Two Dimensional Interpolation using Tensor Product of Chebyshev Systems

[1,2]Lukita Ambarwati and [1]Hendra Gunawan

[1] Analysis and Geometry Group
Institut Teknologi Bandung, Jl. Ganeca 10 Bandung, Indonesia
email: lukita_72@yahoo.com ; hgunawan@math.itb.ac.id

[2] Mathematics Department
Universitas Negeri Jakarta, Jl. Pemuda 10 Rawamangun, Jakarta, Indonesia.

Abstract. In this paper we will discuss about an interpolation problem of a set of data on an $m_1 \times m_2$ grid on $A_1 \times A_2$. We will use tensor product of two Chebyshev systems on A_1 and A_2 as the space of interpolants. The procedure will be explained and some examples will be presented.

Keywords: *interpolation, Chebyshev system.*

1 Introduction

The double Fourier sine series

$$U(x,y) := \sum_{k=1}^{m_1} \sum_{l=1}^{m_2} a_{kl} \sin(k\pi x)\sin(l\pi y) \tag{1}$$

can be used to interpolate a set of points $\{(x_i, y_j, c_{ij}) : i = 1,2,...,m_1;\ j = 1,2,...,m_2\}$ on an $m_1 \times m_2$ grid on $[0,1]^2$ (see [1],[4]). By substituting the given points, we obtain the linear equation system $AX = B$, where $A = \left[\sin(k\pi x_i)\left[\sin(l\pi y_j)\right]\right]$ is a block matrix of size $m_1 m_2 \times m_1 m_2$. Since

$$\begin{aligned}\det(A) &= \det\left(\left[\sin(k\pi x_i)\left[\sin(l\pi y_j)\right]\right]_{m_1m_2\times m_1m_2}\right)\\ &= \det\left(\left[\sin(k\pi x_i)\right]\right)^{m_2} \bullet \det\left(\left[\sin(l\pi y_j)\right]\right)^{m_1} \qquad (2)\\ &\neq 0,\end{aligned}$$

(1) has a unique solution.

In this paper we will generalize the above result. Let A_1 and A_2 be compact Hausdorff topological spaces. Let $\{(x_i, y_j, c_{ij}) : i = 1,2,...,m_1;\ j = 1,2,...,m_2\}$ be a set of points on $A_1 \times A_2 \times \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\{\phi_1, \phi_2, ..., \phi_{m_1}\}$ and $\{\psi_1, \psi_2, ..., \psi_{m_2}\}$ be the set of functions on A_1 and A_2 respectively . We will show that the functions

$$U(x, y) = \sum_{m=1}^{m_1}\sum_{n=1}^{m_2} a_{mn}\phi_m(x)\psi_n(y) \qquad (3)$$

can interpolate the given points only if $\left[\varphi_j(x_i)\right]$ and $\left[\psi_j(y_i)\right]$ are nonsingular. The set of functions $\{\phi_1, \phi_2, ..., \phi_{m_1}\}$ and $\{\psi_1, \psi_2, ..., \psi_{m_2}\}$ which satisfy this condition are known as Chebyshev system. Furthermore, the function (3) can be used to interpolate any set points that is contained on any $m_1 \times m_2$ grid on $A_1 \times A_2$.

2 Two Dimensional Interpolation

2.1 Problem 1: Full Grid

Definition 1 (Chebyshev System) *[2]*
Let A be a compact Hausdorff topological space that contain at least n points. A set of continuous, complex or real value, functions $\{\phi_1, ..., \phi_n\}$ on A is called a Chebyshev system on A if it satisfies the following condition:
For arbitrary n distinct points, $x_1, ..., x_n$, on A then determinant

$$D(x_1, ..., x_n) = det(\phi_j(x_i))_{n\times n} \neq 0$$

where $[\phi_j(x_i)]_{n\times n}$ is a matrix sized $n \times n$ (with $\phi_j(x_i)$ being the element on i^{th}-row and j^{th}-column).

Let A_1 and A_2 be compact Haudorff topological spaces. Let $\{(x_i, y_j, c_{ij}) : i = 1, 2, ..., m_1;\ j = 1, 2, ..., m_2\}$ be a set of points on $m_1 \times m_2$ grid on $A_1 \times A_2 \times \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$

or $\mathbb{C}$. Let $\{\phi_1,\phi_2,...,\phi_{m_1}\}$ and $\{\psi_1,\psi_2,...,\psi_{m_2}\}$ be Chebyshev systems on A_1 and A_2 respectively. Then we will show that

$$U(x,y)=\sum_{m=1}^{m_1}\sum_{n=1}^{m_2}a_{mn}\phi_m(x)\psi_n(y) \tag{4}$$

can interpolate the given points.

Now, substituting the given points to (4), we obtain the linear equation system *AX*=*B*, where

$$A=\begin{pmatrix} \phi_1(x_1)\left[\psi_j(y_i)\right] & \phi_2(x_1)\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_1)\left[\psi_j(y_i)\right] \\ \phi_1(x_2)\left[\psi_j(y_i)\right] & \phi_2(x_2)\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_2)\left[\psi_j(y_i)\right] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_{m_1})\left[\psi_j(y_i)\right] & \phi_2(x_{m_1})\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_{m_1})\left[\psi_j(y_i)\right] \end{pmatrix}_{m_1m_2\times m_1m_2}. \tag{5}$$

The existence of the coefficient a_{mn} on (4) is determined by matrix *A*.

Definition 2 (Kronecker Product) *[3]*
Let A and B be matrices of size $m\times n$ and $k\times l$ respectively. The Kronecker product of A and B, $A\otimes B$ is defined by a block matrix of size $mk\times nl$, given by

$$A\otimes B:=\begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}_{mk\times nl}. \tag{6}$$

Theorem 1 *[3]*
Let A and B be square matrixes of size $m\times m$ and $n\times n$ respectively. Then

$$\det(A\otimes B)=\left(\det(A)\right)^n\times\left(\det(B)\right)^m. \tag{7}$$

Theorem 2
Let $\Phi = \{\phi_1, \phi_2, ..., \phi_{m_1}\}$ *and* $\Psi = \{\psi_1, \psi_2, ..., \psi_{m_2}\}$ *be Chebyshev systems on* A_1 *and* A_2 *respectively. Let* $\{(x_i, y_j, c_{ij}) : i = 1,2,..., m_1;\ j = 1,2,..., m_2\}$ *be a set of points on* $m_1 \times m_2$ *grid on* $A_1 \times A_2$. *Then the system of linear equations*

$$\sum_{m=1}^{m_1}\sum_{n=1}^{m_2} a_{mn}\phi_m(x_i)\psi_n(y_j) = c_{ij},\ i = 1,2,..., m_1;\ j = 1,2,..., m_2 \tag{8}$$

has a unique solution.

Proof. The system of linear equations (8) can be written as $AX = B$, where

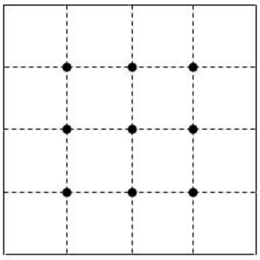
$$A = \begin{pmatrix} \phi_1(x_1)\left[\psi_j(y_i)\right] & \phi_2(x_1)\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_1)\left[\psi_j(y_i)\right] \\ \phi_1(x_2)\left[\psi_j(y_i)\right] & \phi_2(x_2)\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_2)\left[\psi_j(y_i)\right] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_{m_1})\left[\psi_j(y_i)\right] & \phi_2(x_{m_1})\left[\psi_j(y_i)\right] & \cdots & \phi_{m_1}(x_{m_1})\left[\psi_j(y_i)\right] \end{pmatrix}_{m_1m_2\times m_1m_2}. \tag{9}$$

Because Φ and Ψ are Chebyshev systems, we have

$$\begin{aligned} \det(A) &= \det\left(\left[\phi_j(x_i)\right]_{m_1\times m_1} \otimes \left[\psi_j(x_i)\right]_{m_2\times m_2}\right) \\ &= \left\{\det\left[\phi_j(x_i)\right]_{m_1\times m_1}\right\}^{m_2} \times \left\{\det\left[\psi_j(x_i)\right]_{m_2\times m_2}\right\}^{m_1} \\ &\neq 0. \end{aligned} \tag{10}$$

This implies that (8) has a unique solution. □

Example 1. Suppose we want to interpolate $\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)$, $\left(\frac{1}{4},\frac{1}{2},1\right)$, $\left(\frac{1}{4},\frac{3}{4},2\right)$, $\left(\frac{1}{2},\frac{1}{4},1\right)$, $\left(\frac{1}{2},\frac{1}{2},2\right)$, $\left(\frac{1}{2},\frac{3}{4},1\right)$, $\left(\frac{3}{4},\frac{1}{4},\frac{1}{2}\right)$, $\left(\frac{3}{4},\frac{1}{2},1\right)$, $\left(\frac{3}{4},\frac{3}{4},1\right)$ using the product of two Chebyshev systems on [0,1]. The points are given as 3×3 grid:



.

Since we want to interpolate points on 3×3 grid on $[0,1]\times[0,1]$, we use two Chebyshev systems that consist of 3 functions on $[0,1]$.

a. If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as Chebyshev systems, then the general interpolant has form

$$
\begin{aligned}
U(x,y) = \quad & a_{11}\sin \pi x + a_{12}\sin \pi x\cos \pi y + a_{13}\sin \pi x \cos 2\pi y \\
& + a_{21}\sin 2\pi x + a_{22}\sin 2\pi x \cos \pi y + a_{23}\sin 2\pi x \cos 2\pi y \\
& + a_{31}\sin 3\pi x + a_{32}\sin 3\pi x \cos \pi y + a_{33}\sin 3\pi x \cos 2\pi y.
\end{aligned} \tag{11}
$$

Substituting the value from the given points and reducing into a row echelon form, we get

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2}+\frac{1}{2} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-\sqrt{2}}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{2}}{2}-\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}
\end{bmatrix},
$$

so

$$U(x,y) = \left(\frac{\sqrt{2}}{2}+\frac{1}{2}\right)\sin \pi x - \frac{1}{2}\sin \pi x \cos \pi y - \frac{1}{2}\sin \pi x \cos 2\pi y$$
$$+\frac{1}{4}\sin 2\pi x - \frac{\sqrt{2}}{4}\sin 2\pi x \cos \pi y + \frac{1}{4}\sin 2\pi x \cos 2\pi y \qquad (12)$$
$$+\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right)\sin 3\pi x - \frac{1}{2}\sin 3\pi x \cos \pi y + \frac{1}{2}\sin 3\pi x \cos 2\pi y$$

interpolates the given points.

b. If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as Chebyshev systems, then the general interpolant has form

$$U(x,y) = a_{11} + a_{12}y + a_{13}y^2 + a_{21}\cos \pi x + a_{22}y\cos \pi x + a_{23}y^2 \cos \pi x$$
$$+ a_{31}\cos 2\pi x + a_{32}y\cos 2\pi x + a_{33}y^2 \cos 2\pi x. \qquad (13)$$

Substituting the value from the given points and solving linear equation system, we get

$$U(x,y) = 2y + \frac{\sqrt{2}}{2}\cos \pi x - 3\sqrt{2}y\cos \pi x + 4\sqrt{2}y^2 \cos \pi x$$
$$+ 2\cos 2\pi x - 14y\cos 2\pi x + 16y^2 \cos 2\pi x \qquad (14)$$

interpolate the given points.

c. Let we use $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ as Chebyshev systems. Similar to the above, we get

$$U(x,y) = \left(\frac{3\sqrt{2}}{4}-1\right)\sin \pi y + \frac{-5}{2}\sin 2\pi y + \left(\frac{3\sqrt{2}}{4}+1\right)\sin 3\pi y$$
$$+\left(\frac{-\sqrt{2}}{2}+8\right)x\sin \pi y + 9x\sin 2\pi y + \left(\frac{-\sqrt{2}}{2}-8\right)x\sin 3\pi y \qquad (15)$$
$$-8x^2 \sin \pi y - 8x^2 \sin 2\pi y + 8x^2 \sin 3\pi y$$

as an interpolant.

The graph of the functions that interpolate the given points can be seen in Figure 1.

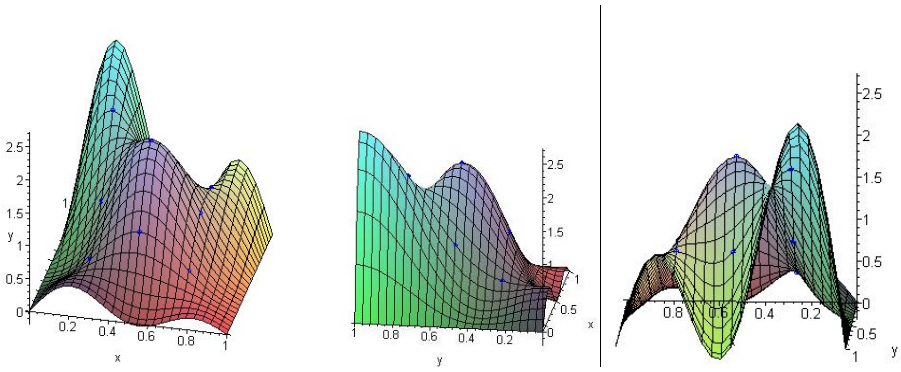
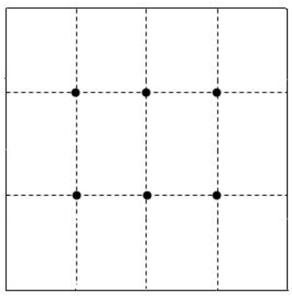


Figure 1: The graph of the interpolants using two Chebyshev systems: $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

Example 2. Suppose we want to interpolate $\left(\frac{1}{4},\frac{1}{3},4\right),\left(\frac{1}{4},\frac{2}{3},1\right),\left(\frac{1}{2},\frac{1}{3},1\right),\left(\frac{1}{2},\frac{2}{3},4\right),$ $\left(\frac{3}{4},\frac{1}{3},4\right),\left(\frac{3}{4},\frac{2}{3},1\right)$ using the product of two Chebyshev systems on [0,1]. The points are given as 3×2 grid:



Since we want to interpolate points on 3×2 grid on $[0,1]\times[0,1]$, we use two Chebyshev systems that consist of 3 functions and 2 functions. Similar to the above example, we have

$$U(x,y) = \left(\frac{5}{4}+\frac{5\sqrt{2}}{4}\right)\sin \pi x + \left(\frac{3\sqrt{2}}{2}-\frac{3}{2}\right)\sin \pi x \cos \pi y + \left(\frac{5\sqrt{2}}{4}-\frac{5}{4}\right)\sin 3\pi x + \left(\frac{3\sqrt{2}}{2}+\frac{3}{2}\right)\sin 3\pi x \cos \pi y \quad (16)$$

as interpolant if we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ as Chebyshev systems. If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ as Chebyshev systems, then we get

$$U(x,y) = 7 - 9y + 9\cos 2\pi x - 18y\cos 2\pi x \quad (17)$$

as an interpolant.

Meanwhile, if we use $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ as Chebyshev systems, then we get

$$U(x,y) = \frac{5\sqrt{3}}{3}\sin \pi y + 7\sqrt{3}\sin 2\pi y - 32\sqrt{3}x\sin 2\pi y + 32\sqrt{3}x^2\sin 2\pi y \quad (18)$$

as an interpolant.

The graph of the functions that interpolate the given points can be seen in Figure 2.

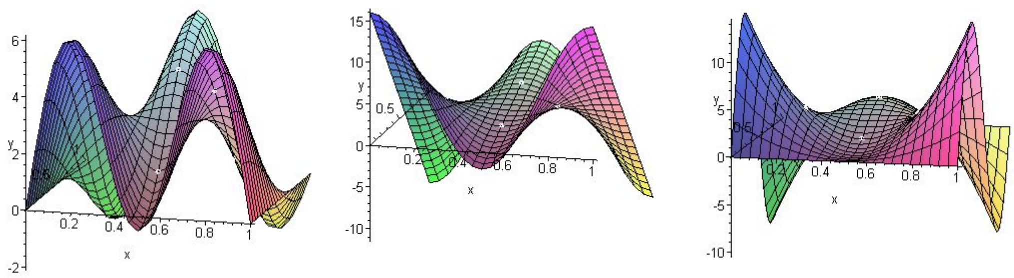


Figure 2: The graph of the interpolants using the product of two Chebyshev systems: $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ (left), $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ (center) and $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ (right).

2.2 Problem 2: Part of Grid

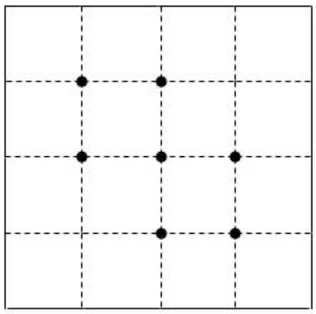
Let $G = \{(x_l, y_l, c_l) : l = 1, 2, ..., k\}$ be any set of points on $A_1 \times A_2 \times \mathbb{F}$. So there is a set $H = \{(x_i, y_j, c_{ij}) : i = 1, 2, ..., m_1; j = 1, 2, ..., m_2\}$ (has a grid form), such that H is a 'minimal' grid that contains G. This implies $k < m_1 \bullet m_2$. Let $\{\phi_1, \phi_2, ..., \phi_{m_1}\}$ and $\{\psi_1, \psi_2, ..., \psi_{m_2}\}$ be Chebyshev systems on on A_1 and A_2 respectively. We can use

$$U(x,y) = \sum_{m=1}^{m_1}\sum_{n=1}^{m_2} a_{mn}\phi_m(x)\psi_n(y) \tag{19}$$

as an interpolant of G.

Substituting the points on G to (19), we obtain the linear equation system $AX=B$. It has k equations and $m_1 \bullet m_2$ variables. Since $k < m_1 \bullet m_2$, $AX=B$ has many solutions. This implies there are many sets of a_{mn} such that (19) interpolate the given points.

Example3. Suppose we want to interpolate $\left(\frac{1}{4},\frac{1}{2},2\right),\left(\frac{1}{4},\frac{3}{4},1\right),\left(\frac{1}{2},\frac{1}{4},2\right),\left(\frac{1}{2},\frac{1}{2},3\right)$ $\left(\frac{1}{2},\frac{3}{4},2\right),\left(\frac{3}{4},\frac{1}{4},1\right),\left(\frac{3}{4},\frac{1}{2},2\right)$ using the product of two Chebyshev systems on [0,1]. The points are given on a subset of a 3×3 grid:



Since the 'minimal' grid that contains the given points is 3×3 grid on $[0,1]\times[0,1]$, we use two Chebyshev systems that consist of 3 functions.

a. If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as the Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) = \; & a_{11}\sin \pi x + a_{12}\sin \pi x \cos \pi y + a_{13}\sin \pi x \cos 2\pi y \\ & + a_{21}\sin 2\pi x + a_{22}\sin 2\pi x \cos \pi y + a_{23}\sin 2\pi x \cos 2\pi y \\ & + a_{31}\sin 3\pi x + a_{32}\sin 3\pi x \cos \pi y + a_{33}\sin 3\pi x \cos 2\pi y. \end{aligned} \tag{20}$$

Substituting the values from the given points and reducing the matrix into a row echelon form, we get

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \sqrt{2}+\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & \sqrt{2}-1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} & \sqrt{2}-\frac{3}{2} \end{bmatrix}.$$

There are many functions that interpolate the given points, one of them is:

$$\begin{aligned} U(x,y) = & \left(\frac{-1}{2}\sqrt{2}+\frac{1}{2}\right)\sin\pi x - \sin\pi x\cos 2\pi y + \left(\sqrt{2}-1\right)\sin 2\pi x \\ & + \left(\sqrt{2}-\frac{3}{2}\right)0.25\sin 2\pi x\cos 2\pi y. \end{aligned} \tag{21}$$

b. If we use $\{1, \cos\pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as the Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) = \; & a_{11} + a_{12}y + a_{13}y^2 + a_{21}\cos\pi x + a_{22}y\cos\pi x + a_{23}y^2\cos\pi x \\ & + a_{31}\cos 2\pi x + a_{32}y\cos 2\pi x + a_{33}y^2\cos 2\pi x. \end{aligned} \tag{22}$$

The same process applied to the above gives

$$U(x,y) = -2 + 16y - 16y^2 - \cos 2\pi x \tag{23}$$

as one of the functions that interpolate the given points.

c. If we use $\{1, x, x^2\}$ and $\{\sin\pi y, \sin 2\pi y, \sin 2\pi y\}$ as the Chebyshev systems, then one of the functions that interpolate the given points is

$$U(x,y) = \left(\frac{-5}{2}+\sqrt{2}\right)\sin \pi y + \left(2-\sqrt{2}\right)\sin 2\pi y + \left(\sqrt{2}-\frac{3}{2}\right)\sin 3\pi y$$
$$+16x\sin \pi y + \left(-4+2\sqrt{2}\right)x\sin 2\pi y - 16x^2 \sin \pi y. \quad (24)$$

The graph of the functions that interpolate the given points can be seen in Figure 3.

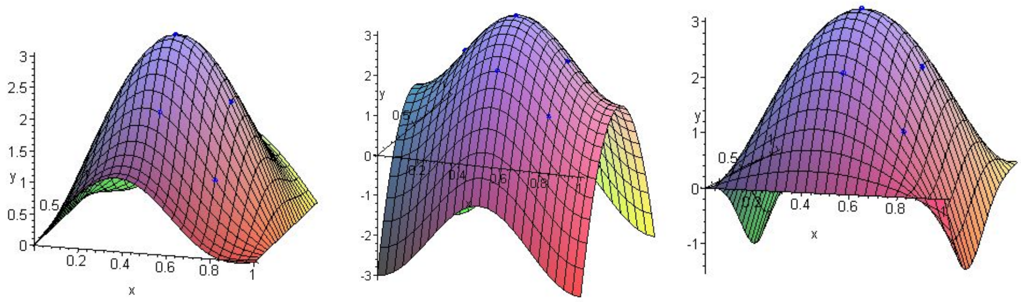


Figure 3: The graph of the interpolants using the product of two Chebyshev systems: $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

Acknowledgement. L. Ambarwati and H. Gunawan are supported by PRI Research Grant 2010/2011.

References

[1] H. Gunawan, E. Rusyaman, L.Ambarwati (2009), *Surfaces with prescribed nodes and minimum energy integral of fractional order*, submitted.

[2] G.B. Lorentz (1966), *Approximation of Function*, AMS Chelsea Publishing, USA.

[3] C.R. Rao and M.B. Rao (1998), *Matrix Algebra and Its Application to Statistics and Econometric*, World Scientific, Singapore.

[4] E. Rusyaman, H. Gunawan, A.K. Supriatna, R.E. Siregar (2010), *Eksistensi interpolan sinusoida berdimensi dua*, to appear in JMS.