Inclusion Properties of Orlicz and Weak Orlicz Spaces

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Abstract

This paper discusses the structure of Orlicz spaces and weak Orlicz spaces on $\mathbb{R}^n$. We obtain some necessary and sufficient conditions for the inclusion property of these spaces. One of the keys is to compute the norm of the characteristic functions of the balls in $\mathbb{R}^n$.

Keywords: Orlicz spaces, weak Orlicz spaces.

1 Introduction

Orlicz spaces were introduced by Z.W. Birnbaum and W. Orlicz in 1931 [9]. Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function, that is, Φ is convex, continuous, $\Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Given a measure space (X, dx), we define the Orlicz space $L_\Phi(X)$ to be the set of measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X \Phi(a|f(x)|)dx < \infty$$

for some $a > 0$. This space is a Banach space equipped with the norm

$$\|f\|_{L_\Phi(X)} := \inf\left\{b > 0 : \int_X \Phi\left(\frac{|f(x|)}{b}\right) dx \leq 1\right\}.$$

Note that, if $\Phi(t) := t^p$ for some $p \geq 1$ and $X := \mathbb{R}^n$, then $L_\Phi(X) = L_p(\mathbb{R}^n)$, the Lebesgue space of p-th integrable functions on $\mathbb{R}^n$ [6]. Thus, Orlicz spaces can be viewed as a generalization of Lebesgue spaces.

Many authors have been culminating important observations about Orlicz spaces. Here we are interested in the inclusion property of these spaces. In 1966, R. Welland [12] proved the following inclusion property: Let X be of finite measure, and Φ, Ψ be two Young functions. If there is $C > 0$ such that $\Phi(t) \leq \Psi(Ct)$ for every $t > 0$, then $L_\Phi(X) \subseteq L_\Psi(X)$. Accordingly, if X is of finite measure, Φ, Ψ are two Young functions, and there is $C > 0$ such that $\Psi(\frac{t}{C}) \leq \Phi(t) \leq \Psi(Ct)$ for every $t > 0$, then we have $L_\Phi(X) = L_\Psi(X)$. A refinement of this result may be found in [2], which states that $L_\Phi(X) \subseteq L_\Psi(X)$ if only if there are $C > 0$ and $T > 0$ such that $\Phi(t) \leq \Psi(Ct)$ for every $t \geq T$. Motivated by these results, the purpose of this paper to study further the inclusion property of Orlicz spaces $L_\Phi(\mathbb{R}^n)$ and extend the results to weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$. (Here $X := \mathbb{R}^n$ has an infinite measure.)

The rest of this paper is organized as follows. The main results are presented in Sections 2 and 3. In Section 2, we state the inclusion property of Orlicz spaces $L_\Phi(\mathbb{R}^n)$ as Theorem 2.5, which contains a necessary and sufficient condition for the inclusion property to hold. An analogous result for the weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ is stated as Theorem 3.3.

To prove the results, we pay attention to the characteristic functions of balls in $\mathbb{R}^n$ and use the inverse function of Φ, namely $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$. The reader will find the following lemma useful.

Lemma 1.1. *[7] Suppose that Φ is a Young function and $\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}$. We have*

(1) $\Phi^{-1}(0) = 0$.

(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ *for* $s_1 \leq s_2$.

(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.

(4) *Let* $C > 0$. *Then* $\Phi_1(t) \leq \Phi_2(Ct)$ *if only if* $C\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.

(5) *Let* $C > 0$. *Then* $\Phi_1(t) \leq C\Phi_2(t)$ *if only if* $\Phi_1^{-1}(Ct) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.

More lemmas (and their proofs) will be presented in the next sections.

2 Inclusion property of Orlicz spaces

Let us first recall several properties of Young functions and the Luxemburg norm.

Lemma 2.1. *[11] Let Φ be a Young function. Then*

(1) $\Phi(\alpha t) \leq \alpha\Phi(t)$ *for* $t > 0$ *and* $0 \leq \alpha \leq 1$.

(2) $\Psi(t) := \frac{\Phi(t)}{t}$ *is increasing.*

(3) $\Phi(t)$ *is strictly increasing.*

Proof. (1) Observe that, $\Phi(\alpha t) = \Phi(\alpha t + (1-\alpha)0) \leq \alpha\Phi(t) + (1-\alpha)\Phi(0) = \alpha\Phi(t)$, for $t > 0$ and $0 \leq \alpha \leq 1$.

(2) Let $0 < t_1 \leq t_2$ (so that $\frac{t_1}{t_2} \leq 1$). Observe that, $\frac{1}{t_1}\Phi(t_1) \leq \frac{1}{t_2}\Phi(t_2)$. Hence $\psi(t) = \frac{\Phi(t)}{t}$ is increasing.

(3) Let $0 < t_1 < t_2$ (so that $\frac{1}{t_2} < \frac{1}{t_1}$). Observe that, $\frac{1}{t_1}\Phi(t_1) \leq \frac{1}{t_2}\Phi(t_2) < \frac{1}{t_1}\Phi(t_2)$, which implies that $\Phi(t_1) < \Phi(t_2)$. Hence $\Phi(t)$ is strictly increasing. □

Lemma 2.2. *[3] Let Φ be a Young function and $f \in L_\Phi(\mathbb{R}^n)$. If $0 < \|f\|_{L_\Phi(\mathbb{R}^n)} < \infty$, then $\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi(\mathbb{R}^n)}}\right) dx \leq 1$. Furthermore, $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq 1$ if only if $\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq 1$.*

Proof. Let $A_\Phi := \left\{b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \leq 1\right\}$. Choose a sequence $b_n \in A_\Phi$ such that $b_n \to \|f\|_{L_\Phi(\mathbb{R}^n)}$ as $n \to \infty$. By using the continuity property of Φ, we have $\Phi\left(\frac{|f(x)|}{b_n}\right) \to$

$\Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi(\mathbb{R}^n)}}\right)$ as $n \to \infty$. By using Fatou's lemma, we have

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi(\mathbb{R}^n)}}\right) dx = \int_{\mathbb{R}^n} \lim_{n\to\infty} \Phi\left(\frac{|f(x)|}{b_n}\right) dx \leq \liminf \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b_n}\right) dx \leq 1.$$

Now, if $0 < \|f\|_{L_\Phi(\mathbb{R}^n)} \leq 1$, then $\frac{1}{\|f\|_{L_\Phi(\mathbb{R}^n)}} \geq 1$. By using Lemma 2.1(3), we obtain

$$\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\|f\|_{L_\Phi(\mathbb{R}^n)}}\right))dx \leq 1.$$

Conversely, if $\int_{\mathbb{R}^n} \Phi(|f(x)|)dx \leq 1$, then by definition we have $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq 1$. □

Corollary 2.3. *Let Φ, Ψ be two Young functions. If there exists $C > 0$ such that $\Phi(t) \leq \Psi(Ct)$ for $t > 0$, then $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ with $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq C\|f\|_{L_\Psi(\mathbb{R}^n)}$ for every $f \in L_\Psi(\mathbb{R}^n)$.*

Proof. Suppose that $f \in L_\Psi(\mathbb{R}^n)$. Observe that

$$\int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{C\|f\|_{L_\Psi(\mathbb{R}^n)}}\right) dx \leq \int_{\mathbb{R}^n} \Psi\left(\frac{C|f(x)|}{C\|f\|_{L_\Psi(\mathbb{R}^n)}}\right) dx = \int_{\mathbb{R}^n} \Psi\left(\frac{|f(x)|}{\|f\|_{L_\Psi(\mathbb{R}^n)}}\right) dx \leq 1.$$

By the definition of $\|f\|_{L_\Phi(\mathbb{R}^n)}$, we have $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq C\|f\|_{L_\Psi(\mathbb{R}^n)}$. This proves that $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$, as desired. □

`Remark`. From Corollary 2.3, we note that if $\Phi \leq \Psi$, then $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ with $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq \|f\|_{L_\Psi(\mathbb{R}^n)}$ for every $f \in L_\Psi(\mathbb{R}^n)$. As we shall see below, the converse of this statement also holds. We need the following lemma, which we adopt from [10].

Lemma 2.4. *Let Φ be a Young function, $a \in \mathbb{R}^n$, and $r > 0$. Then $\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$, where $|B(a,r)|$ denotes the volume of $B(a,r)$.*

Proof. Let $a \in \mathbb{R}^n$ and $r > 0$. Observe that, $\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \inf\left\{b > 0 : \Phi(\frac{1}{b}) \leq \frac{1}{|B(a,r)|}\right\} = \inf A_\Phi$ where $A_\Phi := \left\{b > 0 : \Phi(\frac{1}{b}) \leq \frac{1}{|B(a,r)|}\right\}$. Meanwhile, $\Phi^{-1}(\frac{1}{|B(a,r)|}) = \inf\{r \geq 0 : \Phi(r) > \frac{1}{|B(a,r)|}\} = \inf E_\Phi$, where $E_\Phi := \{r \geq 0 : \Phi(r) > \frac{1}{|B(a,r)|}\}$. Choosing $b = \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$, we have $|B(a,r)|\Phi(\frac{1}{b}) \leq 1$ and $\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} \leq \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$. Now, if $\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} < \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$, then $\frac{1}{\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}} > \Phi^{-1}(\frac{1}{|B(a,r)|})$. By using the definition of $\Phi^{-1}(\frac{1}{|B(a,r)|})$, we can find $r_1 \in E_\Phi$ such that $\frac{1}{\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}} > r_1 \geq \Phi^{-1}(\frac{1}{|B(a,r)|})$. Since $r_1 \in E_\Phi$, we obtain $\frac{1}{r_1} \notin A_\Phi$, and so $\frac{1}{r_1} \leq \|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}$. This contradicts the fact that $\frac{1}{\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)}} > r_1$. Hence we must have $\|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$. □

Theorem 2.5. *Let Φ, Ψ be two Young functions. Then the following statements are equivalent:*

(1) *$\Phi(t) \leq \Psi(t)$ for every $t > 0$.*

(2) *$L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ with $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq \|f\|_{L_\Psi(\mathbb{R}^n)}$ for every $f \in L_\Psi(\mathbb{R}^n)$.*

Proof. We have seen that (1) implies (2). Now assume that (2) holds. By using Lemma 2.4, we have

$$\frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})} = \|\chi_{B(a,r)}\|_{L_\Phi(\mathbb{R}^n)} \leq \|\chi_{B(a,r)}\|_{L_\Psi(\mathbb{R}^n)} = \frac{1}{\Psi^{-1}(\frac{1}{|B(a,r)|})},$$

or $\Phi^{-1}(\frac{1}{|B(a,r)|}) \geq \Psi^{-1}(\frac{1}{|B(a,r)|})$, for every $a \in \mathbb{R}^n$, $r > 0$. By using Lemma 1.1(4), we obtain $\Phi(\frac{1}{|B(a,r)|}) \leq \Psi(\frac{1}{|B(a,r)|})$. Since $r > 0$ is arbitrary, we conclude that $\Phi \leq \Psi$. □

2.1 A special case

One might ask whether the previous result contains the known fact about Lebesgue spaces. The answer is yes, but we need the following lemma.

Lemma 2.6. *Let Φ_1, Φ_2, and Φ_3 be three Young functions such that $\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq \Phi_3^{-1}(t)$ for every $t \geq 0$. If $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$, then $fg \in L_{\Phi_3}(\mathbb{R}^n)$ with $\|fg\|_{L_{\Phi_3}(\mathbb{R}^n)} \leq 2\|f\|_{L_{\Phi_1}(\mathbb{R}^n)}\|g\|_{L_{\Phi_2}(\mathbb{R}^n)}$.*

Proof. Let $s, t \geq 0$. Without loss of generality, suppose that $\Phi_1(s) \leq \Phi_2(t)$. By using Lemma 1.1(3), we obtain

$$st \leq \Phi_1^{-1}(\Phi_1(s))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_1^{-1}(\Phi_2(t))\Phi_2^{-1}(\Phi_2(t)) \leq \Phi_3^{-1}(\Phi_2(t)).$$

Hence $\Phi_3(st) \leq \Phi_3(\Phi_3^{-1}(\Phi_2(t))) \leq \Phi_2(t) \leq \Phi_2(t) + \Phi_1(s)$. Now it follows from Lemmas 2.1 and 2.2 that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi_3\left(\frac{|f(x)g(x)|}{2\|f\|_{L_{\Phi_1}(\mathbb{R}^n)}\|g\|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx &\leq \frac{1}{2}\int_{\mathbb{R}^n} \Phi_3\left(\frac{|f(x)g(x)|}{\|f\|_{L_{\Phi_1}(\mathbb{R}^n)}\|g\|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx \\ &\leq \frac{1}{2}\int_{\mathbb{R}^n} \Phi_1\left(\frac{|f(x)|}{\|f\|_{L_{\Phi_1}(\mathbb{R}^n)}}\right) dx + \frac{1}{2}\int_{\mathbb{R}^n} \Phi_2\left(\frac{|g(x)|}{\|g\|_{L_{\Phi_2}(\mathbb{R}^n)}}\right) dx \\ &\leq \frac{1}{2} + \frac{1}{2} = 1,\end{aligned}$$

whenever $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$. By using the definition of $\|fg\|_{L_{\Phi_3}(\mathbb{R}^n)}$, we have $\|fg\|_{L_{\Phi_3}(\mathbb{R}^n)} \leq 2\|f\|_{L_{\Phi_1}(\mathbb{R}^n)}\|g\|_{L_{\Phi_2}(\mathbb{R}^n)}$, as desired. □

Corollary 2.7. *Let $X := B(a, r_0) \subset \mathbb{R}^n$ for some $a \in \mathbb{R}^n$ and $r_0 > 0$. If Φ_1, Φ_2 are two Young functions and there is a Young function Φ such that*

$$\Phi_1^{-1}(t)\Phi^{-1}(t) \leq \Phi_2^{-1}(t)$$

for every $t \geq 0$, then $L_{\Phi_1}(X) \subseteq L_{\Phi_2}(X)$ with

$$\|f\|_{L_{\Phi_2}(X)} \leq \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})}\|f\|_{L_{\Phi_1}(X)}$$

for $f \in L_{\Phi_1}(X)$.

Proof. Let $f \in L_{\Phi_1}(X)$. By using Lemma 2.4 and choosing $g := \chi_{B(a,r_0)}$, we obtain

$$\|f\chi_{B(a,r_0)}\|_{L_{\Phi_2}(X)} \leq 2\|\chi_{B(a,r_0)}\|_{L_\Phi(X)}\|f\|_{L_{\Phi_1}(X)} = \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})}\|f\|_{L_{\Phi_1}(X)}.$$

This shows that $L_{\Phi_1}(X) \subseteq L_{\Phi_2}(X)$. □

Corollary 2.8. *Let $X := B(a, r_0)$ for some $a \in \mathbb{R}^n$ and $r_0 > 0$. If $1 \le p_2 < p_1 < \infty$, then $L_{p_1}(X) \subseteq L_{p_2}(X)$.*

Proof. Let $\Phi_1(t) := t^{p_1}, \Phi_2(t) := t^{p_2}$, and $\Phi(t) := t^{\frac{p_1 p_2}{p_1 - p_2}}$ $(t \ge 0)$. Since $1 \le p_2 < p_1 < \infty$, we have $\frac{p_1 p_2}{p_1 - p_2} > 1$. Thus, Φ_1, Φ_2, and Φ are three Young functions. Observe that, using the definition of Φ^{-1} and Lemma 1.1, we have

$$\Phi_1^{-1}(t) = t^{\frac{1}{p_1}},\ \Phi_2^{-1}(t) = t^{\frac{1}{p_2}},\ \text{and } \Phi^{-1}(t) = t^{\frac{p_1 - p_2}{p_1 p_2}}.$$

Moreover, $\Phi_1^{-1}(t)\Phi^{-1}(t) = t^{\frac{1}{p_1}} t^{\frac{p_1 - p_2}{p_1 p_2}} = t^{\frac{1}{p_2}} = \Phi_2^{-1}(t)$, and so it follows from Corollary 2.7 that $\|f\|_{L_{p_2}(X)} \le \frac{2}{\Phi^{-1}(\frac{1}{|B(a,r)|})} \|f\|_{L_{p_1}(X)}$, and therefore $L_{p_1}(X) \subseteq L_{p_2}(X)$. □

`Remark`. Of course we can prove the inclusion of property of Lebesgue spaces on a finite measure space directly via Hölder's inequality. What we showed here is that we can obtain the result through the lense of Orlicz spaces.

3 Inclusion property of Weak Orlicz spaces

First, we recall the definition of weak Orlicz spaces [8]. Let Φ be a Young function. We define the weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ to be the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wL_\Phi(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{wL_\Phi(\mathbb{R}^n)} := \inf \left\{ b > 0 : \sup_{t>0} \Phi(t) \left| \{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\} \right| \le 1 \right\}.$$

The relation between weak Orlicz spaces and (strong) Orlicz spaces are clear, as presented in the following theorem.

Theorem 3.1. *Let Φ be a Young function. Then $L_\Phi(\mathbb{R}^n) \subset wL_\Phi(\mathbb{R}^n)$ with $\|f\|_{wL_\Phi(\mathbb{R}^n)} \le \|f\|_{L_\Phi(\mathbb{R}^n)}$ for every $f \in L_\Phi(\mathbb{R}^n)$.*

Proof. Given $f \in L_\Phi(\mathbb{R}^n)$, let $A_\Phi := \left\{ b > 0 : \sup\limits_{t>0} \Phi(t) \left| \{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\} \right| \le 1 \right\}$ and $B_\Phi := \left\{ b > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x|)}{b}\right) dx \le 1 \right\}$. Then $\|f\|_{wL_\Phi(\mathbb{R}^n)} = \inf A_\Phi$ and $\|f\|_{L_\Phi(\mathbb{R}^n)} = \inf B_\Phi$. Observe that, for arbitrary $b \in B_\Phi$ and $t > 0$, we have

$$\begin{aligned} \Phi(t) \left| \{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\} \right| &\le \int_{\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}} \Phi\left(\frac{|f(x)|}{b}\right) dx \\ &\le \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{b}\right) dx \le 1. \end{aligned}$$

Since $t > 0$ is arbitrary, we have $\sup\limits_{t>0} \Phi(t) \mid \{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\} \mid \le 1$, and $B_\Phi \subseteq A_\Phi$. Hence, $f \in wL_\Phi$ with $\|f\|_{wL_\Phi(\mathbb{R}^n)} \le \|f\|_{L_\Phi(\mathbb{R}^n)}$. □

`Remark`. As the strong and weak Orlicz spaces contain the strong and weak Lebesgue spaces respectively, the inclusion in the above theorem is proper. See [1] for a counterexample.

In addition to Lemma 2.4, we have the following lemma for the characteristic functions of balls in weak Orlicz spaces, which we adapt from [4].

Lemma 3.2. *Let Φ be a Young function, $a \in \mathbb{R}^n$, and $r > 0$ be arbitrary. Then we have* $\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$.

Proof. Let $A_\Phi = \Big\{b > 0 : \sup\limits_{t>0} \Phi(t)\Big|\{x \in B(a,r) : \frac{1}{b} > t\}\Big| \le 1\Big\}$. Then

$$\begin{aligned}\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} &= \inf\Big\{b > 0 : \sup_{t>0} \Phi(t)\Big|\{x \in \mathbb{R}^n : \frac{|\chi_{B(a,r)}|}{b} > t\}\Big| \le 1\Big\} \\ &= \inf\Big\{b > 0 : \sup_{t>0} \Phi(t)\Big|\{x \in B(a,r) : \frac{1}{b} > t\}\Big| \le 1\Big\} \\ &= \inf A_\Phi.\end{aligned}$$

One may also observe that $\Phi^{-1}(\frac{1}{|B(a,r)|}) = \inf E_\Phi$ where $E_\Phi = \{r \ge 0 : \Phi(r) > \frac{1}{|B(a,r)|}\}$. By using Theorem 3.1 and Lemma 2.4, we have $\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} \le \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$. Now, suppose that $\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} < \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$ or $\Phi^{-1}(\frac{1}{|B(a,r)|}) < \frac{1}{\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)}}$. By using the definition of Φ^{-1}, there should be $r_1 \in E_\Phi$ such that $\Phi^{-1}(\frac{1}{|B(a,r)|}) \le r_1 < \frac{1}{\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)}}$ and $\Phi(r_1) > \frac{1}{|B(a,r)|}$. Now, take $b \in A_\Phi$, we have $\Phi(t) \le \frac{1}{|B(a,r)|}$ with $\frac{1}{b} > t$. Because $\frac{1}{r_1} > \|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)}$, we obtain $b_1 \in A_\Phi$ such that $\frac{1}{r_1} > b_1 \ge \|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)}$. Choosing $t = r_1$ (so that $r_1 < \frac{1}{b_1}$), we have $\Phi(r_1) \le \frac{1}{|B(a,r)|}$. Thus we obtain a contradiction. Hence, we must have $\|\chi_{B(a,r)}\|_{wL_\Phi(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{|B(a,r)|})}$. □

Now we come to the inclusion property of weak Orlicz spaces.

Theorem 3.3. *Let Φ, Ψ be two Young functions. Then the following statements are equivalent:*

(1) $\Phi(t) \le \Psi(t)$ *for every* $t > 0$.

(2) $wL_\Psi(\mathbb{R}^n) \subseteq wL_\Phi(\mathbb{R}^n)$ *with* $\|f\|_{wL_\Phi(\mathbb{R}^n)} \le \|f\|_{wL_\Psi(\mathbb{R}^n)}$ *for every* $f \in wL_\Psi(\mathbb{R}^n)$.

Proof. Assume that (1) holds, and let $f \in wL_\Psi(\mathbb{R}^n)$. Put

$$A_\Phi = \Big\{b > 0 : \sup_{t>0} \Phi(t)\Big|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big| \le 1\Big\}$$

and

$$A_\Psi = \Big\{b > 0 : \sup_{t>0} \Psi(t)\Big|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big| \le 1\Big\}.$$

Then $\|f\|_{wL_\Phi(\mathbb{R}^n)} = \inf A_\Phi$ and $\|f\|_{wL_\Psi(\mathbb{R}^n)} = \inf A_\Psi$. Observe that, for arbitrary $b \in A_\Psi$ and $t > 0$, we have

$$\Phi(t)\Big|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big| \le \Psi(t)\Big|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big| \le 1.$$

Thus, $\sup\limits_{t>0} \Phi(t)\Big|\{x \in \mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big| \le 1$. Hence it follows that $b \in A_\Phi$, and so we conclude that $A_\Psi \subseteq A_\Phi$. Accordingly, we obtain $\|f\|_{wL_\Phi(\mathbb{R}^n)} = \inf A_\Phi \le \inf A_\Psi = \|f\|_{wL_\Psi(\mathbb{R}^n)}$, which also proves that $wL_\Psi(\mathbb{R}^n) \subset wL_\Phi(\mathbb{R}^n)$.

Assume now that (2) holds. By using Lemma 3.2, we have

$$\frac{1}{\Phi^{-1}(\frac{1}{|B(a,r_0)|})} = \|\chi_{B(a,r_0)}\|_{wL_\Phi(\mathbb{R}^n)} \leq \|\chi_{B(a,r_0)}\|_{wL_\Psi(\mathbb{R}^n)} = \frac{1}{\Psi^{-1}(\frac{1}{|B(a,r_0)|})},$$

or $\Phi^{-1}(\frac{1}{|B(a,r_0)|}) \geq \Psi^{-1}(\frac{1}{|B(a,r_0)|})$, for arbitrary $a \in \mathbb{R}^n$ and $r_0 > 0$. By using Lemma 1.1, we have

$$\Phi\Big(\frac{1}{|B(a,r_0)|}\Big) \leq \Psi\Big(\frac{1}{|B(a,r_0)|}\Big).$$

Since $a \in \mathbb{R}^n$ and $r_0 > 0$ are arbitrary, we conclude that $\Phi \leq \Psi$. □

4 Concluding remarks

We have shown the inclusion property of (strong) Orlicz spaces and of weak Orlicz spaces. Both use the norm of the characteristic functions of the balls in $\mathbb{R}^n$. As our final conclusion, we have the following corollary which states that the inclusion property of (strong) Orlicz spaces are equivalent to that of weak Orlicz spaces, and both can be observed just by comparing the associated Young functions.

Corollary 4.1. *Let Φ, Ψ be two Young functions. Then the following statements are equivalent:*

(1) $\Phi(t) \leq \Psi(t)$ *for every* $t > 0$.

(2) $L_\Psi(\mathbb{R}^n) \subseteq L_\Phi(\mathbb{R}^n)$ *with* $\|f\|_{L_\Phi(\mathbb{R}^n)} \leq \|f\|_{L_\Psi(\mathbb{R}^n)}$ *for every* $f \in L_\Psi(\mathbb{R}^n)$.

(3) $wL_\Psi(\mathbb{R}^n) \subseteq wL_\Phi(\mathbb{R}^n)$ *with* $\|f\|_{wL_\Phi(\mathbb{R}^n)} \leq \|f\|_{wL_\Psi(\mathbb{R}^n)}$ *for every* $f \in wL_\Psi(\mathbb{R}^n)$.

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References

[1] H. Gunawan, D.I. Hakim, K.M. Limanta, and A.A. Masta, ”An inclusion property of generalized Morrey spaces”, submitted.

[2] O. Kufner, O. John, and S. Fučik, *Function Space*, Noordhoff International Publishing, Czechoslovakia, 1977.

[3] C. Léonard, ”Orlicz spaces”, http://cmap.polytechnique.fr/∼leonard/papers/orlicz.pdf.

[4] N. Liu and Y. Ye, ”Weak Orlicz space and its convergence theorems”, *Acta Math. Sci. Ser. B* **30**-5 (2010), 1492–1500.

[5] W.A.J. Luxemburg, *Banach Function Spaces*, Thesis, Technische Hogeschool te Delft, 1955.

[6] A.A. Masta, *On Uniform Orlicz Spaces*, Thesis, Bandung Institute of Technology, 2013.

[7] E. Nakai, ”On Orlicz-Morrey spaces”, http://www.kurims.kyoto-u.ac.jp/∼kyodo/kokyuroku/contents/pdf/1520-10.pdf.

[8] E. Nakai, ”Orlicz-Morrey spaces and some integral operators”, http://www.kurims.kyoto-u.ac.jp/∼kyodo/kokyuroku/contents/pdf/1399-13.pdf.

[9] W. Orlicz, *Linear Functional Analysis (Series in Real Analysis Volume 4)*, World Scientific, Singapore, 1992.

[10] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, Marcel Dekker, Inc., New York, 1991.

[11] Y. Sawano, *A Handbook of Harmonic Analysis*, http://www.comp.tmu.ac.jp/harmonic-analysis/harmonic-analysis-textbook.pdf.

[12] R. Welland, ”Inclusion relations among Orlicz spaces”, *Proc. Amer. Math. Soc.* **17**-1 (1966), 135–139.