HARDY-TYPE AND HEISENBERG'S INEQUALITY IN MORREY SPACES

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Abstract. We use the Morrey norm estimate for the imaginary power of the Laplacian to prove an interpolation inequality for the fractional power of the Laplacian on Morrey spaces. We then prove a Hardy-type inequality and use it together with the interpolation inequality to obtain a Heisenberg-type inequality in Morrey spaces.

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1. Introduction

Inspired by the work of Ciatti, Cowling, and Ricci [1], we are interested in obtaining an estimate for the Morrey norm of the fractional power of the Laplacian, in order to prove Heisenberg's uncertainty inequality in Morrey spaces. To begin with, let $(-\Delta)^{z/2}$ be the complex power of the Laplacian, given by

$$[(-\Delta)^{z/2} f]^{\widehat{}}(\xi) := |\xi|^z \widehat{f}(\xi), \quad \xi \in \mathbb{R}^n, \tag{1.1}$$

for suitable functions f on $\mathbb{R}^n$, where the Fourier transform is defined by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^n.$$

Our first aim here is to show the following Morrey norm estimate for the imaginary power of the Laplacian:

$$\|(-\Delta)^{iu/2} f\|_{\mathcal{M}^p_q} \lesssim (1+|u|)^{\frac{n}{2}} \|f\|_{\mathcal{M}^p_q}, \quad f \in \mathcal{M}^p_q(\mathbb{R}^n), \tag{1.2}$$

for every $u \in \mathbb{R}$, provided that $1 < p \le q < \infty$.

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Recall that, for $1 \le p \le q < \infty$, the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^n, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p \, {\rm d}y\right)^{\frac{1}{p}}$$

is finite. We refer the reader to [14] for various function spaces built on Morrey spaces.

Based on [9], let us explain why $(-\Delta)^{iu/2}$ should be bounded on $\mathcal{M}^p_q(\mathbb{R}^n)$, for $1 < p \le q < \infty$, with bound $C(u) \lesssim (1+|u|)^{n/2}$. We define $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ to be the closure of $C^\infty_{\rm c}(\mathbb{R}^n)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$, or equivalently, $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is the closure of $L^q(\mathbb{R}^n)$ in $\mathcal{M}^p_q(\mathbb{R}^n)$ (see [15, p. 1846]). We know that $(-\Delta)^{iu/2}$ maps $L^q(\mathbb{R}^n)$ boundedly into $L^q(\mathbb{R}^n)$ [2]. We also establish in Lemma 2.1 that, for $f \in C^\infty_{\rm c}(\mathbb{R}^n)$, $\|(-\Delta)^{iu/2}f\|_{\mathcal{M}^p_q} \lesssim C(u)\|f\|_{\mathcal{M}^p_q}$, keeping in mind that $C^\infty_{\rm c}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n) \subset \mathcal{M}^p_q(\mathbb{R}^n)$ and that $(-\Delta)^{iu/2}f$ makes sense for $f \in C^\infty_{\rm c}(\mathbb{R}^n)$ by (1.1). This means that $(-\Delta)^{iu/2} : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ is bounded (see Definition 2.2 and Lemma 2.3). Next, we know that the space $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is the dual of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ (see [11, Theorem 4.3]) if $\frac{1}{p}+\frac{1}{p'} = \frac{1}{q}+\frac{1}{q'} = 1$. Here, we recall that $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^{q'}(\mathbb{R}^n)$ for which

$$f = \sum_{j=1}^{\infty} \lambda_j A_j, \tag{1.3}$$

where $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and $\{A_j\}_{j=1}^\infty$ is a sequence of functions supported on balls with $\|A_j\|_{L^{q'}} \le 1$ for every $j \in \mathbb{N}$. The norm of $f \in \mathcal{H}^{p'}_{q'}$ is defined by

$$\|f\|_{\mathcal{H}^{p'}_{q'}} := \inf\left\{\sum_{j=1}^\infty |\lambda_j| : \{\lambda_j\}_{j=1}^\infty \text{ and } \{A_j\}_{j=1}^\infty \text{ satisfying (1.3)}\right\}.$$

Meanwhile, the dual of $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is $\mathcal{M}^p_q(\mathbb{R}^n)$ [16]. In general, the dual mapping of a bounded linear mapping T from a Banach space X to Y is bounded from Y^* to X^*. So, since $(-\Delta)^{iu/2}$ is formally self-adjoint, we see that the boundedness $(-\Delta)^{iu/2} : \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \to \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ established above entails $(-\Delta)^{iu/2} : \mathcal{H}^{p'}_{q'}(\mathbb{R}^n) \to \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ (see Definition 2.4 and Lemma 2.5), which in turn entails the boundedness of $(-\Delta)^{iu/2} : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathcal{M}^p_q(\mathbb{R}^n)$ (see Definition 2.6 and Proposition 2.7).

We note that $|\cdot|^{iu}\widehat{f}$ does not make sense for some $f \in \mathcal{M}^p_q(\mathbb{R}^n)$. As indicated above, the operator $(-\Delta)^{iu/2}$ which is initially defined on $C^\infty_{\rm c}(\mathbb{R}^n)$ is then defined on $\mathcal{M}^p_q(\mathbb{R}^n)$ by the duality relation

$$\langle(-\Delta)^{iu/2}f, g\rangle = \langle f, (-\Delta)^{-iu/2}g\rangle, \quad g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n),$$

where $\mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ is the space whose dual is $\mathcal{M}^p_q(\mathbb{R}^n)$ (see [16, Proposition 5] and Definition 2.4). We claim that this definition of $(-\Delta)^{iu/2}f$ coincides with the one by the Fourier transform, whenever the Fourier transform of f makes sense. Indeed, we can show that

$$\overline{\psi(\xi)}\mathcal{F}[(-\Delta)^{iu/2}f](\xi) = \overline{\psi(\xi)}|\xi|^{iu}\mathcal{F}f(\xi),$$

for every $\psi \in C^\infty_c(\mathbb{R}^n)$ and $0 \notin \operatorname{supp}\psi$, where $\mathcal{F}$ denotes the Fourier transform. Observe that if $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, then $\mathcal{F}^{-1}[\psi\mathcal{F}g] \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. In fact

$$\mathcal{F}^{-1}[\psi\mathcal{F}g](x) = (2\pi)^n\mathcal{F}^{-1}\psi * g(x) = (2\pi)^n\int_{\mathbb{R}^n}\mathcal{F}^{-1}\psi(y)g(x-y)\,dy.$$

As a result,

$$\begin{aligned}\|\mathcal{F}^{-1}[\psi\mathcal{F}g]\|_{\mathcal{H}^{p'}_{q'}} &\le (2\pi)^n\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\psi(y)|\|g(\cdot - y)\|_{\mathcal{H}^{p'}_{q'}}\,dy\\ &\le (2\pi)^n\int_{\mathbb{R}^n}|\mathcal{F}^{-1}\psi(y)|\|g\|_{\mathcal{H}^{p'}_{q'}}\,dy\\ &= C\|g\|_{\mathcal{H}^{p'}_{q'}} < \infty.\end{aligned}$$

So, $\mathcal{F}^{-1}[\psi\mathcal{F}g] \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$. It follows that

$$\langle(-\Delta)^{iu/2}f, \mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle = \langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle,$$

or equivalently

$$\langle\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]], g\rangle = \langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle.$$

Since $g \in L^{q'}(\mathbb{R}^n)$, we have

$$(-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g] = \mathcal{F}^{-1}[|\cdot|^{-iu}\psi\mathcal{F}g],$$

and hence

$$\langle f, (-\Delta)^{-iu/2}\mathcal{F}^{-1}[\psi\mathcal{F}g]\rangle = \langle f, \mathcal{F}^{-1}[|\cdot|^{-iu}\psi\mathcal{F}g]\rangle = \langle\mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f], g\rangle.$$

We therefore have

$$\langle\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]], g\rangle = \langle\mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f], g\rangle.$$

Since g is arbitrary, $\mathcal{F}^{-1}[\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f]] = \mathcal{F}^{-1}[\overline{\psi}|\cdot|^{iu}\mathcal{F}f]$, so that we obtain $\overline{\psi}\mathcal{F}[(-\Delta)^{iu/2}f] = \overline{\psi}|\cdot|^{iu}\mathcal{F}f$ as claimed.

In the following sections, we prove the Morrey norm estimate for the imaginary power of the Laplacian and its consequence for the fractional power of the Laplacian. We also prove a Hardy-type inequality and use it together with the estimate for the fractional power of the Laplacian to obtain Heisenberg's uncertainty inequality in Morrey spaces.

2. Morrey norm estimates for the fractional power of the Laplacian

For each $u \in \mathbb{R} \setminus \{0\}$, it is known that on $L^p(\mathbb{R}^n)$, for $1 \le p \le 2$, the operator $(-\Delta)^{iu/2}$ (defined by (1.1)) admits an integral kernel K_u given by

$$K_u(x) := \frac{\pi^{-n/2}\Gamma\left(\frac{n+iu}{2}\right)}{2^{-iu}\Gamma\left(\frac{-iu}{2}\right)}|x|^{-n-iu} = C(u)|x|^{-n-iu}, \quad x \in \mathbb{R}^n$$

(see [13, p. 51]). Here $\widehat{K_u}(\xi) = |\xi|^{iu}$ (in the distribution sense). A close inspection of the above constant shows

$$|C(u)| \lesssim (1+|u|)^{\frac{n}{2}}, \quad u \in \mathbb{R}.$$

As shown in [2, 12], we have

$$\|(-\Delta)^{iu/2}f\|_{L^p} \lesssim (1+|u|)^{\left|\frac{n}{p}-\frac{n}{2}\right|}\|f\|_{L^p} \lesssim (1+|u|)^{\frac{n}{2}}\|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),$$

for every $u \in \mathbb{R}$, provided that $1 < p \le 2$. By duality, the same inequality also holds for $2 < p < \infty$.

Based on the discussion in Section 1, we shall now prove that the inequality also holds in Morrey spaces (see [9] for similar results). We need several lemmas and definitions.

Lemma 2.1. *Let $u \in \mathbb{R}$ and $1 < p \le q < \infty$. Then we have*

$$\|(-\Delta)^{iu/2}f\|_{\widetilde{\mathcal{M}}^p_q} \lesssim (1+|u|)^{\frac{n}{2}}\|f\|_{\widetilde{\mathcal{M}}^p_q}$$

for every $f \in C_c^\infty(\mathbb{R}^n)$.

Proof. To prove the inequality, it is sufficient for us to establish

$$|B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\left(\int_{B(a,r)}|(-\Delta)^{iu/2}f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \lesssim (1+|u|)^{\frac{n}{2}}\|f\|_{\mathcal{M}^p_q}$$

for all fixed balls $B = B(a,r)$. To do so, we adopt the technique used in [6]. For a fixed ball $B = B(a,r)$, we decompose $f := f_1 + f_2$, where $f_1 := f\chi_{B(a,2r)}$ and

$f_2 := f - f_1$. Then by the boundedness of $(-\Delta)^{iu/2}$ on $L^p(\mathbb{R}^n)$, we have

$$\begin{aligned}
|B(a,r)|^{\frac{1}{q}-\frac{1}{p}} &\left(\int_{B(a,r)} |(-\Delta)^{iu/2} f_1(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |(-\Delta)^{iu/2} f_1(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\lesssim (1+|u|)^{\frac{n}{2}} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{\mathbb{R}^n} |f_1(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\sim (1+|u|)^{\frac{n}{2}} |B(a,2r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,2r)} |f(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\lesssim (1+|u|)^{\frac{n}{2}} \|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Meanwhile, for each $x \in B$, we have

$$\begin{aligned}
|(-\Delta)^{iu/2} f_2(x)| &\le |C(u)| \int_{\mathbb{R}^n \setminus B(x,r)} \frac{|f(y)|}{|x-y|^n} \,\mathrm{d}y \\
&\le |C(u)| \sum_{k=0}^{\infty} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} \frac{|f(y)|}{|x-y|^n} \,\mathrm{d}y \\
&\lesssim |C(u)| \sum_{k=0}^{\infty} \frac{1}{(2^k r)^n} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(y)| \,\mathrm{d}y \\
&\lesssim |C(u)| \sum_{k=0}^{\infty} \left(\frac{1}{(2^k r)^n} \int_{B(x,2^{k+1}r)\setminus B(x,2^k r)} |f(y)|^p \,\mathrm{d}y\right)^{\frac{1}{p}} \\
&\lesssim |C(u)| \|f\|_{\mathcal{M}^p_q} \sum_{k=0}^{\infty} (2^k r)^{-\frac{n}{q}} \\
&\lesssim r^{-\frac{n}{q}} |C(u)| \|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Thus,

$$\begin{aligned}
|B(a,r)|^{\frac{1}{q}-\frac{1}{p}} &\left(\int_{B(a,r)} |(-\Delta)^{iu/2} f_2(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\lesssim |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} (r^{-\frac{n}{q}} |C(u)| \|f\|_{\mathcal{M}^p_q})^p \,\mathrm{d}y\right)^{\frac{1}{p}} \\
&= |B(a,r)|^{\frac{1}{q}} r^{-\frac{n}{q}} |C(u)| \|f\|_{\mathcal{M}^p_q} \\
&\sim |C(u)| \|f\|_{\mathcal{M}^p_q} \\
&\lesssim (1+|u|)^{\frac{n}{2}} \|f\|_{\mathcal{M}^p_q}.
\end{aligned}$$

Combining the two estimates, we obtain the desired inequality. □

Using Lemma 2.1 and density, we give the following natural definition:

Definition 2.2. Given $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, we define

$$(-\Delta)^{iu/2} f := \lim_{j\to\infty} (-\Delta)^{iu/2} f_j,$$

where $f_j \in C^\infty_{\rm c}(\mathbb{R}^n)$ and $f_j \to f$ in the $\mathcal{M}^p_q$-norm.

A direct consequence of Lemma 2.1 and the above definition is:

Lemma 2.3. *Let $u \in \mathbb{R}$ and $1 < p \le q < \infty$. Then we have*

$$\|(-\Delta)^{iu/2} f\|_{\widetilde{\mathcal{M}}^p_q} \lesssim (1+|u|)^{\frac{n}{2}} \|f\|_{\widetilde{\mathcal{M}}^p_q}$$

for every $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Definition 2.4. For every $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, we define

$$\langle (-\Delta)^{iu/2} g, h\rangle = \langle g, (-\Delta)^{-iu/2} h\rangle,$$

for every $h \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Lemma 2.5. *Let $u \in \mathbb{R}$ and $1 < p \le q < \infty$. Then*

$$\|(-\Delta)^{iu/2} g\|_{\mathcal{H}^{p'}_{q'}} \lesssim (1+|u|)^{\frac{n}{2}} \|g\|_{\mathcal{H}^{p'}_{q'}}$$

for every $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Proof. For every $h \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$, we have

$$\begin{aligned} |\langle (-\Delta)^{iu/2} g, h\rangle| = |\langle g, (-\Delta)^{-iu/2} h\rangle| &\le \|g\|_{\mathcal{H}^{p'}_{q'}} \|(-\Delta)^{-iu/2} h\|_{\widetilde{\mathcal{M}}^p_q} \\ &\lesssim (1+|u|)^{\frac{n}{2}} \|g\|_{\mathcal{H}^{p'}_{q'}} \|h\|_{\widetilde{\mathcal{M}}^p_q}. \end{aligned}$$

Since $(\widetilde{\mathcal{M}^p_q})^*(\mathbb{R}^n) \simeq \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$ [16], we get the desired result. □

We use Lemma 2.5 to give the following definition:

Definition 2.6. For every $f \in \mathcal{M}^p_q(\mathbb{R}^n)$, we define

$$\langle (-\Delta)^{iu/2} f, g\rangle = \langle f, (-\Delta)^{-iu/2} g\rangle,$$

for every $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$.

Proposition 2.7. *Let $u \in \mathbb{R}$ and $1 < p \le q < \infty$. Then*

$$\|(-\Delta)^{iu/2} f\|_{\mathcal{M}^p_q} \lesssim (1+|u|)^{\frac{n}{2}} \|f\|_{\mathcal{M}^p_q}$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^n)$.

Proof. For every $g \in \mathcal{H}^{p'}_{q'}(\mathbb{R}^n)$, we have

$$|\langle(-\Delta)^{iu/2}f, g\rangle| = |\langle f, (-\Delta)^{-iu/2}g\rangle| \le \|f\|_{\mathcal{M}^p_q}\|(-\Delta)^{-iu/2}g\|_{\mathcal{H}^{p'}_{q'}}$$
$$\lesssim (1+|u|)^{\frac{n}{2}}\|f\|_{\mathcal{M}^p_q}\|g\|_{\mathcal{H}^{p'}_{q'}}.$$

Since $(\mathcal{H}^{p'}_{q'})^*(\mathbb{R}^n) \simeq \mathcal{M}^p_q(\mathbb{R}^n)$, we get the desired result. □

As a corollary of Proposition 2.7, we obtain the following result for the fractional power of the Laplacian, which is analogous to the interpolation inequality in [1]. We refer the interested reader to [4] and references therein for the interpolation of Morrey spaces.

Theorem 2.8. *Let $\alpha \ge 0$. Then, for $0 \le \theta \le 1$, we have*

$$\|(-\Delta)^{\alpha\theta/2}f\|_{\mathcal{M}^p_q} \lesssim \|f\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}}\|(-\Delta)^{\alpha/2}f\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}}, \quad f \in C^\infty_c(\mathbb{R}^n), \tag{2.1}$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \tag{2.2}$$

with $1 < p_0 \le q_0 < \infty$ and $1 < p_1 \le q_1 < \infty$.

To prove Theorem 2.8, we use the following observation which is based on [5].

Lemma 2.9. *Let $1 \le w \le \infty$, $v \in [0,1]$, $\alpha \ge 0$, and B be any ball in $\mathbb{R}^n$. Then for every $f \in C^\infty_c(\mathbb{R}^n)$, we have*

$$\|(-\Delta)^{\frac{\alpha v}{2}}f\|_{L^w(B)} \le C,$$

where the constant $C = C(n, \alpha, B, f)$ is independent of w and v.

Proof. Let $N := \lfloor n+\alpha \rfloor + 1$. Then, for every $x \in \mathbb{R}^n$ we have

$$|(-\Delta)^{\frac{\alpha v}{2}}f(x)| \le \int_{\{|\xi|<1\}} |\xi|^{\alpha v}|\hat{f}(\xi)|\,\mathrm{d}\xi + \int_{\{|\xi|\ge 1\}} |\xi|^{\alpha v}|\hat{f}(\xi)|\,\mathrm{d}\xi$$
$$\le \|\hat{f}\|_{L^\infty}|B(0,1)| + \|\mathcal{F}[(-\Delta)^N f]\|_{L^\infty}\int_{\{|\xi|\ge 1\}} |\xi|^{\alpha-2N}\,\mathrm{d}\xi. \tag{2.3}$$

Let $E := \mathrm{supp}(f)$. Observe that

$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1} \le \|f\|_{C^\infty_c(\mathbb{R}^n)}|E| \tag{2.4}$$

and

$$\|\mathcal{F}[(-\Delta)^N f]\|_{L^\infty} \le \|(-\Delta)^N f\|_{L^1} \le \|f\|_{C^\infty_c(\mathbb{R}^n)}|E|. \tag{2.5}$$

Combining (2.3)-(2.5) and $\int_{\{|\xi|\ge 1\}} |\xi|^{\alpha-2N}\,\mathrm{d}\xi = \mathrm{O}\left(\frac{1}{2N-\alpha-n}\right)$, we get

$$\|(-\Delta)^{\frac{\alpha v}{2}}f\|_{L^\infty(B)} \le C_{n,\alpha,f},$$

where

$$C_{n,\alpha,f} := \left(|B(0,1)| + \frac{D}{2N-\alpha-n}\right)\|f\|_{C^\infty_c(\mathbb{R}^n)}|E|$$

with $D \gg 1$. Consequently, for $1 \le w < \infty$, we have

$$\|(-\Delta)^{\frac{\alpha v}{2}} f\|_{L^w(B)} \le C_{n,\alpha,f} |B|^{\frac{1}{w}} \le C_{n,\alpha,f} \max(1, |B|),$$

as desired. □

Now we are ready to prove Theorem 2.8.

Proof of Theorem 2.8. Let $f \in C_{\rm c}^\infty(\mathbb{R}^n)$. We prove (2.1) by showing that

$$\left(\int_B |(-\Delta)^{\alpha\theta/2} f(x)|^p \, {\rm d}x\right)^{\frac{1}{p}} \lesssim |B|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta}, \tag{2.6}$$

for every fixed ball $B = B(a, r)$. Let p_0', p_1', and p' be defined by $\frac{1}{p_0'} := 1 - \frac{1}{p_0}$, $\frac{1}{p_1'} := 1 - \frac{1}{p_1}$, and $\frac{1}{p'} := 1 - \frac{1}{p}$, respectively. We define $S := \{z \in \mathbb{C} : 0 < {\rm Re}(z) < 1\}$ and let $\overline{S}$ be its closure. For every $z \in \overline{S}$ and $x \in \mathbb{R}^n$, we define

$$G(z, x) := \begin{cases} 0, & g(x) = 0, \\ {\rm sgn}(g(x)) |g(x)|^{p'\left(\frac{1-z}{p_0'} + \frac{z}{p_1'}\right)}, & g(x) \ne 0, \end{cases}$$

where g is an arbitrary simple function with $\|g\|_{L^{p'}(B)} = 1$. We shall apply the Three Lines Theorem to the function $F(z)$, defined by

$$F(z) := e^{z^2} \int_B (-\Delta)^{\alpha z/2} f(x) G(z, x) \, {\rm d}x.$$

Note that F is continuous on $\overline{S}$ and holomorphic in S. Let $z = v + iu$ where $v \in [0, 1]$ and $u \in \mathbb{R}$. Define $\frac{1}{w} := 1 - \frac{1-v}{p_0'} - \frac{v}{p_1'}$. Then

$$|F(v + iu)| \lesssim e^{-u^2} (1 + \alpha|u|)^{\frac{n}{2}} \|(-\Delta)^{\alpha v/2} f\|_{L^w(B)} \|G(v + iu, \cdot)\|_{L^{w'}(B)}. \tag{2.7}$$

Here we have used the boundedness of $(-\Delta)^{i\alpha u/2}$ on $L^w(B)$ and the fact that

$$(-\Delta)^{\alpha z/2} = (-\Delta)^{i\alpha u/2} (-\Delta)^{\alpha v/2}.$$

Combining (2.7), Lemma 2.9, and

$$\|G(v + iu, \cdot)\|_{L^{w'}(B)} = \left\| |g|^{p'\left(\frac{1-v}{p_0'} + \frac{v}{p_1'}\right)} \right\|_{L^{w'}(B)} = \|g\|_{L^{p'}(B)}^{\frac{p'}{w'}} = 1,$$

we have $\sup_{z \in \overline{S}} |F(z)| < \infty$, that is, F is bounded on $\overline{S}$. Next, we observe that

$$\begin{aligned} |F(iu)| &\lesssim e^{-u^2} \|(-\Delta)^{i\alpha u/2} f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{\frac{1}{p_0}-\frac{1}{q_0}} \|G(iu, \cdot)\|_{L^{p_0'}(B)} \\ &\lesssim e^{-u^2} (1 + \alpha|u|)^{\frac{n}{2}} \|f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{\frac{1}{p_0}-\frac{1}{q_0}} \||g|^{p'/p_0'}\|_{L^{p_0'}(B)} \\ &\lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}} |B|^{\frac{1}{p_0}-\frac{1}{q_0}} \end{aligned}$$

and similarly

$$|F(1 + iu)| \lesssim \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}} |B|^{\frac{1}{p_1}-\frac{1}{q_1}}.$$

It thus follows from the Three Lines Theorem that

$$\begin{aligned} |F(\theta)| &\le \sup_{u\in\mathbb{R}} |F(\theta+iu)| \\ &\le \left(\sup_{u\in\mathbb{R}} |F(iu)|\right)^{1-\theta} \cdot \left(\sup_{u\in\mathbb{R}} |F(1+iu)|\right)^{\theta} \\ &\lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta} |B|^{\frac{1}{p}-\frac{1}{q}}, \end{aligned}$$

for $0 \le \theta \le 1$. Accordingly, we obtain

$$\left| \int_B (-\Delta)^{\alpha\theta/2} f(x) g(x)\,\mathrm{d}x \right| = e^{-\theta^2} |F(\theta)| \lesssim \|f\|_{\mathcal{M}^{p_0}_{q_0}}^{1-\theta} \|(-\Delta)^{\alpha/2} f\|_{\mathcal{M}^{p_1}_{q_1}}^{\theta} |B|^{\frac{1}{p}-\frac{1}{q}}.$$

Since g is any simple function of $L^{p'}(B)$-norm 1, we conclude that (2.6) holds. □

3. A Hardy-type inequality and a Heisenberg-type inequality

We shall now prove a Hardy-type inequality and Heisenberg's uncertainty inequality in Morrey spaces. According to [10], we have

$$\|W \cdot (-\Delta)^{-\alpha/2} f\|_{\mathcal{M}^p_q} \lesssim \|W\|_{\mathcal{M}^u_v} \|f\|_{\mathcal{M}^p_q}, \quad f \in \mathcal{M}^p_q(\mathbb{R}^n), \tag{3.1}$$

where $0 < \alpha < n,\ 1 < p \le q < \frac{n}{\alpha},\ u = \frac{np}{\alpha q},\ v = \frac{n}{\alpha}$. This inequality goes back to the work of Olsen [8], so we call it Olsen's inequality. Note that the inequality follows from Hölder's inequality and the boundedness of the fractional integral operator $I_\alpha := (-\Delta)^{-\alpha/2}$ from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $\mathcal{M}^s_t(\mathbb{R}^n)$ for $0 < \alpha < n,\ 1 < p \le q < \frac{n}{\alpha}$, $\frac{1}{s} = \frac{1}{p} - \frac{\alpha q}{np}$, and $\frac{s}{t} = \frac{p}{q}$ (see also [3]). Note that through its Fourier transform, one may recognize $(-\Delta)^{-\alpha/2}$ as the convolution operator whose kernel is a multiple of $|\cdot|^{\alpha-n}$, which is initially defined on $C^\infty_{\rm c}(\mathbb{R}^n)$ (see [13]).

As a consequence of the inequality (3.1), we have:

Proposition 3.1. *Let $1 < p \le q < \infty$ and $0 < \alpha < \frac{n}{q}$. Then we have*

$$\||\cdot|^{-\alpha} g\|_{\mathcal{M}^p_q} \lesssim \|(-\Delta)^{\alpha/2} g\|_{\mathcal{M}^p_q} \tag{3.2}$$

for every $g \in C^\infty_c(\mathbb{R}^n)$.

Remark 3.2. The inequality (3.2) may be viewed as a Hardy-type inequality in Morrey spaces.

To prove the proposition, we need some lemmas.

Lemma 3.3. *Let $0 < \alpha < n$. If $g \in C^\infty_{\rm c}(\mathbb{R}^n)$, then we have*

$$|(-\Delta)^{\alpha/2} g(x)| \lesssim \min(1, |x|^{-\alpha-n}).$$

In particular, $f = (-\Delta)^{\alpha/2} g \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$.

Proof. We have already seen that $|(-\Delta)^{\alpha/2}g(x)| \lesssim 1$ in the proof of Lemma 2.9. Now let $\psi \in C_{\rm c}^\infty(\mathbb{R}^n)$ be such that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$, where $B(r)$ denotes the ball centered at the origin of radius r. Define $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$. We decompose

$$(-\Delta)^{\alpha/2}g(x) = \mathcal{F}^{-1}[|\cdot|^\alpha(1-\psi)\mathcal{F}g](x) + \sum_{j=-\infty}^{0} \mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F}g](x).$$

Since $h = \mathcal{F}^{-1}[|\cdot|^\alpha(1-\psi)\mathcal{F}g]$ belongs to $\mathcal{S}(\mathbb{R}^n)$, we need to handle the second term. Using a crude estimate $\mathcal{F}g \in L^\infty(\mathbb{R}^n)$, we get

$$|\mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F}g](x)| \lesssim 2^{j\alpha}\||2^{-j}\cdot|^\alpha \varphi_j \mathcal{F}g\|_{L^1} \sim 2^{j(\alpha+n)}.$$

Let $N \in \mathbb{N}$ be large enough. Then as before,

$$\begin{aligned}|x|^{2N}|\mathcal{F}^{-1}[|\cdot|^\alpha \varphi_j \mathcal{F}g](x)| &= |\mathcal{F}^{-1}[\Delta^N[|\cdot|^\alpha \varphi_j \mathcal{F}g]](x)| \\ &\lesssim \sum_{\beta \in (\mathbb{N}\cup\{0\})^n, |\beta|=2N} \|\partial^\beta[|\cdot|^\alpha \varphi_j \mathcal{F}g]\|_{L^1}.\end{aligned}$$

Here and below let β be such that $|\beta| = 2N$. Then

$$|\partial^\beta[|\xi|^\alpha \varphi_j(\xi)\mathcal{F}g(\xi)]| \lesssim \sum_{\beta_1+\beta_2+\beta_3=\beta} |\partial^{\beta_1}[|\xi|^\alpha]||\partial^{\beta_2}\varphi_j(\xi)||\partial^{\beta_3}\mathcal{F}g(\xi)|.$$

Noting that $\varphi_j(\xi)$ vanishes outside $\{2^{j-2} \le |\xi| \le 2^{j+2}\}$, we have

$$\partial^{\beta_1}[|\xi|^\alpha] = {\rm O}(|\xi|^{\alpha-|\beta_1|}), \quad \partial^{\beta_2}\varphi_j(\xi) = {\rm O}(|\xi|^{-|\beta_2|}), \quad |\partial^{\beta_3}\mathcal{F}g(\xi)| \lesssim 1 \lesssim 2^{-j|\beta_3|},$$

as $\xi \to 0$. Thus,

$$\begin{aligned}|\partial^\beta[|\xi|^\alpha \varphi_j(\xi)\mathcal{F}g(\xi)]| &\lesssim \sum_{\beta_1+\beta_2+\beta_3=\beta} |\xi|^{\alpha-|\beta_1|}|\xi|^{-|\beta_2|}2^{-j|\beta_3|}\chi_{\{2^{j-2}\le|\xi|\le 2^{j+2}\}}(\xi) \\ &\lesssim 2^{j(\alpha-2N)}\chi_{\{|\xi|\le 2^{j+2}\}}(\xi)\end{aligned}$$

and hence

$$\|\partial^\beta[|\cdot|^\alpha \varphi_j \mathcal{F}g]\|_{L^1} = {\rm O}(2^{j(\alpha+n-2N)})$$

as $j \to -\infty$. As a result,

$$\begin{aligned}|(-\Delta)^{\alpha/2}g(x)| &\lesssim |x|^{-\alpha-n} + \sum_{j=-\infty}^{0} \min(|x|^{-2N}2^{j(\alpha+n-2N)}, 2^{j(\alpha+n)}) \\ &\le |x|^{-\alpha-n} + |x|^{-\alpha-n}\sum_{j=-\infty}^{\infty} \min(|x|^{\alpha+n-2N}2^{j(\alpha+n-2N)}, |x|^{\alpha+n}2^{j(\alpha+n)}).\end{aligned}$$

Noticing that

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \min(|x|^{\alpha+n-2N}2^{j(\alpha+n-2N)}, |x|^{\alpha+n}2^{j(\alpha+n)}) \\
&\quad \le \sum_{j=-\infty;2^j|x|\le 1}^{\infty} (2^j|x|)^{\alpha+n} + \sum_{j=-\infty;2^j|x|>1}^{\infty} (2^j|x|)^{\alpha+n-N} \\
&\quad \lesssim \sum_{j=-\infty;2^j|x|\le 1}^{\infty} \int_{2^j|x|}^{2^{j+1}|x|} t^{\alpha+n-1}\,dt + \sum_{j=-\infty;2^j|x|>1}^{\infty} \int_{2^{j-1}|x|}^{2^j|x|} t^{\alpha+n-N-1}\,dt \\
&\quad \le \int_0^2 t^{\alpha+n-1}\,dt + \int_{1/2}^{\infty} t^{\alpha+n-N-1}\,dt \lesssim 1,
\end{aligned}$$

we conclude that

$$|(-\Delta)^{\alpha/2}g(x)| \lesssim |x|^{-\alpha-n},$$

as desired. □

Lemma 3.4. *Let $1 \le p \le q < \infty$ and $0 < \alpha < n$. For $g \in C_c^\infty(\mathbb{R}^n)$, define $f := (-\Delta)^{\alpha/2}g$. Then $f \in \mathcal{M}_q^p(\mathbb{R}^n)$ and $(-\Delta)^{-\alpha/2}f = g$ pointwise.*

Proof. We have proved that $f \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Consequently,

$$\|f\|_{\mathcal{M}_q^p} \le \|f\|_{L^q} \le \|f\|_{L^\infty}^{1-\frac{1}{q}} \|f\|_{L^1}^{\frac{1}{q}} < \infty.$$

[This justifies the right-hand side of (3.2).] Next, $|\cdot|^\alpha \widehat{g} \in L^1(\mathbb{R}^n)$ and $f = \mathcal{F}^{-1}(|\cdot|^\alpha \widehat{g}) \in L^1(\mathbb{R}^n)$. Hence $\widehat{f} = |\cdot|^\alpha \widehat{g}$ pointwise, and so $|\cdot|^{-\alpha}\widehat{f} = \widehat{g}$ pointwise. This tells us that $(-\Delta)^{-\alpha/2}f = g$ pointwise. □

Now we come to the proof of Proposition 3.1.

Proof of Proposition 3.1. For $1 < p < q < \infty$ and $0 < \alpha < \frac{n}{q}$, we have $u = \frac{np}{\alpha q} < \frac{n}{\alpha} = v$. By computing directly its Morrey norm, we obtain that $W(\cdot) := |\cdot|^{-\alpha} \in \mathcal{M}_v^u(\mathbb{R}^n)$. Hence, for $g \in C_c^\infty(\mathbb{R}^n)$, we take $f := (-\Delta)^{\alpha/2}g$, which is a function in $\mathcal{M}_q^p(\mathbb{R}^n)$ by Lemma 3.4. Moreover, $g = (-\Delta)^{-\alpha/2}f \in \mathcal{M}_t^s(\mathbb{R}^n)$ where $\frac{1}{s} = \frac{1}{p} - \frac{\alpha q}{np}$ and $\frac{s}{t} = \frac{p}{q}$, so that Olsen's inequality (3.1) gives

$$\||\cdot|^{-\alpha}g\|_{\mathcal{M}_q^p} \lesssim \|W\|_{\mathcal{M}_v^u}\|(-\Delta)^{\alpha/2}g\|_{\mathcal{M}_q^p}.$$

For $1 \le p = q < \frac{n}{\alpha}$, we use the fact that $f \in L^q(\mathbb{R}^n)$ and that $g = (-\Delta)^{-\alpha/2}f \in wL^t(\mathbb{R}^n)$ for $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}$ with $\|(-\Delta)^{-\alpha/2}f\|_{wL^t} \lesssim \|f\|_{L^q}$ (where $wL^t(\mathbb{R}^n)$ denotes the weak Lebesgue space of exponent t). It thus follows from [7, Proposition 4.1] that

$$\||\cdot|^{-\alpha}g\|_{wL^q} = \|W(-\Delta)^{-\alpha/2}f\|_{wL^q} \lesssim \|W\|_{wL^v}\|(-\Delta)^{-\alpha/2}f\|_{wL^t} \lesssim \|W\|_{wL^v}\|f\|_{L^q},$$

where $v = \frac{n}{\alpha}$ (as above). This inequality holds for every $1 \le q < \frac{n}{\alpha}$. By the Marcinkiewicz interpolation theorem, we obtain

$$\||\cdot|^{-\alpha} g\|_{L^q} \lesssim \|W\|_{wL^v} \|f\|_{L^q} = \|W\|_{wL^v} \|(-\Delta)^{\alpha/2} g\|_{L^q},$$

for $1 < q < \frac{n}{\alpha}$. This completes the proof. ☐

As a corollary of Proposition 3.1, we obtain the following result (which is analogous to [1, Corollary 5.2]).

Theorem 3.5. *Let $1 < p \le q < \infty$, $1 \le p_2 \le q_2 < \infty$, $\beta > 0$, and $0 < \gamma < \frac{n}{q}$. If $\frac{\beta+\gamma}{p_0} = \frac{\beta}{p} + \frac{\gamma}{p_2}$ and $\frac{\beta+\gamma}{q_0} = \frac{\beta}{q} + \frac{\gamma}{q_2}$, then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\gamma/2} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^{p}_{q}}$$

for every $g \in C_c^\infty(\mathbb{R}^n)$.

Proof. Write $g(x) = \left[|x|^\beta g(x)\right]^{\gamma/(\beta+\gamma)} \left[|x|^{-\gamma} g(x)\right]^{\beta/(\beta+\gamma)}$. By Hölder's inequality and Proposition 3.1, we obtain

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \le \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \||\cdot|^{-\gamma} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^{p}_{q}} \lesssim \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\gamma/2} g\|^{\beta/(\beta+\gamma)}_{\mathcal{M}^{p}_{q}},$$

as desired. ☐

Finally, we use our estimate for the fractional power of the Laplacian in Theorem 2.8 to get the following Heisenberg's uncertainty inequality (which is analogous to [1, Theorem 5.4]).

Theorem 3.6. *Let $1 < p_1 \le q_1 < \infty$, $1 \le p_2 \le q_2 < \infty$, and $\beta, \delta > 0$. If $\frac{\beta+\delta}{p_0} = \frac{\beta}{p_1} + \frac{\delta}{p_2}$ and $\frac{\beta+\delta}{q_0} = \frac{\beta}{q_1} + \frac{\delta}{q_2}$, then*

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \||\cdot|^{\beta} g\|^{\delta/(\beta+\delta)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta/(\beta+\delta)}_{\mathcal{M}^{p_1}_{q_1}}$$

for every $g \in C_c^\infty(\mathbb{R}^n)$.

Proof. The idea of the proof is the same as in [1]. If $\delta < \frac{n}{q_1}$, we do not have to do anything – the inequality is the same as in Theorem 3.5. Otherwise, we set $\gamma = \delta\theta$ and apply the interpolation inequality

$$\|(-\Delta)^{\delta\theta/2} g\|_{\mathcal{M}^{p}_{q}} \lesssim \|g\|^{1-\theta}_{\mathcal{M}^{p_0}_{q_0}} \|(-\Delta)^{\delta/2} g\|^{\theta}_{\mathcal{M}^{p_1}_{q_1}}$$

for $0 < \theta < \frac{n}{\delta q_1}$, so that the inequality in Theorem 3.5 becomes

$$\|g\|_{\mathcal{M}^{p_0}_{q_0}} \lesssim \||\cdot|^{\beta} g\|^{\gamma/(\beta+\gamma)}_{\mathcal{M}^{p_2}_{q_2}} \|(-\Delta)^{\delta/2} g\|^{\beta\theta/(\beta+\gamma)}_{\mathcal{M}^{p_1}_{q_1}} \|g\|^{\beta(1-\theta)/(\beta+\gamma)}_{\mathcal{M}^{p_0}_{q_0}}.$$

Rearranging the expression, we get the desired inequality. ☐

Remark 3.7. Note that the value of δ in the above proposition can be as large as possible. This is the benefit from the interpolation inequality for the fractional power of the Laplacian.

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