NORM ESTIMATES FOR BESSEL-RIESZ OPERATORS ON GENERALIZED MORREY SPACES

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Abstract. We revisit the properties of Bessel-Riesz operators and present a different proof of the boundedness of these operators on generalized Morrey spaces. We also obtain an estimate for the norm of these operators on generalized Morrey spaces in terms of the norm of their kernels on an associated Morrey space. As a consequence of our results, we reprove the boundedness of fractional integral operators on generalized Morrey spaces, especially of exponent 1, and obtain a new estimate for their norm.

Key words: *Bessel-Riesz operators, fractional integral operators, generalized Morrey spaces.*

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1 Introduction

Integral operators such as maximal operators and fractional integral operators have been studied extensively in the last four decades. Here we are interested in Bessel-Riesz operators, which are related to fractional integral operators. Let $0 < \alpha < n$ and $\gamma \geq 0$. The operator $I_{\alpha,\gamma}$ which maps every $f \in L^p_{\rm loc}(\mathbb{R}^n),\ 1 \leq p < \infty$, to

$$I_{\alpha,\gamma} f\,(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y) f(y) dy = K_{\alpha,\gamma} * f(x), \quad x \in \mathbb{R}^n,$$

where $K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^{\gamma}}$, is called *Bessel-Riesz operator*, and the kernel $K_{\alpha,\gamma}$ is called *Bessel-Riesz kernel*. The boundedness of these operators on Morrey spaces and on generalized Morrey spaces was studied in [10, 11].

Let $1 \leq p < \infty$ and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be of class $\mathcal{G}_p$, that is ϕ is almost decreasing [$\exists\, C > 0$ such that $\phi(r) \geq C\phi(s)$ for $r \leq s$] and $\phi^p(r)r^n$ is almost increasing [$\exists\, C > 0$ such that $\phi^p(r)r^n \leq C\phi^p(s)s^n$ for

$r \le s]$. Clearly if ϕ is of class $\mathcal{G}_p$, then ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that $\frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C$ whenever $1 \le \frac{r}{s} \le 2$. We define the *generalized Morrey space* $L^{p,\phi}(\mathbb{R}^n)$ to be the set of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{L^{p,\phi}} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

where $|B|$ denotes the Lebesgue measure of B. (Recall that the Lebesgue measure of $B = B(a,r)$ is $|B(a,r)| = C_n r^n$ for every $a \in \mathbb{R}^n$ and $r > 0$, where $C_n > 0$ depends only on n.)

If $1 \le p \le q < \infty$ and $\phi(r) := C_n r^{-\frac{n}{q}}$ $(r > 0)$, then $L^{p,\phi}(\mathbb{R}^n)$ is the classical Morrey space $L^{p,q}(\mathbb{R}^n)$, which is equipped by

$$\|f\|_{L^{p,q}} := \sup_{B=B(a,r)} |B|^{\frac{1}{q}-\frac{1}{p}} \left(\int_B |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Particularly, for $p = q$, $L^{p,p}(\mathbb{R}^n)$ is the Lebesgue space $L^p(\mathbb{R}^n)$.

In [11], we know that for $\gamma > 0$, $K_{\alpha,\gamma}$ is a member of $L^t(\mathbb{R}^n)$ spaces for some values of t depending on α and γ. It follows from Young's inequality [4] that

$$\|I_{\alpha,\gamma} f\|_{L^q} \le \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),$$

whenever $1 \le p < t'$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{t'}$ (where t' denotes the dual exponent of t) and $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. This tells us that $I_{\alpha,\gamma}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $\|I_{\alpha,\gamma}\|_{L^p \to L^q} \le \|K_{\alpha,\gamma}\|_{L^t}$. In [10], it is also shown that $I_{\alpha,\gamma}$ is bounded on generalized Morrey spaces but without a good estimate for its norm as on Morrey spaces. We shall now refine the results, by estimating the norms of the operators more carefully through the membership of K_α in Morrey spaces.

Note that for $\gamma = 0$, $I_{\alpha,0} = I_\alpha$ is the *fractional integral operator* with kernel $K_\alpha(x) := |x|^{\alpha-n}$. Hardy-Littlewood [7, 8] and Sobolev [17] proved the boundedness of I_α on Lebesgue spaces. The boundedness of I_α on Morrey spaces is proved by Spanne [16], and improved by Adams [1] and Chiarenza-Frasca [2]. Later, Nakai [13] obtained the boundedness of I_α on generalized Morrey spaces, which can be viewed as an extension of Spanne's result. In 2009, Gunawan-Eridani [5] proved the boundedness of I_α on generalized Morrey spaces which extends Adams' and Chiarenza-Frasca's results.

In this paper, we give a new proof of the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces. At the same time, an upper bound for the norm of the operators is obtained. As a consequence of our result, we have an estimate for the norm of I_α (from a generalized Morrey space to another) in terms of the norm of K_α on the associated Morrey space. A lower bound for the norm of the operators is discussed in §3.

2 The Boundedness of $I_{\alpha,\gamma}$ on Generalized Morrey Spaces

We begin with a lemma about the membership of K_α in some Morrey spaces. Note that throughout this paper, the letters C and C_k denote constants which may change from line to line.

Lemma 2.1 *If $0 < \alpha < n$, then $K_\alpha \in L^{s,t}(\mathbb{R}^n)$ where $1 \le s < t = \frac{n}{n-\alpha}$.*

Proof. Let $0<\alpha<n$. Take an arbitrary $B=B(a,R)$ where $a\in\mathbb{R}^n$ and $R>0$. For $1\le s<t=\frac{n}{n-\alpha}$, we observe that

$$|B|^{\frac{s}{t}-1}\int_B K_\alpha^s(x)dx \le |B(0,R)|^{\frac{s}{t}-1}\int_{B(0,R)}|x|^{(\alpha-n)s}dx \le C\,R^{n(\frac{s}{t}-1)}R^{n(1-\frac{s}{t})}=C.$$

By taking the supremum over $B=B(a,R)$, we obtain $\|K_\alpha\|_{L^{s,t}}^s\le C$. Hence $K_\alpha\in L^{s,t}(\mathbb{R}^n)$. ∎

`Remark`. For $0<\alpha<n$ and $\gamma>0$, we know that $K_{\alpha,\gamma}\in L^t(\mathbb{R}^n)$ for $\frac{n}{n+\gamma-\alpha}<t<\frac{n}{n-\alpha}$ [11]. By the inclusion property of Morrey spaces (see [6]), we have $K_{\alpha,\gamma}\in L^t(\mathbb{R}^n)=L^{t,t}(\mathbb{R}^n)\subseteq L^{s,t}(\mathbb{R}^n)$, for $1\le s\le t$ and $\frac{n}{n+\gamma-\alpha}<t<\frac{n}{n-\alpha}$. Moreover, because $K_{\alpha,\gamma}(x)\le K_\alpha(x)$ for every $x\in\mathbb{R}^n$, $K_{\alpha,\gamma}$ is also contained in $L^{s,t}(\mathbb{R}^n)$ for $1\le s<t=\frac{n}{n-\alpha}$.

As a counterpart of the results in [10, 11], we have the following theorem on the boundedness of $I_{\alpha,\gamma}$ on Morrey spaces. Note particularly that the estimate holds for $p_1=1$.

Theorem 2.2 *If $0<\alpha<n$ and $\gamma\ge 0$, then $I_{\alpha,\gamma}$ is bounded from $L^{p_1,q_1}(\mathbb{R}^n)$ to $L^{p_2,q_2}(\mathbb{R}^n)$ with*

$$\|I_{\alpha,\gamma}f\|_{L^{p_2,q_2}}\le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}}\|f\|_{L^{p_1,q_1}},\quad f\in L^{p_1,q_1}(\mathbb{R}^n),$$

whenever $1\le p_1\le q_1<\frac{n}{\alpha}$, $\frac{1}{p_2}=\frac{1}{p_1}-\frac{1}{s'}$, and $\frac{1}{q_2}=\frac{1}{q_1}-\frac{1}{t'}$, with $1\le s<t=\frac{n}{n-\alpha}$ (for $\gamma\ge 0$) or $1\le s\le t$ and $\frac{n}{n+\gamma-\alpha}<t<\frac{n}{n-\alpha}$ (for $\gamma>0$).

Theorem 2.2 is in fact a special case of the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces, which is stated in the following theorem.

Theorem 2.3 *Let $0<\alpha<n$ and $\gamma\ge 0$. If $\phi:\mathbb{R}^+\to\mathbb{R}^+$ is of class $\mathcal{G}_{p_1}$ such that $\int_R^\infty \phi(r)r^{\frac{n}{t'}-1}dr\le C\,\phi(R)R^{\frac{n}{t'}}$ for every $R>0$, then $I_{\alpha,\gamma}$ is bounded from $L^{p_1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\psi}(\mathbb{R}^n)$ where $\psi(r):=\phi(r)r^{\frac{n}{t'}}$, with*

$$\|I_{\alpha,\gamma}f\|_{L^{p_2,\psi}}\le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}}\|f\|_{L^{p_1,\phi}},\quad f\in L^{p_1,\phi}(\mathbb{R}^n),$$

whenever $1\le p_1<\frac{n}{\alpha}$ and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{1}{s'}$, with $1\le s<t=\frac{n}{n-\alpha}$ (for $\gamma\ge 0$) or $1\le s\le t$ and $\frac{n}{n+\gamma-\alpha}<t<\frac{n}{n-\alpha}$ (for $\gamma>0$).

Proof. Suppose that $\gamma>0$ and all the hypotheses hold. For $f\in L^{p_1,\phi}(\mathbb{R}^n)$ and $B=B(a,R)$ where $a\in\mathbb{R}^n$ and $R>0$, write

$$f:=f_1+f_2:=f\chi_{\widetilde{B}}+f\chi_{\widetilde{B}^c},$$

where $\widetilde{B}=B(a,2R)$ and $\widetilde{B}^c$ denotes its complement. To estimate $I_{\alpha,\gamma}f_1$, we observe that for every $x\in B$, Hölder's inequality gives

$$\begin{aligned}|I_{\alpha,\gamma}f_1(x)| &\le \int_{\widetilde{B}} K_{\alpha,\gamma}(x-y)|f(y)|dy\\ &= \int_{\widetilde{B}} K_{\alpha,\gamma}^{\frac{s}{p_2}}(x-y)|f(y)|^{\frac{p_1}{p_2}}K_{\alpha,\gamma}^{\frac{p_2-s}{p_2}}(x-y)|f(y)|^{\frac{p_2-p_1}{p_2}}dy\\ &\le \left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)|f(y)|^{p_1}dy\right)^{\frac{1}{p_2}}\left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{\frac{p_2-s}{p_2-1}}(x-y)|f(y)|^{\frac{p_2-p_1}{p_2-1}}dy\right)^{\frac{1}{p_2'}}.\end{aligned}$$

Meanwhile, we have

$$\int_{\widetilde{B}} K_{\alpha,\gamma}^{\frac{p_2-s}{p_2-1}}(x-y)|f(y)|^{\frac{p_2-p_1}{p_2-1}}dy \leq \left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)dy\right)^{p_2'(\frac{1}{s}-\frac{1}{p_2})} \left(\int_{\widetilde{B}} |f(y)|^{p_1}dy\right)^{\frac{p_2'}{s'}}.$$

Therefore we obtain

$$\begin{aligned}|I_{\alpha,\gamma}f_1(x)| &\leq \left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)|f(y)|^{p_1}dy\right)^{\frac{1}{p_2}} \left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)dy\right)^{\frac{1}{s}-\frac{1}{p_2}} \left(\int_{\widetilde{B}} |f(y)|^{p_1}dy\right)^{\frac{1}{s'}} \\ &\leq \left(\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)|f(y)|^{p_1}dy\right)^{\frac{1}{p_2}} \times C\,R^{n(1-\frac{s}{t})(\frac{1}{s}-\frac{1}{p_2})+\frac{n}{s'}}\phi^{\frac{p_1}{s'}}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{1-\frac{s}{p_2}}\|f\|_{L^{p_1,\phi}}^{\frac{p_1}{s'}}.\end{aligned}$$

We take the p_2-th power and integrate both sides over B to get

$$\begin{aligned}\int_B |I_{\alpha,\gamma}f_1(x)|^{p_2}dx &\leq \int_B\int_{\widetilde{B}} K_{\alpha,\gamma}^{s}(x-y)|f(y)|^{p_1}dy\,dx \\ &\quad\times\left(C\,R^{n(1-\frac{s}{t})(\frac{1}{s}-\frac{1}{p_2})+\frac{n}{s'}}\phi^{\frac{p_1}{s'}}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{1-\frac{s}{p_2}}\|f\|_{L^{p_1,\phi}}^{\frac{p_1}{s'}}\right)^{p_2}.\end{aligned}$$

By Fubini's theorem, we have

$$\begin{aligned}\int_B |I_{\alpha,\gamma}f_1(x)|^{p_2}dx &\leq \int_{\widetilde{B}} |f(y)|^{p_1}\left(\int_B K_{\alpha,\gamma}^{s}(x-y)dx\right)dy \\ &\quad\times\left(C\,R^{n(1-\frac{s}{t})(\frac{1}{s}-\frac{1}{p_2})+\frac{n}{s'}}\phi^{\frac{p_1}{s'}}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{1-\frac{s}{p_2}}\|f\|_{L^{p_1,\phi}}^{\frac{p_1}{s'}}\right)^{p_2} \\ &\leq C\,R^{n(1-\frac{s}{t})}\|K_{\alpha,\gamma}\|_{L^{s,t}}^{s}\int_{\widetilde{B}}|f(y)|^{p_1}dy \\ &\quad\times\left(R^{n(1-\frac{s}{t})(\frac{1}{s}-\frac{1}{p_2})+\frac{n}{s'}}\phi^{\frac{p_1}{s'}}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{1-\frac{s}{p_2}}\|f\|_{L^{p_1,\phi}}^{\frac{p_1}{s'}}\right)^{p_2} \\ &\leq C\,R^{n(1-\frac{s}{t})+n}\phi^{p_1}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{s}\|f\|_{L^{p_1,\phi}}^{p_1} \\ &\quad\times\left(R^{n(1-\frac{s}{t})(\frac{1}{s}-\frac{1}{p_2})+\frac{n}{s'}}\phi^{\frac{p_1}{s'}}(2R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{1-\frac{s}{p_2}}\|f\|_{L^{p_1,\phi}}^{\frac{p_1}{s'}}\right)^{p_2} \\ &\leq C\,|B|\,\psi^{p_2}(R)\|K_{\alpha,\gamma}\|_{L^{s,t}}^{p_2}\|f\|_{L^{p_1,\phi}}^{p_2},\end{aligned}$$

whence

$$\frac{1}{\psi(R)}\left(\frac{1}{|B|}\int_B |I_{\alpha,\gamma}f_1(x)|^{p_2}dx\right)^{\frac{1}{p_2}} \leq C\,\|K_{\alpha,\gamma}\|_{L^{s,t}}\|f\|_{L^{p_1,\phi}}.$$

Next, we estimate $I_{\alpha,\gamma}f_2$. For every $x\in B=B(a,R)$, we observe that

$$\begin{aligned}|I_{\alpha,\gamma}f_2(x)| &\leq \int_{\widetilde{B}^c} K_{\alpha,\gamma}(x-y)|f(y)|dy \\ &\leq \int_{|x-y|\geq R} K_{\alpha,\gamma}(x-y)|f(y)|dy \\ &= \sum_{k=0}^{\infty}\int_{2^kR\leq|x-y|<2^{k+1}R} K_{\alpha,\gamma}(x-y)|f(y)|dy \\ &\leq \sum_{k=0}^{\infty} K_{\alpha,\gamma}(2^kR)\int_{2^kR\leq|x-y|<2^{k+1}R}|f(y)|dy \\ &\leq C\sum_{k=0}^{\infty} K_{\alpha,\gamma}(2^kR)(2^kR)^{\frac{n}{p_1'}}\left(\int_{2^kR\leq|x-y|<2^{k+1}R}|f(y)|^{p_1}dy\right)^{\frac{1}{p_1}} \\ &\leq C\,\|f\|_{L^{p_1,\phi}}\sum_{k=0}^{\infty} K_{\alpha,\gamma}(2^kR)(2^kR)^{n}\phi(2^kR).\end{aligned}$$

For every $k \in \mathbb{Z}$, we have

$$K_{\alpha,\gamma}(2^kR) \le C\,(2^kR)^{-\frac{n}{s}} \left(\int_{2^kR \le |x-y| < 2^{k+1}R} K^s_{\alpha,\gamma}(x-y)dy \right)^{\frac{1}{s}} \le C\,(2^kR)^{-\frac{n}{t}} \, \|K_{\alpha,\gamma}\|_{L^{s,t}}.$$

Since $\int_R^\infty \phi(r) r^{\frac{n}{t'}-1} dr \le C\,\phi(R) R^{\frac{n}{t'}}$, we get

$$\begin{aligned} |I_{\alpha,\gamma} f_2(x)| &\le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^kR)^{\frac{n}{t'}} \phi(2^kR) \\ &\le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \int_R^\infty \phi(r) r^{\frac{n}{t'}-1} dr \\ &\le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \phi(R) R^{\frac{n}{t'}} \\ &= C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \psi(R). \end{aligned}$$

Raising to the p_2-th power and integrating over B, we obtain

$$\int_B |I_{\alpha,\gamma} f_2(x)|^{p_2} dx \le C\left(\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \right)^{p_2} \psi^{p_2}(R) |B|,$$

whence

$$\frac{1}{\psi(R)} \left(\frac{1}{|B|} \int_B |I_{\alpha,\gamma} f_2(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}.$$

Combining the two estimates for $I_{\alpha,\gamma} f_1$ and $I_{\alpha,\gamma} f_2$, we obtain

$$\frac{1}{\psi(R)} \left(\frac{1}{|B|} \int_B |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{\frac{1}{p_2}} \le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}.$$

Since this inequality holds for every $a \in \mathbb{R}^n$ and $R > 0$, it follows that

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C\,\|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}},$$

as desired.

We may repeat the same argument and use Lemma 2.1 to obtain the same inequality for the case where $\gamma = 0$ and $1 \le s < t = \frac{n}{n-\alpha}$. ■

`Remark`. Theorems 2.2 and 2.3 give us upper estimates for the norm of the Bessel-Riesz operators (from one Morrey space to another). In particular, for $\gamma = 0$, we have an estimate for the norm of the fractional integral operator I_α in terms of the norm of its kernel (on the associated Morrey space), which follows from the inequality

$$\|I_\alpha f\|_{L^{p_2,\psi}} \le C\,\|K_\alpha\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}},$$

for $1 \le p_1 < \frac{n}{\alpha}$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{s'}$, with $1 \le s < t = \frac{n}{n-\alpha}$.

In the following section, we discuss lower estimates for the norm of the operators in terms of the norm of the Bessel-Riesz kernel (on some Morrey spaces).

3 An Estimate for the Norm of the Operators

Recall that if $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces and that $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is a linear operator, then the norm of T (from X to Y) is defined by

$$\|T\|_{X\to Y} := \sup_{f\neq 0} \frac{\|Tf\|_Y}{\|f\|_X}.$$

Knowing that the Bessel-Riesz operator $I_{\alpha,\gamma}$ is a linear operator on Morrey spaces, we would like to estimate the norm of $I_{\alpha,\gamma}$ from a (generalized) Morrey space to another. We obtain the following result.

Theorem 3.1 *Let $0 < \alpha < n$, $\gamma \geq 0$, and ϕ is of class $\mathcal{G}_{p_1}$ where $1 \leq p_1 < \frac{n}{\alpha}$. If $\phi(r)r^n$ is almost increasing and for every $R > 0$ we have (i) $\int_R^\infty \phi(r)r^{\frac{n}{t'}-1}\,dr \leq C_1\phi(R)R^{\frac{n}{t'}}$, (ii) $\int_0^R \phi^{p_1}(r)r^{n-1}\,dr \leq C_2\,\phi^{p_1}(R)R^n$, and (iii) $\int_0^R \frac{r^{n-1}}{\phi^{s'}(r)r^{ns'}}\,dr \leq \frac{C_3 R^n}{\phi^{s'}(R)R^{ns'}}$, where $1 \leq p_1 < t$ and $1 < s < t = \frac{n}{n-\alpha}$ (for $\gamma \geq 0$) or $1 \leq p_1 \leq t$, $1 < s \leq t$, and $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$ (for $\gamma > 0$), then we have*

$$C_4\|K_{\alpha,\gamma}\|_{L^{p_1,t}} \leq \|I_{\alpha,\gamma}\|_{L^{p_1,\phi}\to L^{p_2,\psi}} \leq C_5\|K_{\alpha,\gamma}\|_{L^{s,t}},$$

whenever $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{s'}$ and $\psi(r) := \phi(r)r^{\frac{n}{t'}}$. In particular, for $\gamma = 0$, $1 \leq p_1 < t$, and $1 < s < t = \frac{n}{n-\alpha}$, we have

$$C_4\|K_\alpha\|_{L^{p_1,t}} \leq \|I_\alpha\|_{L^{p_1,\phi}\to L^{p_2,\psi}} \leq C_5\|K_\alpha\|_{L^{s,t}},$$

whenever $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{s'}$ and $\psi(r) := \phi(r)r^{\frac{n}{t'}}$.

Proof. Suppose that $\gamma > 0$ and all the hypotheses hold. By Theorem 2.3, we already have

$$\|I_{\alpha,\gamma}\|_{L^{p_1,\phi}\to L^{p_2,\psi}} \leq C\,\|K_{\alpha,\gamma}\|_{L^{s,t}}.$$

To prove the lower estimate, put $\rho(r) := \phi(r)r^n$. Let $B = B(a,R)$ where $a \in \mathbb{R}^n$ and $R > 0$. By our assumptions on ϕ, we have

$$|B|^{\frac{1}{t}}\psi(R)\left(\frac{1}{|B|}\int_B \rho^{-s'}(|x|)\,dx\right)^{\frac{1}{s'}} \leq C\,\phi(R)R^{\frac{n}{s'}}\left(\int_0^R \frac{r^{n-1}}{\phi^{s'}(r)r^{ns'}}\,dr\right)^{\frac{1}{s'}} \leq C.$$

Now take $f_0(x) := \phi(|x|)$. Here $\|f_0\|_{L^{p_1,\phi}} \approx 1$. Moreover, one may compute that

$$I_{\alpha,\gamma}f_0(x) \geq \int_{B(x,2|x|)} K_{\alpha,\gamma}(x-y)f_0(y)dy \geq C\,K_{\alpha,\gamma}(2x)\phi(|x|)|x|^n = C\,\rho(|x|)K_{\alpha,\gamma}(x),$$

for every $x \in \mathbb{R}^n$. It follows that

$$\|\rho(|\cdot|)K_{\alpha,\gamma}\|_{L^{p_2,\psi}} \leq C\,\|I_{\alpha,\gamma}f_0\|_{L^{p_2,\psi}} \leq C\,\|I_{\alpha,\gamma}\|_{L^{p_1,\phi}\to L^{p_2,\psi}}.$$

Next, by Hölder's inequality, we have

$$\left(\int_B K_{\alpha,\gamma}^{p_1}(x)dx\right)^{\frac{1}{p_1}} \leq \left(\int_B \rho^{-s'}(|x|)\,dx\right)^{\frac{1}{s'}}\left(\int_B \left[\rho(|x|)K_{\alpha,\gamma}(x)\right]^{p_2}dx\right)^{\frac{1}{p_2}},$$

whence

$$|B|^{\frac{1}{t}-\frac{1}{p_1}}\left(\int_B K^{p_1}_{\alpha,\gamma}(x)dx\right)^{\frac{1}{p_1}} \le |B|^{\frac{1}{t}}\psi(R)\left(\frac{1}{|B|}\int_B \rho^{-s'}(|x|)\,dx\right)^{\frac{1}{s'}}$$
$$\times\frac{1}{\psi(R)}\left(\frac{1}{|B|}\int_B\big[\rho(|x|)K_{\alpha,\gamma}(x)\big]^{p_2}dx\right)^{\frac{1}{p_2}}$$
$$\le C\,\|I_{\alpha,\gamma}\|_{L^{p_1,\phi}\to L^{p_2,\psi}}.$$

By taking the supremum over $B = B(a,R)$, we conclude that

$$C\,\|K_{\alpha,\gamma}\|_{L^{p_1,t}} \le \|I_{\alpha,\gamma}\|_{L^{p_1,\phi}\to L^{p_2,\psi}},$$

as desired.

The same argument applies for the case where $\gamma = 0$, with $1 \le p_1 < t$ and $1 < s < t = \frac{n}{n-\alpha}$. ■

Remark. One may observe that the constants C_4 and C_5 in Theorem 3.1 depend on ϕ, n, p_1, s, and t, but not on α and γ. Although the lower and the upper bound are not comparable, we may still get useful information from these estimates, especially for the norm of the operator I_α from $L^{p_1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\psi}(\mathbb{R}^n)$. Observe that for $1 \le p_1 < t = \frac{n}{n-\alpha}$, we have $\|K_\alpha\|^{p_1}_{L^{p_1,t}} = \frac{C}{(\alpha-n)p_1+n} \ge \frac{C}{\alpha}$. Hence, if all the hypotheses in Theorem 3.1 hold for the case where $\gamma = 0$, then we obtain $\|I_\alpha\|_{L^{p_1,\phi}\to L^{p_2,\psi}} \ge \frac{C}{\alpha}$, which blows up when $\alpha \to 0^+$. For $\phi(r) := r^{-\frac{n}{q_1}}$ with $1 \le p_1 < q_1 < \min\{s, \frac{n}{\alpha}\}$, our result reduces to the estimate $\|I_\alpha\|_{L^{p_1,q_1}\to L^{p_2,q_2}} \ge \frac{C}{\alpha}$ where $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{s'}$ and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$. A similar behavior of the norm of I_α from $L^{p_1}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ for $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$ when $\alpha \to 0^+$ is observed in [12, Chapter 4].

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