infty}$ is a Cauchy sequence with respect to $\mathcal{A}$. Since E is complete, B is closed, and $\{x_m\}_{m=0}^{\infty}$ is in B, we obtain $x_m \xrightarrow{\mathcal{A}} x$, for some $x \in B$.

We will show that x is a fixed point of T. By the triangle inequality and φ-contraction mapping (2), we see that

$$\begin{aligned}\|Tx - x, a_{i_2}, \ldots, a_{i_n}\| & \leq \|Tx - x_m, a_{i_2}, \ldots, a_{i_n}\| + \|x_m - x, a_{i_2}, \ldots, a_{i_n}\| \\ & \leq \varphi(\|x - x_{m-1}, a_{i_2}, \ldots, a_{i_n}\|) + \|x_m - x, a_{i_2}, \ldots, a_{i_n}\|\end{aligned}$$

for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$. For any ϵ, we can find m sufficiently large such that the right hand side is less that ϵ. Thus $x_m \xrightarrow{\mathcal{A}} x$. We conclude that $\|Tx - x, a_{i_2}, \ldots, a_{i_n}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$, and by Lemma 1 we have $Tx = x$.

Now, we prove that T has no other fixed points. Let $x' \in X$ be another fixed point of T, so that $Tx' = x'$. Then

$$\begin{aligned}\|x - x', a_{i_2}, \ldots, a_{i_n}\| & = \|Tx - Tx', a_{i_2}, \ldots, a_{i_n}\| \\ & \leq \varphi(\|x - x', a_{i_2}, \ldots, a_{i_n}\|),\end{aligned}$$

from which we can derive that $\|x - x', a_{i_2}, \ldots, a_{i_n}\| = 0$ for every $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$, using Lemma 2. By Lemma 1, we must have $x = x'$. □

4 Concluding Remark

We have presented a new definition of boundedness with respect to a linearly independent set (of n vectors) in order to correct Kir and Kiziltunc's definition. Using this new notion of boundedness, we have reproved the fixed point theorems for contraction mappings and φ-contraction mappings with respect to the chosen linearly independent set. In fact, our results are independent of the choice of the linearly independent set that we use to define bounded sets in the n-normed space, since a set is bounded with respect to a linearly independent set if and only if it is bounded with respect to another linearly independent set (of n vectors as well). See [12] for a related result in this direction.

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References

[1] V. Berinde, *Iteration Approximation of Fixed Points*, Springer-Verlag, Berlin, (2007).

[2] S. Ekariani, H. Gunawan, and M. Idris, "A contractive mapping theorem on the n-normed space Of p-summable sequences", *J. Math. Analysis* **4** (2013), 1–7.

[3] S. Gähler, "Lineare 2-normierte Räume", *Math. Nachr.* **28** (1964), 1–43.

[4] S. Gähler, "Untersuchungen uber verallgemeinerte m-metrische räume I", *Math. Nachr.* **40** (1969), 165–189.

[5] S. Gähler, "Untersuchungen uber verallgemeinerte m-metrische räume II", *Math. Nachr.* **40** (1969), 229–264.

[6] S. Gähler, "Untersuchungen uber verallgemeinerte m-metrische räume III", *Math. Nachr.* **41** (1970), 23–26.

[7] H. Gunawan, "The space of p-summable sequences and its natural n-norm", *Bull. Austral. Math. Soc.* **64** (2001), 137–147.

[8] H. Gunawan and Mashadi, "On finite dimensional 2-normed spaces", *Soochow J. Math.* **27** (2001), 321–329.

[9] P. K. Harikrishnan and K. T. Ravindran, "Some properties of accretive operators in linear 2-normed spaces", *Internat. Math. Forum* **6** (2011), 2941–2947.

[10] K. Iseki, "Mathematics on two normed spaces", *Bull. Korean. Math. Soc.* **3** (1976), 127–135.

[11] M. Kir and H. Kiziltunc, "On fixed point theorems for contraction mappings in n-normed spaces", *Applied Math. Information Sci. Letters* **2** (2014), 59–64.

[12] S. Konca, H. Gunawan, and M. Basarir, "Some remarks on ℓ^p as an n-normed space", *Math. Sci. Appl. E-Notes* **2** (2015), 45–50.