Nonsmooth atomic decompositions for generalized Orlicz-Morrey spaces

Yoshihiro Sawano, Denny Ivanal Hakim, and Hendra Gunawan

Abstract

In this paper, we shall consider the non-smooth decomposition of generalized Orlicz-Morrey spaces. As an application, we consider the boundedness of the bilinear operator, which is called the Olsen inequality nowadays. To obtain a sharp norm estimate, we first investigate the predual space, which is even new, and we make full advantage of the vector-valued inequality of the Hardy-Littlewood maximal operator.

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1 Introduction

In 1938, C. Morrey proposed an embedding result to handle the regularity of the solutions of the elliptic differential equations [23]. Later based on this result, many authors studied the space $\mathcal{M}^p_q(\mathbb{R}^n)$ intensively. Recall that the space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(y)|^q \, dy \right)^{1/q}$$

is finite. Here we adopt the standard notations. All "cubes" in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes. Denote by $\mathcal{Q}$ the set of all cubes. For a cube $Q \in \mathcal{Q}$, the symbol $\ell(Q)$ stands for its side-length; $\ell(Q) \equiv |Q|^{\frac{1}{n}}$. Usually, we consider the case when $1 \le q \le p < \infty$. However, recent results show that $\mathcal{M}^p_q(\mathbb{R}^n)$ is not enough to tackle some problems in harmonic analysis. For example, the Hardy-Littlewood maximal operator fails to be bounded in $\mathcal{M}^p_1(\mathbb{R}^n)$. A remedy is to modify the parameter q in some sense [39, Lemma 3.5]. Also, the operator $(1-\Delta)^{-n/2p}$ fails to be bounded from $\mathcal{M}^p_q(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. One of the effective ways is to change the parameter p to the extent that we do not lose much information [46, Theorem 5.1]. To have a better understanding of these situations, generalized Orlicz-Morrey spaces can be used. Although generalized Orlicz-Morrey spaces are complicated, the atomic decomposition which we present in the present paper makes things more transparent.

Generalized Orlicz-Morrey spaces are equipped with two functions ϕ and Φ instead of parameters p and q. Denote by $\mathcal{G}_1$ the set of all increasing functions $\phi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto t^{-n}\phi(t) \in (0,\infty)$ is decreasing. Denote by Δ_2 the set of all convex bijections $\Phi : [0,\infty) \to [0,\infty)$ such that the doubling condition

$$\Phi(2t) \le C\Phi(t) \quad (t \ge 0) \tag{1}$$

holds for some constant $C \ge 2$ and by ∇_2 the set of all convex functions $\Phi : [0,\infty) \to [0,\infty)$ such that

$$2C'\Phi(t) \le \Phi(2t) \quad (t \ge 0) \tag{2}$$

for some $C' > 1$. Note that C in (1) exceeds 2 due to (2). Let Q be a cube and f a measurable function. Define the $(\Phi; Q)$-average of f by;

$$\|f\|_{\Phi;Q} \equiv \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\right\},$$

when we are given a cube $Q \in \mathcal{Q}$ and a measurable function $f : \mathbb{R}^n \to \mathbb{C}$.

The generalized Orlicz-Morrey space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ is defined as the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \equiv \sup_{Q\in\mathcal{Q}} \phi(\ell(Q))\|f\|_{\Phi;Q}$$

is finite. Note that, if $\Phi(r) = r^p, 1 \le p < \infty$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) =: \mathcal{M}_{\phi,p}(\mathbb{R}^n)$ is the generalized Morrey space discussed in [24, 39]. Observe that, if $\Phi(r) = r^p$ and $\phi(r) = r^{\frac{n}{p}}$, then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. Other definitions of generalized Orlicz-Morrey spaces can be found in [25, 26, 27]. The class $\mathcal{G}_1$ is of importance; in [39, Remark 2.4(1)], we have shown that, for any function $\phi : (0,\infty) \to (0,\infty)$ and $\Phi \in \Delta_2 \cap \nabla_2$, there exists $\rho \in \mathcal{G}_1$ such that $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) = \mathcal{M}_{\rho,\Phi}(\mathbb{R}^n)$ with norm equivalence.

We aim here to prove the following decomposition result about the functions in the above Morrey spaces. This decomposition result is made up of decomposition and synthesis.

Theorem 1.1 (Decomposition). *Let $L \in \mathbb{N} \cup \{0\}$, $\Phi \in \Delta_2 \cap \nabla_2$ and $\phi \in \mathcal{G}_1$. Assume that there exists a constant $C > 0$ such that*

$$\int_r^\infty \frac{ds}{\phi(s)s} \le \frac{C}{\phi(r)} \quad (r > 0).$$

Then, for every $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n) \cap L^1_{\rm loc}(\mathbb{R}^n)$ and that

$$|a_j| \le \chi_{Q_j}, \tag{3}$$

that for all $v > 0$ there exists a constant $C_v > 0$ such that

$$\left\| \left(\sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}} \le C_v \|f\|_{\mathcal{M}_{\phi,\Phi}}, \tag{4}$$

and that

$$\int_{\mathbb{R}^n} x^\beta a_j(x)\, dx = 0 \tag{5}$$

for all multi-indices β with $|\beta| \le L$.

To describe the synthesis result, we need to consider Orlicz spaces. Define

$$\|f\|_{L^\Phi} \equiv \inf \left\{ \lambda > 0 \,:\, \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda} \right) dx \le 1 \right\}$$

for a measurable function f. The space $L^\Phi(\mathbb{R}^n)$ is the set of all measurable functions f for which the quantity $\|f\|_{L^\Phi}$ is finite. Define $L^\Phi_{\rm loc}(\mathbb{R}^n)$ as the set of all measurable functions f for which $f\chi_Q \in L^\Phi(\mathbb{R}^n)$ for all cubes Q. Furthermore, we define $\mathcal{Q}_x(\mathbb{R}^n)$ to be a collection of all cubes that contain $x \in \mathbb{R}^n$. To formulate our next result, we need the following definition.

Definition 1.2. Let $\Phi : [0,\infty) \to [0,\infty)$ be a convex function and assume $\Phi \in \Delta_2 \cap \nabla_2$. The maximal operator M^Φ is defined by

$$M^\Phi f(x) \equiv \sup_{Q \in \mathcal{Q}_x(\mathbb{R}^n)} \|f\|_{\Phi;Q}$$

if $f : \mathbb{R}^n \to \mathbb{C}$ is measurable.

With this definition in mind, let us formualte our second main result.

Theorem 1.3 (Synthesis)**.** *Let $\Phi, \Theta \in \Delta_2 \cap \nabla_2$. Define*

$$\Psi(t) \equiv \sup_{s>0}(st - \Phi(s)), \quad \Delta(t) \equiv \sup_{s>0}(st - \Theta(s))$$

for $t \ge 0$. Assume that $\phi, \eta \in \mathcal{G}_1$ satisfy

$$\int_r^\infty \frac{\eta(s)}{\phi(s)s}\, ds \le C \frac{\eta(r)}{\phi(r)} \quad (r > 0) \tag{6}$$

for some constant $C > 0$ and that Ψ and Δ satisfy

$$\|M^\Delta g\|_{\Psi;Q} \le C \|g\|_{\Psi;Q}. \tag{7}$$

Assume also that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset \mathcal{M}_{\eta,\Theta}(\mathbb{R}^n) \cap \mathcal{M}_{\eta,\Phi}(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$\|a_j\|_{\mathcal{M}_{\eta,\Phi}} \le \eta(\ell(Q_j)), \quad \|a_j\|_{\mathcal{M}_{\eta,\Theta}} \le \eta(\ell(Q_j)), \tag{8}$$

and

$$\operatorname{supp}(a_j) \subset Q_j, \quad \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}} < \infty. \tag{9}$$

Then $f \equiv \sum_{j=1}^{\infty} \lambda_j a_j$ *converges in* $\mathcal{S}'(\mathbb{R}^n) \cap L^{\Phi}_{\rm loc}(\mathbb{R}^n)$ *and satisfies*

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \le C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi,\Phi}}. \tag{10}$$

We know that (7) is equivalent to

$$\int_1^t \Delta\left(\frac{t}{s}\right) \Theta(s)\, ds \le \Theta(Ct) \quad (t > 1)$$

for some $C > 0$, see [39, Proposition 2.17].

To prove Theorem 1.1, we consider the "generalized Hardy-Orlicz-Morrey" space $H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, which extends the Hardy space $H^p(\mathbb{R}^n)$ with $0 < p \le \infty$. The Hardy space $H^p(\mathbb{R}^n)$ can be more informative than the Lebesgue space $L^p(\mathbb{R}^n)$ when we discuss the boundedness of some operators appearing in harmonic analysis. One of the earliest real variable definitions of Hardy spaces was based on the grand maximal operator, which is discussed in [48] and references therein. In fact, one can give an equivalent definition for the Hardy space $H^p(\mathbb{R}^n)$ by using the atomic decomposition. This definition states that any elements of Hardy spaces can be represented as the series of atoms. An atom is a compactly supported function which enjoys the size condition (3) and the moment condition (5). The concept of the atomic decomposition in Hardy spaces can be developed to other function spaces. Some of these works are the atomic decomposition of Hardy-Morrey space $H\mathcal{M}^p_q(\mathbb{R}^n)$ [18], the one of Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent [36], the one of Orlicz-Hardy space $H^{\Phi}(\mathbb{R}^n)$ [29] and the one of Morrey spaces $\mathcal{M}^p_q(\mathbb{R}^n)$ [17]. In [17], the authors dealt with the space $\mathcal{M}^p_q(\mathbb{R}^n)$. In our current research, we investigate the atomic decomposition for generalized Orlicz-Morrey spaces.

Unlike Orlicz-Hardy spaces and Hardy spaces with variable exponent, a problem arises for Morrey spaces and generalized Orlicz-Morrey spaces; there is not good density result for these spaces. Let $1 < q < p < \infty$. The function $f(x) \equiv |x|^{-n/p} \in \mathcal{M}^p_q(\mathbb{R}^n)$ symbolizes one of the main difficulties when we handle Morrey spaces. In fact,

$$\|\chi_{\{|y|<R\}} f\|_{\mathcal{M}^p_q} = \|\chi_{\{|y|>R\}} f\|_{\mathcal{M}^p_q} = \|f\|_{\mathcal{M}^p_q} \tag{11}$$

for any $R > 0$. The equality (11) shows that it is hard to approximate the function f with compactly supported functions. To overcome this problem, we propose an estimate in Lemma 4.4. This estimate will finally allow us to expand f in some sense, which we do in Lemma 5.7 by using Lemma 4.4.

While Theorem 1.1 is located as a counterpart of various researches about decompositions of functions in many function spaces, Theorem 1.3 has an independent interest. To explain this, let us formulate a counterpart to Theorem 1.3 as follows:

Proposition 1.4. *Let $\Phi \in \Delta_2 \cap \nabla_2$. Assume that $\phi \in \mathcal{G}_1$ satisfies*

$$\int_r^\infty \frac{1}{\phi(s)s}\,ds \le C\frac{1}{\phi(r)}. \tag{12}$$

Assume also that $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$, $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ and $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$ fulfill

$$|a_j| \le \chi_{Q_j}, \quad \left\|\sum_{j=1}^\infty \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}} < \infty. \tag{13}$$

Then $f \equiv \sum_{j=1}^\infty \lambda_j a_j$ converges in $\mathcal{S}'(\mathbb{R}^n) \cap L^\Phi_{\rm loc}(\mathbb{R}^n)$ and satisfies

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \le C\left\|\sum_{j=1}^\infty \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}. \tag{14}$$

So, we can say that $L^\infty(\mathbb{R}^n)$ took place of $\mathcal{M}_{\eta,\Theta}(\mathbb{R}^n) \cap \mathcal{M}_{\eta,\Phi}(\mathbb{R}^n)$. As we will see in Section 6, the space $L^\infty(\mathbb{R}^n)$ is not enough. In our current setting, we claim that the space $\mathcal{M}_{\eta,\Theta}(\mathbb{R}^n) \cap \mathcal{M}_{\eta,\Phi}(\mathbb{R}^n)$ is a correct space to formulate the size condition. Of course, as we have explained, we can replace (8) with

$$\|a_j\|_{L^\infty} \le 1. \tag{15}$$

Note that (8) is weaker than (15). Under this weaker condition, Theorem 1.3 is proved by using the duality and the vector-valued inequality; Section 3 considers the predual space of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ and Section 4 intends as investigation of the Fefferman-Stein inequality.

Finally, we describe how we organize the present paper. In Section 2, we investigate some fundamental properties of the space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. As auxiliary steps, as we have mentioned just above, in Sections 3 and 4 we obtain some Hölder type inequality as well as the duality result and the vector-valued maximal inequalities. We collect the proofs of Theorems 1.1 and 1.3 in Section 5. Finally, we apply Theorems 1.1 and 1.3 to obtain a bilinear estimate of I_α in Section 6, which is nowadays called the Olsen inequality [30, Theorem 2]. Recall that we define the fractional integral operator I_α with $0 < \alpha < n$ by;

$$I_\alpha f(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy$$

for all suitable functions f on $\mathbb{R}^n$. Olsen's inequality is the one of the form

$$\|g \cdot I_\alpha f\|_Z \le C\|f\|_X\|g\|_Y,$$

where X, Y, Z are suitable Banash spaces. There is a vast amount of literatures on Olsen inequalities; see [3, 39, 37, 38, 40, 41, 42, 49, 52] for theoretical aspects and [6, 7, 8, 30] for applications to PDEs.

2 Preliminaries

2.1 Structure of generalized Morrey spaces

We first investigate the growth of the norm of the indicator functions of cubes.

Lemma 2.1. *Let $a, r > 0$ and $\Phi \in \Delta_2 \cap \nabla_2$. Then*

$$\|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n} = \min\left(\frac{1}{\Phi^{-1}(1)}, \frac{1}{\Phi^{-1}(a^{-n}r^n)}\right). \tag{16}$$

Proof. Note that if we set $\lambda \equiv \|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n}$, then λ solves

$$\frac{a^n}{r^n}\Phi\left(\frac{1}{\lambda}\right) = 1$$

when $a < r$ and

$$\Phi\left(\frac{1}{\lambda}\right) = 1$$

when $a \geq r$. Thus, we obtain (16). □

Proposition 2.2. *Let $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. Then*

$$\|\chi_{[0,a]^n}\|_{\mathcal{M}_{\phi,\Phi}} = \sup_{r \geq a} \frac{\phi(r)}{\Phi^{-1}(a^{-n}r^n)} = \sup_{r \geq 1} \frac{\phi(ar)}{\Phi^{-1}(r^n)}. \tag{17}$$

In particular, the following are equivalent;

1. *there exists a constant $C > 0$ such that $\sup_{r \geq 1} \frac{\phi(ar)}{\Phi^{-1}(r^n)} \leq C\phi(a)$ for all $a > 0$,*
2. *$\chi_{[0,a]^n} \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ for any $a > 0$ and $\phi(a) \leq \|\chi_{[0,a]^n}\|_{\mathcal{M}_{\phi,\Phi}} \leq C\phi(a)$.*

Proof. A geometric observation shows that

$$\|\chi_{[0,a]^n}\|_{\mathcal{M}_{\phi,\Phi}} = \sup_{r>0} \phi(r)\|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n}.$$

Since ϕ is assumed increasing, Lemma 2.1 yields

$$\begin{aligned}\|\chi_{[0,a]^n}\|_{\mathcal{M}_{\phi,\Phi}} &= \max\left(\sup_{0<r\leq a} \phi(r)\|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n}, \sup_{r>a} \phi(r)\|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n}\right) \\ &= \max\left(\frac{\phi(a)}{\Phi^{-1}(1)}, \sup_{r>a} \phi(r)\|\chi_{[0,a]^n}\|_{\Phi;[0,r]^n}\right) = \sup_{r \geq a} \frac{\phi(r)}{\Phi^{-1}(a^{-n}r^n)}.\end{aligned}$$

(17) is thus proved. □

In the next lemma, we adopt the following definition of the Hardy-Littlewood maximal operator to estimate some integrals.

Definition 2.3. Let $\mathcal{Q}_x(\mathbb{R}^n)$ be a collection of all cubes that contain $x \in \mathbb{R}^n$. The Hardy-Littlewood maximal operator M is defined by

$$Mf(x) \equiv \sup_{Q \in \mathcal{Q}_x(\mathbb{R}^n)} \frac{1}{|Q|} \int_Q |f(y)| dy, \tag{18}$$

for a locally integrable function f.

We remark that M is bounded on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ if $\Phi \in \Delta_2 \cap \nabla_2$; see [39, Corollary 2.17].

One of our strategies in this paper is to view $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ as the Hardy type space. As a preliminary step, let us recall the definition of $\mathcal{S}'(\mathbb{R}^n)$ As an auxliary step, let us prove that our function space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ can be readily embedded into $\mathcal{S}'(\mathbb{R}^n)$.

Definition 2.4.

1. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the set of all $C^\infty(\mathbb{R}^n)$-functions φ for which the quantity
$$\rho_N(\varphi) \equiv \sum_{|\alpha| \le N} \sup_{x \in \mathbb{R}^n} (1+|x|)^N |\partial^\alpha \varphi(x)|.$$
is finite for all $N \in \mathbb{N}$. We define the topology on $\mathcal{S}(\mathbb{R}^n)$ with the norm $\{\rho_N\}_{N \in \mathbb{N}}$.
2. The space $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$.

Lemma 2.5. *Let $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. Then, for all $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ and all $\kappa \in \mathcal{S}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\kappa(x) f(x)|\, dx \le C \|f\|_{\mathcal{M}_{\phi,\Phi}} \sup_{x \in \mathbb{R}^n} (1+|x|)^{2n+1} |\kappa(x)|, \tag{19}$$

where the constant C is independent of f and κ.

Proof. We decompose the left-hand side as follows:

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\kappa(x) f(x)|\, dx \\
&\le \int_{[-1,1]^n} |\kappa(x) f(x)|\, dx + \sum_{j=1}^\infty \int\limits_{[-(j+1),(j+1)]^n \setminus [-j,j]^n} |\kappa(x) f(x)|\, dx \\
&\le \left(\sup_{x \in [-1,1]^n} |\kappa(x)| \right) \|f\|_{L^1([-1,1]^n)} + \sum_{j=1}^\infty \int\limits_{[-(j+1),(j+1)]^n \setminus [-j,j]^n} \frac{\left|x^{2n+1}\kappa(x)\right| |f(x)|}{j^{2n+1}}\, dx \\
&\le \left(\sup_{x \in \mathbb{R}^n} (1+|x|)^{2n+1} |\kappa(x)| \right) \left(\|f\|_{L^1([-1,1]^n)} + \sum_{j=1}^\infty \int\limits_{[-(j+1),(j+1)]^n \setminus [-j,j]^n} \frac{|f(x)|}{j^{2n+1}}\, dx \right)
\end{aligned}$$

By the definition of the maximal operator M, for all $j \in \mathbb{N}$, we have

$$\frac{1}{|[-j,j]^n|} \int_{[-j,j]^n} |f(y)|\, dy \le Mf(x), \quad x \in [-1,1]^n.$$

Thus, by virtue of the boundedness of M on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, we obtain

$$\phi(2)\left\|\frac{1}{|[-j,j]^n|}\int_{[-j,j]^n}|f(y)|\,dy\right\|_{\Phi;[-1,1]^n} \le \|Mf\|_{\mathcal{M}_{\phi,\Phi}} \le C\|f\|_{\mathcal{M}_{\phi,\Phi}}.$$

This means

$$\int_{[-j,j]^n}|f(y)|\,dy \le Cj^n\|f\|_{\mathcal{M}_{\phi,\Phi}}.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^n}|\kappa(x)f(x)|dx &\le C\|f\|_{\mathcal{M}_{\phi,\Phi}}\sup_{x\in\mathbb{R}^n}(1+|x|)^{2n+1}|\kappa(x)|\left(1+\sum_{j=1}^{\infty}\frac{(j+1)^n}{j^{2n+1}}\right)\\ &= C\|f\|_{\mathcal{M}_{\phi,\Phi}}\sup_{x\in\mathbb{R}^n}(1+|x|)^{2n+1}|\kappa(x)|.\end{aligned}$$

So we are done. □

By rewriting Lemma 2.5 in terms of ρ_{2n+1}, we obtain;

Corollary 2.6. *Let $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. Then $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ is embedded into $\mathcal{S}'(\mathbb{R}^n)$.*

We also need the following estimate of the type of Lebesgue convergence theorem for the $(\Phi; Q)$-average for later considerations.

Lemma 2.7. *Let $\Phi \in \Delta_2 \cap \nabla_2$. Let f be a measurable function such that*

$$\int_{\mathbb{R}^n}\Phi(|f(x)|)\,dx < \infty.$$

Define $f^j \equiv f\chi_{\{|f|\le j\}}$. Then we have

$$\lim_{j\to\infty}\|f-f^j\|_{\Phi;Q} = 0 \tag{20}$$

for any cube Q.

Proof. Let us set

$$\rho_j \equiv \frac{1}{|Q|}\int_Q \Phi(|f(x)-f^j(x)|)\,dx.$$

By the bounded convergence theorem, $\lim_{j\to\infty}\rho_j = 0$. Let $j_0 \in \mathbb{N}$ be such that $j \ge j_0$ implies $\rho_j \le 1$. For $j \ge j_0$, we choose a natural number k_j so that $C^{-k_j-1} \le \rho_j \le C^{-k_j}$, where C is a doubling constant of Φ. Then

$$\begin{aligned}1 &= \frac{1}{\rho_j|Q|}\int_Q \Phi(|f(x)-f^j(x)|)\,dx\\ &\ge \frac{1}{|Q|}\int_Q C^{k_j}\Phi(|f(x)-f^j(x)|)\,dx\\ &\ge \frac{1}{|Q|}\int_Q \Phi\left(2^{k_j}|f(x)-f^j(x)|\right)\,dx.\end{aligned}$$

Thus, it follows that $\|f-f^j\|_{\Phi;Q} \le 2^{-k_j}$. Since $k_j \to \infty$ as $j \to \infty$, (20) follows. □

To conclude this section, we collect some helpful estimates.

Lemma 2.8. *Let $\Phi \in \Delta_2 \cap \nabla_2$. There exist constants $C_0 > 0$ and $p > 1$ such that*

$$\Psi(t) \le \frac{1}{2} + C_0 t^p \tag{21}$$

for all $t \ge 0$.

Proof. Observe that $\Psi(t) \le C' t^p, t \ge 1$ for some $C' > 0$ and $p > 1$. In fact, let $C > 2$ be the doubling constant for Ψ. Then we can choose

$$p = \log_2 C, \quad C' = C\Psi(1).$$

Note also that $\Psi(0) = 0$ since $\Phi(0) = 0$. As a consequence by replacing constant C' with a large constant C_0, we obtain (21). □

Lemma 2.9. *Let p and C_0 be constants in Lemma 2.8. Then for all $g \in L^p(\mathbb{R}^n)$, we have*

$$\|g\|_{\Psi;Q} \le (2C_0)^{1/p} \frac{\|\chi_Q g\|_{L^p}}{|Q|^{1/p}} \tag{22}$$

for each cube Q.

Proof. A calculation shows that

$$\begin{aligned}
\|g\|_{\Psi;Q} &= \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \Psi\left(\frac{|g(x)|}{\lambda}\right) dx \le 1\right\} \\
&\le \inf\left\{\lambda > 0 \,:\, \frac{1}{|Q|}\int_Q \left(\frac{1}{2} + C_0\left(\frac{|g(x)|}{\lambda}\right)^p\right) dx \le 1\right\} \\
&= \inf\left\{\lambda > 0 \,:\, \frac{C_0}{|Q|}\int_Q \left(\frac{|g(x)|}{\lambda}\right)^p dx \le \frac{1}{2}\right\} \\
&= (2C_0)^{1/p} \frac{\|g\|_{L^p(Q)}}{|Q|^{1/p}}.
\end{aligned}$$

Hence, we have (22). □

2.2 A convolution estimate

By the use of the cube-based maximal operator given by (18), we can prove the following inequality.

Proposition 2.10. [2, Proposition 2.7] *Suppose that τ is a radial decreasing positive and integrable function on $\mathbb{R}^n$, so that there exists a decreasing function ρ satisfying*

$$\rho(|x|) = \tau(x) \text{ for all } x \in \mathbb{R}^n. \tag{23}$$

Define

$$\tau_t(x) \equiv t^{-n} \tau\left(\frac{x}{t}\right) \quad (x \in \mathbb{R}^n)$$

for $t>0$. Then we have

$$\tau_t * [|f|](x) \le c_n \|\tau\|_{L^1} Mf(x). \tag{24}$$

for all $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and for all $t>0$.

3 Predual space of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$

A Banach space X is said to admit a predual if there exists a Banach space Y such that X is isomorphic to Y^*. We aim here to establish that $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ admits a predual if $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. The method relies upon the paper by Zorko [60]. See [9, 19] for other constructions of predual spaces of classical Morrey spaces.

3.1 The conjugate of Φ and the Hölder inequality

Let $\Phi \in \Delta_2 \cap \nabla_2$. Then define the conjugate function Ψ of Φ by;

$$\Psi(t) \equiv \sup\{st - \Phi(s) \,:\, s \ge 0\} \quad (t \ge 0). \tag{25}$$

Note that Ψ satisfies the same condition as Φ; $\Psi \in \Delta_2 \cap \nabla_2$. Observe also that

$$st \le \Phi(s) + \Psi(t) \tag{26}$$

for all $s, t \ge 0$ from the definition of Ψ.

Lemma 3.1. *For all $Q \in \mathcal{Q}$ and for all measurable functions $f, g : Q \to \mathbb{C}$,*

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|\,dx \le 2\|f\|_{\Phi;Q}\|g\|_{\Psi;Q}. \tag{27}$$

Proof. Let us suppose that $\|f\|_{\Phi;Q} = \|g\|_{\Psi;Q} = 1$ without loss of generality. Then we have

$$\frac{1}{|Q|}\int_Q \Phi(|f(x)|)\,dx = \frac{1}{|Q|}\int_Q \Psi(|f(x)|)\,dx = 1$$

since Φ and Ψ are assumed doubling. Since $\Phi(|f(x)|) + \Psi(|g(x)|) \ge |f(x)g(x)|$ from (26), (27) follows. □

We can generalize the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$ to generalized Orlicz-Morrey spaces by using Lemma 3.1.

Lemma 3.2. *There exists a constant $C>1$ depending on Φ such that; for all $Q \in \mathcal{Q}$ and for all measurable functions $f : Q \to \mathbb{C}$,*

$$C^{-1}\|f\|_{\Phi;Q} \le \frac{1}{|Q|}\int_Q |f(x)g(x)|\,dx \le 2\|f\|_{\Phi;Q}.$$

for some measurable function $g : Q \to \mathbb{C}$ such that $\|g\|_{\Psi;Q} = 1$.

Proof. The right inequality is a consequence of Lemma 3.1; it suffices to prove the left inequality. To this end, we may assume that f is a simple function, which is justified by a simple approximation procedure. Also, homogeneity allows us to assume that $\|f\|_{\Phi;Q} = 1$. We can choose a simple function g such that;

$$|f(x)g(x)| = \Phi(|f(x)|) + \Psi(|g(x)|) \le \Phi(2|f(x)|).$$

Thus,

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|\,dx \ge \frac{1}{|Q|}\int_Q \Phi(|f(x)|)\,dx = 1.$$

On the other hand,

$$\frac{1}{|Q|}\int_Q \Psi(|g(x)|)\,dx \le \frac{1}{|Q|}\int_Q (\Phi(2|f(x)|) - \Phi(|f(x)|))\,dx \le \frac{C}{|Q|}\int_Q \Phi(|f(x)|)\,dx = C.$$

Hence, $\|g\|_{\Psi;Q} \le C$. If we normalize g, then we obtain the desired function. □

To conclude this section, we generalize Lemma 3.1 for later consideration.

Lemma 3.3. *Let $\Phi_1, \Phi_2, \Phi_3 \in \Delta_2 \cap \nabla_2$. If Φ_1, Φ_2 and Φ_3 satisfy*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \quad (t \ge 0).$$

1. *$\Phi_3(st) \le \Phi_1(s) + \Phi_2(t)$ for every $s, t \in [0, \infty)$.*
2. *Let Q be a cube. Then, for all meausurable functions $f, g : Q \to \mathbb{C}$, we have*

$$\|f \cdot g\|_{\Phi_3;Q} \le 2\|f\|_{\Phi_1;Q}\|g\|_{\Phi_1;Q}. \tag{28}$$

Proof.

1. This is somehow well known. Here for the sake of completeness, we supply the brief proof. Without loss of generality, assume that $\Phi_1(s) \le \Phi_2(t)$, then

$$st = \Phi_1^{-1}(\Phi_1(s))\Phi_2^{-1}(\Phi_2(t)) \le \Phi_1^{-1}(\Phi_2(t))\Phi_2^{-1}(\Phi_2(t)) \le \Phi_3^{-1}(\Phi_2(t)).$$

Thus, $\Phi_3(st) \le \Phi_2(t) \le \Phi_1(s) + \Phi_2(t)$.

2. Observe that for any $\varepsilon > 0$, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \Phi_3\left(\frac{|f(x)g(x)|}{2(\|f\|_{\Phi_1;Q}+\varepsilon)(\|g\|_{\Phi_2;Q}+\varepsilon)}\right)\,dx \\
&\le \frac{1}{2|Q|}\int_Q \Phi_3\left(\frac{|f(x)g(x)|}{(\|f\|_{\Phi_1;Q}+\varepsilon)(\|g\|_{\Phi_2;Q}+\varepsilon)}\right)\,dx \\
&\le \frac{1}{2}\left(\frac{1}{|Q|}\int_Q \Phi_1\left(\frac{|f(x)|}{\|f\|_{\Phi_2;Q}+\varepsilon}\right)\,dx + \frac{1}{|Q|}\int_Q \Phi_2\left(\frac{|g(x)|}{\|g\|_{\Phi_2;Q}+\varepsilon}\right)\,dx\right) \\
&\le \frac{1}{2}(1+1) = 1.
\end{aligned}$$

Thus, $\|f \cdot g\|_{\Phi_3;Q} \le 2(\|f\|_{\Phi_1;Q}+\varepsilon)(\|g\|_{\Phi_1;Q}+\varepsilon)$. The number $\varepsilon > 0$ being arbitary, we obtain the desired result.

□

Theorem 3.4. *Let $\phi_1, \phi_2, \phi_3 \in \mathcal{G}_1$ and let $\Phi_1, \Phi_2, \Phi_3 \in \Delta_2 \cap \nabla_2$. If $\phi_1, \phi_2, \phi_3, \Phi_1, \Phi_2$ and Φ_3 satisfy*

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \quad (t \ge 0)$$

and

$$\phi_3(t) \le \phi_1(t)\phi_2(t) \quad (t > 0), \tag{29}$$

then for every $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n)$, we have

$$\|f \cdot g\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le 2\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}}\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}. \tag{30}$$

Proof. Let Q be a cube. If we combine (28) and (29), then we have

$$\phi_3(\ell(Q))\ \|f \cdot g\|_{\Phi_3;Q} \le 2\ \phi_1(\ell(Q))\ \|f\|_{\Phi_1;Q}\ \phi_2(\ell(Q))\ \|f\|_{\Phi_2;Q} \le 2\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}}\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}.$$

So, we obtain (30). □

See [47] for the classical case.

3.2 Predual space

In this section, we characterize the predual space of $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ by the method of Zorko [60]. Again here denote by Ψ the conjugate function of Φ given by (25).

Definition 3.5. A (ϕ, Ψ)-block is a measurable function A supported on a cube Q satisfying

$$\|A\|_{\Psi;Q} \le |Q|^{-1}\phi(\ell(Q)). \tag{31}$$

A function f is said to belong to $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ if there exist a summable sequence $\{\lambda_j\}_{j=1}^{\infty}$ and a sequence $\{A_j\}_{j=1}^{\infty}$ of (ϕ, Ψ)-blocks such that

$$f(x) = \sum_{j=1}^{\infty} \lambda_j A_j(x) \tag{32}$$

for almost every $x \in \mathbb{R}^n$, where the convergence takes place absolutely. The norm of f is given by;

$$\|f\|_{\mathcal{B}_{\phi,\Psi}} \equiv \inf \sum_{j=1}^{\infty} |\lambda_j|,$$

where the infimum is taken over all admissible expressions (32).

About this definition, we have two helpful examples.

Example 3.6. Let Q be a cube and $g : Q \to \mathbb{C}$ a locally integrable function such that $\|g\|_{\Psi;Q} = 1$. Then $|Q|^{-1}\phi(\ell(Q))\chi_Q g$ is a (ϕ,Ψ)-block. In fact,

$$\mathrm{supp}(|Q|^{-1}\phi(\ell(Q))\chi_Q g) \subset Q, \quad \||Q|^{-1}\phi(\ell(Q))\chi_Q g\|_{\Psi;Q} = |Q|^{-1}\phi(\ell(Q)).$$

Example 3.7. Let p and C_0 be constants from Lemma 2.8. When $g \in L^p(\mathbb{R}^n) \setminus \{0\}$,

$$h \equiv \frac{1}{(2C_0)^{1/p}\|g\|_{L^p}}|Q|^{1/p-1}\phi(\ell(Q))\chi_Q g$$

is a (ϕ,Ψ)-block. In fact,

$$\mathrm{supp}(h) \subset Q, \quad \|h\|_{\Psi;Q} = \frac{1}{(2C_0)^{1/p}\|g\|_{L^p}}|Q|^{1/p-1}\phi(\ell(Q))\|g\|_{\Psi;Q} \le \frac{\phi(\ell(Q))}{|Q|}.$$

The next lemma will justify the assertion that the dual of $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ is $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ as well as the fact that the convergence (32) takes place absolutely for almost all $x \in \mathbb{R}^n$.

Lemma 3.8. *Let $\{\lambda_j\}_{j=1}^\infty$ be a summable sequence and let $\{A_j\}_{j=1}^\infty$ be a sequence of (ϕ,Ψ)-blocks. Then, for all $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, we have*

$$\int_{\mathbb{R}^n} |f(x)| \left(\sum_{j=1}^\infty |\lambda_j A_j(x)|\right) dx \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}} \sum_{j=1}^\infty |\lambda_j|. \tag{33}$$

Proof. Let Q_j be a cube such that $\mathrm{supp}(A_j) \subset Q_j$ and that

$$\|A_j\|_{\Psi;Q_j} \le |Q_j|^{-1}\phi(\ell(Q_j)).$$

Then we have

$$\int_{\mathbb{R}^n} |f(x)A_j(x)|\,dx \le 2\|f\|_{\Phi;Q_j}\|A_j\|_{\Psi;Q_j}|Q_j| \le 2\|f\|_{\Phi;Q_j}\phi(\ell(Q_j)) \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}} \tag{34}$$

from (27) and (31). If we add (34) over $j = 1, 2, \ldots$, then we obtain (33). □

Corollary 3.9.

1. *Let $\{\lambda_j\}_{j=1}^\infty$ be a summable sequence and let $\{A_j\}_{j=1}^\infty$ be a sequence of (ϕ,Ψ)-blocks. Then the series*
$$f(x) \equiv \sum_{j=1}^\infty \lambda_j A_j(x) \tag{35}$$
converges absolutely for almost all $x \in \mathbb{R}^n$.

2. *Let $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Then the mapping*
$$L_f : g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C}$$
defines a bounded linear functional on $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.

3. Let $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then the mapping

$$M_g : f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{C}$$

defines a bounded linear functional on $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proof. Lemma 3.8 guarantees the absolute convergence of the right-hand side of (35). So we are left with the task of verifying that L_f and M_g make sense. This is reduced to establishing;

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}}\|g\|_{\mathcal{B}_{\phi,\Psi}}. \tag{36}$$

Note that (36) implies that $f \cdot g \in L^1(\mathbb{R}^n)$ and that;

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| \le \int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}}\|g\|_{\mathcal{B}_{\phi,\Psi}}.$$

Let $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then we have an expression;

$$g(x) = \sum_{j=1}^{\infty} \lambda_j a_j(x), \quad \sum_{j=1}^{\infty} |\lambda_j| \le (1+\varepsilon)\|g\|_{\mathcal{B}_{\phi,\Psi}}.$$

Then we have

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\,dx &\le \sum_{j=1}^{\infty} |\lambda_j| \int_{\mathbb{R}^n} |f(x)a_j(x)|\,dx \\ &\le 2\|f\|_{\mathcal{M}_{\phi,\Phi}} \sum_{j=1}^{\infty} |\lambda_j| \\ &\le 2(1+\varepsilon)\|f\|_{\mathcal{M}_{\phi,\Phi}}\|g\|_{\mathcal{B}_{\phi,\Psi}}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude (36). Thus, we see that L_f and M_g define bounded linear functionals. □

We give additional information about the density of the predual space $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.

Proposition 3.10. *Denote by $L^{\infty}_{\text{comp}}(\mathbb{R}^n)$ the set of all compactly supported functions which are essentially bounded. Then $L^{\infty}_{\text{comp}}(\mathbb{R}^n)$ is dense in $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.*

Proof. Let $f \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then, f has an expression:

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where each a_j is a (ϕ, Ψ)-block and

$$\sum_{j=1}^{\infty} |\lambda_j| < \infty. \tag{37}$$

Set $f_k \equiv \sum_{j=1}^k \lambda_j a_j$. Then

$$\|f - f_k\|_{\mathcal{B}_{\phi,\Psi}} \le \sum_{j=k+1}^{\infty} |\lambda_j| \to 0$$

as $k \to \infty$. This means that we can suppose that f is expressed as a finite linear combination of (ϕ, Ψ)-blocks or even that f itself is a (ϕ, Ψ)-block.

Let f be a (ϕ, Ψ)-block such that

$$\text{supp}(f) \subset Q, \quad \|f\|_{\Psi;Q} \le \frac{\phi(\ell(Q))}{|Q|}.$$

Let us set $f^j \equiv f\chi_{\{|f| \le j\}}$. Then by virtue of Lemma 2.7

$$\limsup_{j\to\infty} \|f - f^j\|_{\mathcal{B}_{\phi,\Psi}} \le \frac{|Q|}{\phi(l(Q))} \limsup_{j\to\infty} \|f - f^j\|_{\Psi;Q} = 0.$$

Since $f^j \in L^\infty_{\text{comp}}(\mathbb{R}^n)$, we conclude that $L^\infty_{\text{comp}}(\mathbb{R}^n)$ is dense in $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. □

With these facts in mind, we shall show that $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ can be regarded as the dual of $\mathcal{B}_{\psi,\Psi}(\mathbb{R}^n)$.

Theorem 3.11. *Let $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. Any bounded linear functional L on $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ is realized as $L = L_f$ with some $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Furthermore, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{\mathcal{M}_{\phi,\Phi}} \le \|L_f\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}} \le 2\|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{38}$$

Proof. The two-sided estimate (38) is a consequence of Lemma 3.2. The heart of the matter is to establish that any bounded linear functional L can be realized with some $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Let p be a constant from Lemma 2.8.

From (22), for each cube Q, we can define a bounded linear functional

$$g \in L^p(\mathbb{R}^n) \mapsto L(\chi_Q g) \in \mathbb{C}.$$

Let $g \in L^p(\mathbb{R}^n) \setminus \{0\}$. Keep to the notation of Example 3.7. According to Example 3.7, we have

$$\|h\|_{\mathcal{B}_{\phi,\Psi}} \le 1.$$

Since L is bounded on $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$, we see that

$$|L(h)| \le \|L\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}},$$

which implies

$$|L(\chi_Q \cdot g)| \le \frac{(2C_0)^{1/p}}{|Q|^{1/p-1}\phi(\ell(Q))} \|L\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}} \|g\|_{L^p} \quad (g \in L^p(\mathbb{R}^n) \setminus \{0\}). \tag{39}$$

Note that (39) is trivially valid for $g=0$. By the duality $L^p(\mathbb{R}^n)$-$L^{p'}(\mathbb{R}^n)$, we can find $f_Q \in L^{p'}(\mathbb{R}^n)$ such that

$$\mathrm{supp}(f_Q) \subset Q, \quad L(\chi_Q g) = \int_{\mathbb{R}^n} g(x) f_Q(x)\, dx \tag{40}$$

for all $g \in L^p(\mathbb{R}^n)$ and that

$$\|f_Q\|_{L^{p'}} \le \frac{(2C_0)^{1/p}}{|Q|^{1/p-1}\phi(\ell(Q))}\|L\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}.$$

Let Q be a cube and R a cube that engulfs Q. The left-hand side of the second expression in (40) vanishes if g is supported outside Q. In addition, we have

$$L(\chi_Q g) = L(\chi_Q \chi_R g) = \int_{\mathbb{R}^n} \chi_Q(x) f_R(x) g(x)\, dx. \tag{41}$$

Thus, $\chi_Q(x) f_R(x) = f_Q(x)$ for almost every $x \in \mathbb{R}^n$. This implies that the limit $f(x) \equiv \lim_{j\to\infty} f_{[-j,j]^n}(x)$ exists for almost every $x \in \mathbb{R}^n$.

Let us check that the function f belongs to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. To this end, we fix a cube Q once again. Let C be a constant from Lemma 3.2. Then we have

$$C^{-1}\|f\|_{\Phi;Q} \le \frac{1}{|Q|}\int_Q |f(x)| \cdot |g(x)|\, dx$$

for some measurable function g such that $\|g\|_{\Psi;Q} = 1$. Write $f(x)g(x) = |f(x)g(x)|e^{i\phi(x)}$ by using a real valued measurable function $\phi(x)$. If we define $\tilde{g}(x) \equiv g(x)e^{-i\phi(x)}$, then $\|\tilde{g}\|_{\Psi;Q} = \|g\|_{\Psi;Q} = 1$ and

$$C^{-1}\|f\|_{\Phi;Q} \le \frac{1}{|Q|}\int_Q f(x) \cdot \tilde{g}(x)\, dx \tag{42}$$

Thus, by Example 3.6, (41) and (42), we have

$$\|f\|_{\Phi;Q} \le \frac{C}{|Q|} L(\chi_Q \tilde{g}) \le \frac{C}{|Q|}\|\chi_Q \tilde{g}\|_{\mathcal{B}_{\phi,\Psi}}\|L\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}} \le \frac{C\|L\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}}{\phi(\ell(Q))}.$$

Hence $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Finally, let us check that $L = L_f$. To this end, we need to prove $L(g) = L_f(g)$ for all $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ but by Proposition 3.10, we can suppose that $g \in L^\infty_{\mathrm{comp}}(\mathbb{R}^n)$. Since g is compactly supported, we can take $j_0 \in \mathbb{N}$ so that $\mathrm{supp}(g) \subset [-j_0, j_0]^n$. Then

$$L(g) = L(\chi_{[-j,j]^n} g) = \int_{\mathbb{R}^n} f_{[-j,j]^n}(x) g(x)\, dx = \int_{\mathbb{R}^n} f(x) g(x)\, dx = L_f(g).$$

As a consequence, $L(g) = L_f(g)$ for all $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. □

In the same way as Lemma 2.5, we can prove the following:

Proposition 3.12. *Let $\phi \in \mathcal{G}_1$ and $\Psi \in \nabla_2 \cap \Delta_2$. Then, $\mathcal{S}(\mathbb{R}^n)$ is embedded into $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$.*

4 Vector-valued maximal inequality

Our aim is to extend the Fefferman-Stein vector-valued inequality for M, which is proved in [5];

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^u \right)^{\frac{1}{u}} \right\|_{L^p} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{L^p}. \tag{43}$$

More precisely, we prove the following vector-valued maximal inequality.

Theorem 4.1. *Let $1 < u \le \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. Assume that ϕ satisfies*

$$\int_r^\infty \frac{dt}{\phi(t)t} \le \frac{C}{\phi(r)} \tag{44}$$

for every $r > 0$. Then

$$\left\| \left(\sum_{j=1}^{\infty} M f_j{}^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\phi,\Phi}} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

To prove Thoerem 4.1, we need to collect some auxiliary estimates. The first one is standard but for the sake of convenience for readers we supply the detailed proof. For $c > 0$ and $Q \in \mathcal{Q}$, cQ denotes a cube concentric to Q with sidelength $c\,\ell(Q)$.

Lemma 4.2. *For any cube Q and a function f, we have*

$$M[\chi_{\mathbb{R}^n \setminus 3Q} f](x) \le \sum_{k=1}^{\infty} \frac{10^n}{|6 \cdot 2^k Q|} \int_{6 \cdot 2^k Q} |f(y)|\, dy$$

for all $x \in Q$.

Proof. By the dyadic decomposition, we have

$$M[\chi_{\mathbb{R}^n \setminus 3Q} f](x) = \sup_{Q' \in \mathcal{Q}_x(\mathbb{R}^n)} \frac{1}{|Q'|} \int_{(\cup_{k=1}^{\infty}((6 \cdot 2^k Q) \setminus (3 \cdot 2^k Q)) \cap Q'} |f(y)| dy.$$

In order that $Q' \cap \mathbb{R}^n \setminus 3Q \neq \emptyset$ and $x \in Q \cap Q'$, we have $\ell(Q') \ge \ell(Q)$. Hence,

$$M[\chi_{\mathbb{R}^n \setminus 3Q} f](x) = \sup_{Q' \in \mathcal{Q}_x(\mathbb{R}^n); \ell(Q') > \ell(Q)} \frac{1}{|Q'|} \int_{(\cup_{k=1}^{\infty} 6 \cdot 2^k Q \setminus 3 \cdot 2^k Q) \cap Q'} |f(y)| dy.$$

For any $Q' \in \mathcal{Q}_x(\mathbb{R}^n)$, choose Q^* be a cube such that

1. $\ell(Q^*) = \frac{5}{2}\ell(Q')$
2. Q^* and Q have the same center.

Thus, $Q' \subset Q^*$. Consequently,

$$\begin{aligned} M[\chi_{\mathbb{R}^n \setminus 3Q} f](x) &\le \sup_{Q^* \in \mathcal{Q}_x(\mathbb{R}^n); \ell(Q^*) > \ell(Q)} \frac{5^n}{2^n |Q^*|} \int_{(\cup_{k=1}^{\infty} 6 \cdot 2^k Q \setminus 3 \cdot 2^k Q) \cap Q^*} |f(y)| dy \\ &\le \sum_{k=1}^{\infty} \sup_{Q^* \in \mathcal{Q}_x(\mathbb{R}^n); \ell(Q^*) > \ell(Q)} \frac{5^n}{2^n |Q^*|} \int_{(6 \cdot 2^k Q \setminus 3 \cdot 2^k Q) \cap Q^*} |f(y)| dy \\ &\le \sum_{k=1}^{\infty} \frac{10^n}{|6 \cdot 2^k Q|} \int_{6 \cdot 2^k Q} |f(y)| dy. \end{aligned}$$

So we are done. □

The next lemma shows that Φ satisfies the "so-called" $\mathbb{Z}^{\gamma} \cap \mathbb{Z}_{\gamma'}$-condition.

Lemma 4.3. *There exist $q_1, q_2 \in (1, \infty)$ such that*

$$\int_0^{l_1} \frac{\Phi(t)}{t^{q_1+1}} dt \le C \frac{\Phi(l_1)}{l_1^{q_1}} \quad \text{and} \quad \int_{l_2}^{\infty} \frac{\Phi(t)}{t^{q_2+1}} dt \le C \frac{\Phi(l_2)}{l_2^{q_2}}$$

for all $l_1, l_2 \in (0, \infty)$.

Proof. Let C' be a constant from (2). Define

$$q_1 \equiv \begin{cases} \frac{1}{2} + \log_2 C', & C' \ge 2, \\ 1 + \frac{1}{2} \log_2 C', & 1 < C' < 2, \end{cases} \quad \text{and } z \equiv \begin{cases} \frac{1}{2}, & C' \ge 2, \\ 1 - \frac{1}{2} \log_2 C', & 1 < C' < 2. \end{cases}$$

For all $0 < t \le s < \infty$, we choose a number $k \in \mathbb{N} \cup \{0\}$ such that $2^{-k-1} s \le t \le 2^{-k} s$. By using the ∇_2 condition, we get

$$\Phi(s) \ge \Phi\left(2^k t\right) \ge (2C')^k \Phi(t) = 2^{(1+\log_2 C')k} \Phi(t) \ge 2^{-(1+\log_2 C')} \left(\frac{s}{t}\right)^{1+\log_2 C'} \Phi(t),$$

or equivalently,

$$\frac{\Phi(t)}{t^{q_1+1-z}} \le 2^{q_1+1-z} \frac{\Phi(s)}{s^{q_1+1-z}}.$$

Since $z \in (0, 1)$, we have

$$\int_0^{l_1} \frac{\Phi(t)}{t^{q_1+1}} dt = \int_0^{l_1} \frac{\Phi(t)}{t^{q_1+1-z}} \frac{dt}{t^z} \le C \frac{\Phi(l_1)}{l_1^{q_1+1-z}} \int_0^{l_1} \frac{1}{t^z} dt = C \frac{\Phi(l_1)}{l_1^{q_1}}.$$

Let $C > 2$ be a doubling constant of Φ. Take $q_2 \equiv 1 + \log_2 C > 1$. By using convexity and Δ_2 condition of Φ, we get

$$\frac{\Phi(s)}{s^{q_2-1}} \le \frac{\Phi(2^{k+1} t)}{2^{k(q_2-1)} t^{q_2-1}} \le 2^{-k(q_2-1)} C^{k+1} \frac{\Phi(t)}{t^{q_2-1}} = C 2^{-k(q_2-1)+k \log_2 C} \frac{\Phi(t)}{t^{q_2-1}} \le C \frac{\Phi(t)}{t^{q_2-1}}.$$

Thus,

$$\int_{l_2}^{\infty} \frac{\Phi(t)}{t^{q_2+1}} dt = \int_{l_2}^{\infty} \frac{\Phi(t)}{t^{q_2-1}} \frac{dt}{t^2} \le C \frac{\Phi(l_2)}{l_2^{q_2-1}} \int_{l_2}^{\infty} \frac{1}{t^2} dt = C \frac{\Phi(l_2)}{l_2^{q_2}}.$$

Thus, the proof is complete. □

We now prove Theorem 4.1.

Proof. Let Q be a fixed cube. We need to establish:

$$\phi(\ell(Q)) \left\| \left(\sum_{j=1}^{\infty} M f_j{}^u \right)^{\frac{1}{u}} \right\|_{\Phi;Q} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\phi,\Phi}} \tag{45}$$

with the constant C independent of Q.

To simplify, we normalize the right-hand side;

$$\left\| \left(\sum_{j=1}^{\infty} |f_j|^u \right)^{\frac{1}{u}} \right\|_{\mathcal{M}_{\phi,\Phi}} = 1.$$

We decompose the estimate (45) into (46) and (47), where

$$\phi(\ell(Q)) \left\| \left(\sum_{j=1}^{\infty} M[\chi_{\mathbb{R}^n \setminus 3Q} f_j]^u \right)^{\frac{1}{u}} \right\|_{\Phi;Q} \le C \tag{46}$$

and

$$\phi(\ell(Q)) \left\| \left(\sum_{j=1}^{\infty} M[\chi_{3Q} f_j]^u \right)^{\frac{1}{u}} \right\|_{\Phi;Q} \le C. \tag{47}$$

As for (46), we use Lemma 4.2. Denote by u' the conjugate exponent of u; $u' = \frac{u}{u-1}$. Also, we can find a positive constant a_j for $j \in \mathbb{N}$ such that

$$\left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \int_{6\cdot 2^k Q} |f_j(y)| dy \right)^u \right)^{\frac{1}{u}} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_j \int_{6\cdot 2^k Q} |f_j(y)| dy,$$

and that

$$\sum_{j=1}^{\infty} a_j{}^{u'} = 1. \tag{48}$$

By using Lemma 4.2, we obtain

$$\begin{aligned} &\phi(\ell(Q)) \left\| \left(\sum_{j=1}^{\infty} M[\chi_{\mathbb{R}^n \setminus 3Q} f_j]^u \right)^{\frac{1}{u}} \right\|_{\Phi;Q} \\ &\le C\phi(\ell(Q)) \left(\sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{1}{|6\cdot 2^k Q|} \int_{6\cdot 2^k Q} |f_j(y)| dy \right)^u \right)^{\frac{1}{u}} \\ &= C\phi(\ell(Q)) \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{|6\cdot 2^k Q|} \int_{6\cdot 2^k Q} a_j |f_j(y)| dy. \end{aligned}$$

Thus, by (44) and (48),

$$
\begin{aligned}
&\phi(\ell(Q))\left\|\left(\sum_{j=1}^{\infty} M[\chi_{\mathbb{R}^n\setminus 3Q}f_j]^u\right)^{\frac{1}{u}}\right\|_{\Phi;Q}\\
&\le C\sum_{k=1}^{\infty}\frac{\phi(\ell(Q))}{|6\cdot 2^kQ|}\int_{6\cdot 2^kQ}\left(\sum_{j=1}^{\infty}|f_j(y)|^u\right)^{\frac{1}{u}} dy\\
&\le C\sum_{k=1}^{\infty}\frac{\phi(\ell(Q))}{\phi(\ell(6\cdot 2^kQ))}\frac{\phi(\ell(6\cdot 2^kQ))}{|6\cdot 2^kQ|}\int_{6\cdot 2^kQ}\left(\sum_{j=1}^{\infty}|f_j(y)|^u\right)^{\frac{1}{u}} dy\\
&\le C\sum_{k=1}^{\infty}\frac{\phi(\ell(Q))}{\phi(\ell(6\cdot 2^kQ))}\le C\int_{12\ell(Q)}^{\infty}\frac{\phi(\ell(Q))}{\phi(t)}\frac{dt}{t}\le C.
\end{aligned}
$$

It remains to prove (47). To this end, we consider

$$
\mathrm{I}\equiv\frac{1}{|Q|}\int_Q\Phi\left(\phi(\ell(Q))\left(\sum_{j=1}^{\infty}M[\chi_{3Q}f_j](x)^u\right)^{\frac{1}{u}}\right) dx.
$$

Also, we set

$$
f(x)\equiv\phi(\ell(Q))\left(\sum_{j=1}^{\infty}|f_j(x)|^u\right)^{\frac{1}{u}}\quad (x\in\mathbb{R}^n),\quad E_t\equiv\{x\in\mathbb{R}^n\,:\,f(x)>t\}\quad (t>0).
$$

and we abbreviate;

$$
\begin{aligned}
F_Q^1(x)&\equiv\phi(\ell(Q))\left(\sum_{j=1}^{\infty}M[\chi_{3Q\cap E_t}f_j](x)^u\right)^{\frac{1}{u}},\\
F_Q^2(x)&\equiv\phi(\ell(Q))\left(\sum_{j=1}^{\infty}M[\chi_{3Q\setminus E_t}f_j](x)^u\right)^{\frac{1}{u}}
\end{aligned}
$$

and

$$
F_Q(x)\equiv\phi(\ell(Q))\left(\sum_{j=1}^{\infty}M[\chi_{3Q}f_j](x)^u\right)^{\frac{1}{u}}.
$$

Note that Φ is a convex function and hence Φ is absolutely continuous. Denote by $\Phi'(t)$ the (right) derivative;

$$
\Phi'(t)=\lim_{s\downarrow t}\frac{\Phi(t)-\Phi(s)}{t-s}.
$$

By using distribution formula, we have

$$
\mathrm{I}=\frac{1}{|Q|}\int_0^{\infty}\Phi'(t)\,|\{x\in Q\,:\,F_Q(x)>t\}|\,dt.
$$

Since $\chi_{3Q} f_j = \chi_{3Q\cap E_t} f_j + \chi_{3Q\setminus E_t} f_j$, we have

$$M[\chi_{3Q} f_j](x) \le M[\chi_{3Q\cap E_t} f_j](x) + M[\chi_{3Q\setminus E_t} f_j](x).$$

Thus,

$$F_Q(x) \le \phi(\ell(Q)) \left(\sum_{j=1}^{\infty} \left(M[\chi_{3Q\cap E_t} f_j](x) + M[\chi_{3Q\setminus E_t} f_j](x) \right)^u \right)^{\frac{1}{u}} \le F_Q^1(x) + F_Q^2(x).$$

By the Chebychev inequality and (43), we have

$$\begin{aligned}\frac{1}{|Q|} \int_0^\infty \Phi'(t) \left| \left\{ x \in Q \,:\, F_Q^1(x) > \frac{t}{2} \right\} \right| dt &\le C \frac{1}{|Q|} \int_0^\infty \frac{\Phi'(t)}{t^{q_1}} (\|F_Q^1\|_{L^{q_1}})^{q_1}\, dt \\ &\le C \frac{1}{|Q|} \int_0^\infty \frac{\Phi'(t)}{t^{q_1}} (\|\chi_{3Q\cap E_t} f\|_{L^{q_1}})^{q_1}\, dt.\end{aligned}$$

By changing the order of integration, we obtain

$$\frac{1}{|Q|} \int_0^\infty \Phi'(t) \left| \left\{ x \in Q \,:\, F_Q^1(x) > \frac{t}{2} \right\} \right| dt \le C \frac{1}{|Q|} \left(\int_{3Q} \int_0^{f(x)} \frac{\Phi'(t)}{t^{q_1}}\, dt \right) f(x)^{q_1}\, dx.$$

Observe also that

$$\frac{\Phi(t)}{t} \le \Phi'(t) \le \frac{\Phi(2t)}{t} \quad (t > 0).$$

By using Lemma 4.3, we obtain

$$\int_0^{f(x)} \frac{\Phi'(t)}{t^{q_1}}\, dt \le \int_0^{f(x)} \frac{\Phi(2t)}{t^{q_1+1}} dt \le C \int_0^{f(x)} \frac{\Phi(t)}{t^{q_1+1}} dt \le C \frac{\Phi(f(x))}{f(x)^{q_1}}.$$

As a result,

$$\frac{1}{|Q|} \int_0^\infty \Phi'(t) \left| \left\{ x \in Q \,:\, F_Q^1(x) > \frac{t}{2} \right\} \right| dt \le C \frac{1}{|Q|} \int_{3Q} \Phi(f(x))\, dx \le C \qquad (49)$$

since $\|f\|_{\Phi;Q} \le \phi(\ell(Q))^{-1}$.

Likewise, by the Chebychev inequality and (43), we have

$$\begin{aligned}\frac{1}{|Q|} \int_0^\infty \Phi'(t) \left| \left\{ x \in Q \,:\, F_Q^2(x) > \frac{t}{2} \right\} \right| dt &\le C \frac{1}{|Q|} \int_0^\infty \frac{\Phi'(t)}{t^{q_2}} (\|F_Q^2\|_{L^{q_2}})^{q_2}\, dt \\ &\le C \frac{1}{|Q|} \int_0^\infty \frac{\Phi'(t)}{t^{q_2}} (\|\chi_{3Q\setminus E_t} f\|_{L^{q_2}})^{q_2}\, dt.\end{aligned}$$

By changing the order of integration, we obtain

$$\frac{1}{|Q|} \int_0^\infty \Phi'(t) \left| \left\{ x \in Q \,:\, F_Q^2(x) > \frac{t}{2} \right\} \right| dt \le C \frac{1}{|Q|} \left(\int_{3Q} \int_{f(x)}^\infty \frac{\Phi'(t)}{t^{q_2}}\, dt \right) f(x)^{q_2}\, dx.$$

By using Lemma 4.3 once again, we obtain

$$\int_{f(x)}^{\infty} \frac{\Phi'(t)}{t^{q_2}}\, dt \le \int_{f(x)}^{\infty} \frac{\Phi(2t)}{t^{q_2+1}}\, dt \le C\int_{f(x)}^{\infty} \frac{\Phi(t)}{t^{q_2+1}}\, dt \le C\frac{\Phi(f(x))}{f(x)^{q_2}}.$$

As a result,

$$\frac{1}{|Q|}\int_0^{\infty} \Phi'(t) \left|\left\{x \in Q \,:\, F_Q^2(x) > \frac{t}{2}\right\}\right| dt \le C\frac{1}{|Q|}\int_{3Q} \Phi(f(x))\, dx \le C \tag{50}$$

since $\|f\|_{\Phi;Q} \le \phi(\ell(Q))^{-1}$.

$$\begin{aligned} \mathrm{I} &\le \frac{1}{|Q|}\int_0^{\infty} \Phi'(t) \left|\left\{x \in Q \,:\, F_Q^1(x) > \frac{t}{2}\right\}\right| dt \\ &\quad + \frac{1}{|Q|}\int_0^{\infty} \Phi'(t) \left|\left\{x \in Q \,:\, F_Q^2(x) > \frac{t}{2}\right\}\right| dt \\ &\le C. \end{aligned}$$

If we combine (49) and (50), then we obtain (47). So, we are done. □

As we annouced in Section 1, the next lemma is necessary when we consider the decomposition result.

Lemma 4.4. *Let $\kappa > 1$. Let $\{E_k\}_{k\in K}$ be a collection of measurable sets in $\mathbb{R}^n$. Then*

$$\left\|\sum_{k\in K}(M\chi_{E_k})^{\kappa}\right\|_{L^1([-1,1]^n)} \le C\left[\sum_{l=1}^{\infty}\left\{\sum_{k\in K}\frac{1}{2^{ln}}\int_{[-2^l,2^l]^n}\chi_{E_k}(y)\, dy\right\}^{\frac{1}{\kappa}}\right]^{\kappa}.$$

Proof. We decompose the estimate into two parts:

$$\left\|\sum_{k\in K}(M\chi_{E_k\cap[-3,3]^n})^{\kappa}\right\|_{L^1([-1,1]^n)} \le C\left[\sum_{l=1}^{\infty}\left\{\sum_{k\in K}\frac{1}{2^{ln}}\int_{[-2^l,2^l]^n}\chi_{E_k}(y)\, dy\right\}^{\frac{1}{\kappa}}\right]^{\kappa}, \tag{51}$$

$$\left\|\sum_{k\in K}(M\chi_{E_k\setminus[-3,3]^n})^{\kappa}\right\|_{L^1([-1,1]^n)} \le C\left[\sum_{l=1}^{\infty}\left\{\sum_{k\in K}\frac{1}{2^{ln}}\int_{[-2^l,2^l]^n}\chi_{E_k}(y)\, dy\right\}^{\frac{1}{\kappa}}\right]^{\kappa}. \tag{52}$$

For the proof of (51), we calculate as follows: first we expand the integration domain;

$$\left\|\sum_{k\in K}(M\chi_{E_k\cap[-3,3]^n})^{\kappa}\right\|_{L^1([-1,1]^n)} \le \left\|\sum_{k\in K}(M\chi_{E_k\cap[-3,3]^n})^{\kappa}\right\|_{L^1}$$

We write out the right-hand side out in full and use the $L^\kappa(\mathbb{R}^n)$-boundedness of M to obtain;

$$\begin{aligned}
\left\| \sum_{k\in K} (M\chi_{E_k\cap[-3,3]^n})^\kappa \right\|_{L^1([-1,1]^n)} &\le \int_{\mathbb{R}^n} \sum_{k\in K} M\chi_{E_k\cap[-3,3]^n}(x)^\kappa\,dx \\
&= \sum_{k\in K}\int_{\mathbb{R}^n} M\chi_{E_k\cap[-3,3]^n}(x)^\kappa\,dx \\
&\le C\sum_{k\in K}\int_{\mathbb{R}^n} \chi_{E_k\cap[-3,3]^n}(x)\,dx \\
&= C\int_{\mathbb{R}^n}\sum_{k\in K} \chi_{E_k\cap[-3,3]^n}(x)\,dx.
\end{aligned}$$

(51) is thus proved.

It remains to prove (52). By (4.2), we have

$$M\chi_{E_k\setminus[-3,3]^n}(x) \le C\sum_{l=1}^\infty \frac{1}{(6\cdot 2^l)^n}\int_{[-6\cdot 2^l,6\cdot 2^l]^n} \chi_{E_k}(y)\,dy$$

for all $x\in[-1,1]^n$. Thus, we obtain

$$\begin{aligned}
\sum_{k\in K} M\chi_{E_k\setminus[-3,3]^n}(x)^\kappa &\le C\sum_{k\in K}\left(\sum_{l=1}^\infty \frac{1}{(6\cdot 2^l)^n}\int_{[-6\cdot 2^l,6\cdot 2^l]^n} \chi_{E_k}(y)\,dy\right)^\kappa \\
&= C\left[\left\{\sum_{k\in K}\left(\sum_{l=1}^\infty \frac{1}{(6\cdot 2^l)^n}\int_{[-6\cdot 2^l,6\cdot 2^l]^n} \chi_{E_k}(y)\,dy\right)^\kappa\right\}^{\frac{1}{\kappa}}\right]^\kappa
\end{aligned}$$

for all $x\in[-1,1]^n$. By the triangle inequality of $\ell^\kappa(\mathbb{N})$,

$$\begin{aligned}
\sum_{k\in K} M\chi_{E_k\setminus[-3,3]^n}(x)^\kappa &\le C\left[\sum_{l=1}^\infty\left\{\sum_{k\in K}\left(\frac{1}{(6\cdot 2^l)^n}\int_{[-6\cdot 2^l,6\cdot 2^l]^n} \chi_{E_k}(y)\,dy\right)^\kappa\right\}^{\frac{1}{\kappa}}\right]^\kappa \\
&\le C\left[\sum_{l=1}^\infty\left\{\sum_{k\in K}\frac{1}{(6\cdot 2^l)^n}\int_{[-6\cdot 2^l,6\cdot 2^l]^n} \chi_{E_k}(y)\,dy\right\}^{\frac{1}{\kappa}}\right]^\kappa \\
&\le C\left[\sum_{l=1}^\infty\left\{\sum_{k\in K}\frac{1}{2^{ln}}\int_{[-2^l,2^l]^n} \chi_{E_k}(y)\,dy\right\}^{\frac{1}{\kappa}}\right]^\kappa
\end{aligned}$$

for all $x\in[-1,1]^n$. (52) is thus proved. □

Remark 4.5. See [43, 50] for the case when $\Phi(t)=t^q$ and $\varphi(t)=t^{n/p}$ for $t\ge 0$.

5 Proof of main theorems

5.1 Proof of Theorem 1.3

We prove Theorem 1.3 now.

Proof. By a dyadic cube, we mean a set of the form $2^{-j}m + [0, 2^{-j}]^n$ for some $m \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$. By decomposing Q_j into cubes of equivalent length, we may suppose each Q_j is dyadic.

To prove (10), we resort to the duality. For the time being, we assume that there exists $N \in \mathbb{N}$ such that $\lambda_j = 0$ whenever $j \geq N$. Let us assume in addition that each a_j is non-negative without loss of generality. Fix a positive (ϕ, Ψ)-block $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ with the associated cube Q. Again by decomposing Q, we can assume that Q is a dyadic cube as well.

Assume first that each Q_j contains Q as a proper subset. If we group j's such that Q_j are identical, we can assume that Q_j is a dyadic cube containing Q and satisfying $|Q_j| = 2^{jn}|Q|$ for each $j \in \mathbb{N}$. Then we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq \sum_{j=1}^{\infty} \lambda_j \int_Q |a_j(x)g(x)|\, dx \leq 2\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{\Phi;Q}\|g\|_{\Psi;Q}|Q|.$$

By the size condition of a_j and g, we obtain

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq 2\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{\Phi;Q}\|g\|_{\Psi;Q}|Q| \leq 2\sum_{j=1}^{\infty} \lambda_j \frac{\eta(\ell(Q_j))\phi(\ell(Q))}{\eta(\ell(Q))}.$$

Note that

$$\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}} \geq \left\|\lambda_{j_0}\chi_{Q_{j_0}}\right\|_{\mathcal{M}_{\phi,\Phi}} \geq \phi(\ell(Q_{j_0}))\lambda_{j_0}$$

for each j_0. Consequently, it follows from (6) that

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq 2\sum_{j=1}^{\infty} \frac{\eta(\ell(Q_j))\phi(\ell(Q))}{\phi(\ell(Q_j))\eta(\ell(Q))} \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}} \leq C\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}.$$

So, (10) is proved.

Conversely assume that Q contains each Q_j. Then we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq \sum_{j=1}^{\infty} \lambda_j \int_{Q_j} |a_j(x)g(x)|\, dx \leq 2\sum_{j=1}^{\infty} \lambda_j \|a_j\|_{\Theta;Q_j}\|g\|_{\Delta;Q_j}|Q_j|.$$

By the condition of a_j, we obtain

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le 2\sum_{j=1}^{\infty} \lambda_j \|g\|_{\Delta;Q_j} |Q_j|.$$

Thus, in terms of the maximal operator M^{Δ}, we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\,dx &\le 2\sum_{j=1}^{\infty} \lambda_j |Q_j| \times \inf_{y\in Q_j} M^{\Delta}g(y)\\ &\le 2\int_Q \left(\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}(y)\right) M^{\Delta}g(y)\,dy\\ &\le 2|Q| \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\Phi;Q} \left\|M^{\Delta}g\right\|_{\Phi;Q}\end{aligned}$$

By using (7), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} |f(x)g(x)|\,dx &\le C|Q| \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\Phi;Q} \|g\|_{\Phi;Q}\\ &\le C|Q| \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\Phi;Q} \phi(l(Q))|Q|^{-1}\\ &\le C \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}.\end{aligned}$$

Let $g \in \mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$. Then we have an expression; $g = \sum_{j=1}^{\infty} \lambda_j g_j$ where each g_j is a (ϕ,Ψ)-block and $\sum\limits_{j=1}^{\infty} |\lambda_j| < \infty$. Thus,

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \sum_{j=1}^{\infty} |\lambda_j| \int_{\mathbb{R}^n} |f(x)g_j(x)|dx \le C\|g\|_{\mathcal{B}_{\phi,\Psi}} \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}.$$

This implies

$$\|L_f\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}} \left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Since $\|f\|_{\mathcal{M}_{\phi,\Phi}} \le C\|L_f\|_{\mathcal{B}_{\phi,\Psi}\to\mathbb{C}}$, we have the desired result. □

5.2 Proof of Theorem 1.1

We write $\mathcal{P}_d(\mathbb{R}^n)$ for the set of all polynomials of degree less than d.

Let $t > 0$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. Then define

$$e^{t\Delta} f(x) \equiv \int_{\mathbb{R}^n} \frac{1}{\sqrt{(4\pi t)^n}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y) dy \quad (x \in \mathbb{R}^n).$$

We say that $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$ and $\sup_{t>0} |e^{t\Delta} f| \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. We define

$$\|f\|_{H\mathcal{M}_{\phi,\Phi}} \equiv \left\| \sup_{t>0} |e^{t\Delta} f| \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

The next proposition characterizes the space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Proposition 5.1. *Let $\Phi \in \Delta_2 \cap \nabla_2$ and suppose that ϕ satisfies the inequality* (44).

1. *$\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \hookrightarrow H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ in the sense of continous embedding.*
2. *If $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then f is represented by a function $g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*

If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then

$$C^{-1} \|f\|_{\mathcal{M}_{\phi,\Phi}} \le \|f\|_{H\mathcal{M}_{\phi,\Phi}} \le C \|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{53}$$

Proof.

1. Let $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. As we have seen in Corollary 2.6, $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$. Also, we have $\sup_{t>0} |e^{t\Delta} f| \le CMf$ by virtue of Proposition 2.10. Again by the boundedness of the Hardy-Littlewood maximal operator, we see that $f \in H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ and that the right inequality in (53) follows.
2. Recall that the dual of $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ is isomorphic to $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ as we have established in Theorem 3.11. Let $L : f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n) \mapsto L_f \in (\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$ be an isomorphism in Theorem 3.11. So, we are using a general result due to Banach and Alaoglu: If X is a Banach space, then the unit ball of X^* is weak-* (sequentially) compact. By assumption $\{e^{t\Delta} f\}_{t>0}$ forms a bounded set in $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Consider $\{t_j\}_{j=1}^\infty$ which decreases to 0. Then $\{L_{e^{t_j\Delta} f}\}_{j=1}^\infty$ forms a bounded set in $(\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$. Thus, by the Banach Alaoglu theorem, there exists a positive sequence $\{t_j\}_{j=1}^\infty$ which decreases to 0 such that $L_{e^{t_j\Delta} f}$ is convergent to $G = L_g \in (\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n))^*$ for some $g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ in the weak-* sense. Meanwhile, $e^{t_j\Delta} f$ is known to converge to $f \in \mathcal{S}'(\mathbb{R}^n)$. Thus, we conclude $\mathcal{S}'(\mathbb{R}^n) \ni f = g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ from Proposition 3.12.

   The left inequality in (53) follows since $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ is isomorphic to the dual of $\mathcal{B}_{\phi,\Psi}(\mathbb{R}^n)$ (note that we can not replace "isomorphic" to "isometry" because of the general structure induced by Φ). Thus,

$$\begin{aligned}\|f\|_{\mathcal{M}_{\phi,\Phi}} &\le C\|L_f\|_{(\mathcal{B}_{\phi,\Psi})^*} \le C\liminf_{j\to\infty}\|L_{e^{t_j\Delta}f}\|_{(\mathcal{B}_{\phi,\Psi})^*}\\ &\le C\liminf_{j\to\infty}\|e^{t_j\Delta}f\|_{\mathcal{M}_{\phi,\Phi}}\\ &\le C\left\|\sup_{t>0}|e^{t\Delta}f|\right\|_{\mathcal{M}_{\phi,\Phi}}\\ &= C\,\|f\|_{H\mathcal{M}_{\phi,\Phi}}.\end{aligned}$$

So we are done. □

If we reexamine the proof, then we obtain the following precise estimate.

Remark 5.2. If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then the Lebesgue differentiation theorem shows that

$$|f(x)| \le \sup_{t>0}|e^{t\Delta}f(x)| \quad (x \in \mathbb{R}^n).$$

Thus, (53) is a little improved;

$$\|f\|_{\mathcal{M}_{\phi,\Phi}} \le \|f\|_{H\mathcal{M}_{\phi,\Phi}} \le C\|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{54}$$

We now introduce the grand maximal operator by way of Definition 2.4.

Definition 5.3.

1. Let $N \in \mathbb{N}$. Then define $\mathcal{F}_N \equiv \{\varphi \in \mathcal{S}(\mathbb{R}^n) : \rho_N(\varphi) \le 1\}$.
2. The grand maximal operator $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) \equiv \sup\{|t^{-n}\varphi(t^{-1}\cdot) * f(x)| \,:\, t>0, \varphi \in \mathcal{F}_N\} \quad (x \in \mathbb{R}^n)$$

for $f \in \mathcal{S}'(\mathbb{R}^n)$.

The next proposition characterizes the space $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ analogously to Proposition 5.1.

Proposition 5.4. *Let $\Phi \in \Delta_2 \cap \nabla_2$ and ϕ satisfies the inequality* (44). *Let N in Definition* 5.3 *be sufficiently large.*

1. *If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then $\mathcal{M}f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*
2. *If $\mathcal{M}f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then f is represented by a function $g \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.*

If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then

$$C^{-1}\|f\|_{\mathcal{M}_{\phi,\Phi}} \le \|\mathcal{M}f\|_{\mathcal{M}_{\phi,\Phi}} \le C\|f\|_{\mathcal{M}_{\phi,\Phi}}. \tag{55}$$

Proof. If $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, again by virtue of Proposition 2.10 we have

$$\mathcal{M}f(x) \leq CMf(x).$$

So, if we go through the same argument as the former half of Proposition 5.1, then we obtain $\mathcal{M}f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

If $\mathcal{M}f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, then a pointwise estimate

$$\sup_{t>0} |e^{t\Delta} f(x)| \leq C\mathcal{M}f(x) \quad (x \in \mathbb{R}^n)$$

allows us to invoke the latter half of Proposition 5.1; we first obtain

$$\sup_{t>0} |e^{t\Delta} f| \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$$

and then the latter half of Proposition 5.1 yields $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. □

We invoke the following lemma. By $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ we denote the set of all compactly supported smooth functions in $\mathbb{R}^n$. We refer to [48] for the proof.

Lemma 5.5. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$, $d \in \{0, 1, 2, \cdots\}$ and $j \in \mathbb{Z}$. Then there exist collections of cubes $\{Q^*_{j,k}\}_{k\in K_j}$ and functions $\{\eta_{j,k}\}_{k\in K_j} \subset C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, which are all indexed by a set K_j for every j, and a decomposition*

$$f = g_j + b_j, \quad b_j = \sum_{k\in K_j} b_{j,k},$$

such that:

(0) $g_j, b_j, b_{j,k} \in \mathcal{S}'(\mathbb{R}^n)$.

(i) *Define $\mathcal{O}_j \equiv \{y \in \mathbb{R}^n : \mathcal{M}f(y) > 2^j\}$ and consider its Whitney decomposition. Then the cubes $\{200Q^*_{j,k}\}_{k\in K_j}$ have the bounded intersection property, and*

$$\mathcal{O}_j = \bigcup_{k\in K_j} Q^*_{j,k} = \bigcup_{k\in K_j} 200Q^*_{j,k}. \tag{56}$$

(ii) *Consider the partition of unity $\{\eta_{j,k}\}_{k\in K_j}$ with respect to $\{Q^*_{j,k}\}_{k\in K_j}$. Then each function $\eta_{j,k}$ is supported in $Q^*_{j,k}$ and*

$$\sum_{k\in K_j} \eta_{j,k} = \chi_{\{y\in\mathbb{R}^n : \mathcal{M}f(y)>2^j\}}, \quad 0 \leq \eta_{j,k} \leq 1.$$

(iii) *The distribution g_j satisfies the inequality:*

$$\mathcal{M}g_j(x) \leq C\left(\mathcal{M}f(x)\chi_{\mathcal{O}_j{}^c}(x) + 2^j \sum_{k\in K_j} \frac{{\ell_{j,k}}^{n+d+1}}{(\ell_{j,k} + |x - x_{j,k}|)^{n+d+1}}\right) \tag{57}$$

for all $x \in \mathbb{R}^n$.

(iii)' *If f is a locally integrable function, then we can arrange even that g_j be a bounded function with $|g_j(x)| \le 2^{-j}$ for a.e. $x \in \mathbb{R}^n$.*

(iv) *Each distribution $b_{j,k}$ is given by $b_{j,k} = (f - c_{j,k})\eta_{j,k}$ with a certain polynomial $c_{j,k} \in \mathcal{P}_d(\mathbb{R}^n)$ satisfying*

$$\int_{\mathbb{R}^n} b_{j,k}(x)q(x)\,dx = 0 \text{ for all } q \in \mathcal{P}_d(\mathbb{R}^n),$$

and

$$\mathcal{M}b_{j,k}(x) \le C\left(\mathcal{M}f(x)\chi_{Q^*_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}}\chi_{\mathbb{R}^n\setminus Q^*_{j,k}}(x)\right) \tag{58}$$

for all $x \in \mathbb{R}^n$.

In the above, $x_{j,k}$ and $\ell_{j,k}$ denote the center and the side-length of $Q^*_{j,k}$, respectively, and the constants are dependent only on n.

We modify Lemma 4.4 to our current setting:

Lemma 5.6. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Keep to the same notation as Lemma 5.5. Then we have*

$$|\langle b_j, \varphi\rangle| \le C_\varphi \left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\left\|\mathcal{M}f \cdot \chi_{\mathcal{O}_j}\right\|_{L^1([-2^l,2^l]^n)}\right)^{n/(n+d+1)}\right\}^{(n+d+1)/n}, \tag{59}$$

where the constant C_φ in (59) depends on φ but not on j.

Proof. For some large constant $M = M_\varphi$, we have $\psi_x \equiv M^{-1}\varphi(x - \cdot) \in \mathcal{F}_N$ for all $x \in [-1,1]^n$, so that

$$|\langle b_j, \varphi\rangle| = |b_j * \psi_x(z)|_{z=x} \le M \inf_{x\in[-1,1]^n} \mathcal{M}b_j(x).$$

Thus, we have

$$|\langle b_j, \varphi\rangle| \le C \inf_{x\in[-1,1]^n} \mathcal{M}b_j(x) \le C \inf_{x\in[-1,1]^n} \sum_{k\in K_j} \mathcal{M}b_{j,k}(x).$$

Observe also that

$$CM\chi_Q(x) \ge \frac{|Q|}{|Q| + |x - x_Q|^n} \ge \frac{|Q|}{|x - x_Q|^n}\chi_{\mathbb{R}^n\setminus Q}(x) \quad (x \in \mathbb{R}^n),$$

if Q is a cube centered at x_Q. It follows from (58) that

$$\begin{aligned}\sum_{k\in K_j} \mathcal{M}b_{j,k}(x) &\le C\sum_{k\in K_j}\left(\mathcal{M}f(x)\chi_{Q^*_{j,k}}(x) + 2^j \cdot \frac{{\ell_{j,k}}^{n+d+1}}{|x - x_{j,k}|^{n+d+1}}\chi_{\mathbb{R}^n\setminus Q^*_{j,k}}(x)\right)\\ &\le C\left(\mathcal{M}f(x)\chi_{\mathcal{O}_j}(x) + 2^j\sum_{k\in K_j} M\chi_{Q^*_{j,k}}(x)^{\frac{n+d+1}{n}}\right).\end{aligned}$$

We apply Lemma 5.7 with $\kappa = (n+d+1)/n$. From this pointwise estimate, we deduce

$$\begin{aligned}
\|\mathcal{M}b_j\|_{L^1([-1,1]^n)} &\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j} + 2^j\sum_{k\in K_j}(M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^1([-1,1]^n)} \\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1([-1,1]^n)} + C\left\|2^j\sum_{k\in K_j}(M\chi_{Q^*_{j,k}})^{\frac{n+d+1}{n}}\right\|_{L^1([-1,1]^n)} \\
&\le C\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1([-1,1]^n)} \\
&\quad + C\left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1([-2^l,2^l]^n)}\right)^{n/(n+d+1)}\right\}^{(n+d+1)/n}.
\end{aligned}$$

So, we are done. □

The key observation in this paper is the following; despite the failure of the density result for $\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$, we can approximate $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ in some sense.

Lemma 5.7. *Let $\varphi \in \mathcal{G}_1$. Assume that there exists a constant C_0 such that*

$$\int_r^\infty \frac{ds}{\varphi(s)s} \le \frac{C_0}{\varphi(r)} \quad (r>0).$$

Let $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$. Under the notation of Lemma 5.5, in the topology of $\mathcal{S}'(\mathbb{R}^n)$, we have $g_j \to 0$ as $j \to -\infty$ and $b_j \to 0$ as $j \to \infty$. In particular,

$$f = \sum_{j=-\infty}^{\infty}(g_{j+1}-g_j)$$

in the topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof. We begin with a preparatory observation; according to [24], for all $0<\varepsilon<1/C_0$,

$$\int_r^\infty \frac{ds}{\varphi(s)s^{1+\varepsilon}} \le \frac{C_0}{(1-C_0\varepsilon)\varphi(r)r^\varepsilon} \quad (r>0).$$

By the doubling condition of φ, we obtain $\varphi(r)r^\varepsilon \le C\varphi(R)R^\varepsilon$ for $0<r\le R$. Let us show that $b_j \to 0$ as $j \to \infty$ in $\mathcal{S}'(\mathbb{R}^n)$. Once this is proved, then we have $f = \lim_{j\to\infty} g_j$ in $\mathcal{S}'(\mathbb{R}^n)$. Let us choose a test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Then we have

$$|\langle b_j,\varphi\rangle| \le C\inf_{x\in[-1,1]^n}\mathcal{M}b_j(x) \le C\|\mathcal{M}b_j\|_{L^1([-1,1]^n)},$$

where C does depend on φ. Observe that

$$\begin{aligned}
&\left\{\sum_{l=0}^{\infty}\left(\frac{1}{2^{ln}}\left\|\mathcal{M}f\cdot\chi_{\mathcal{O}_j}\right\|_{L^1([-2^l,2^l]^n)}\right)^{n/(n+d+1)}\right\}^{(n+d+1)/n} \\
&\le C\left(\sum_{l=0}^{\infty}\frac{1}{\varphi(2^l)^{n/(n+d+1)}}\right)^{(n+d+1)/n}\|f\|_{\mathcal{M}_{\phi,\Phi}} \le C\|\mathcal{M}f\|_{\mathcal{M}_{\phi,\Phi}}.
\end{aligned}$$

Hence it follows from (59) that $\langle b_j, \varphi \rangle \to 0$ as $j \to \infty$. Meanwhile, $g_j \to 0$ in $L^\infty(\mathbb{R}^n)$ as $j \to -\infty$ by Lemma 5.5(iii)'. Consequently, it follows that

$$f = \lim_{j \to \infty} g_j = \lim_{j,k \to \infty} \sum_{l=-k}^{j} (g_{l+1} - g_l)$$

in $\mathcal{S}'(\mathbb{R}^n)$. □

Now let us conclude the proof of Theorem 1.1.

Proof. For each $j \in \mathbb{Z}$, consider the level set

$$\mathcal{O}_j \equiv \{x \in \mathbb{R}^n : \mathcal{M}f(x) > 2^j\}. \tag{60}$$

Then it follows immediately from the definition that

$$\mathcal{O}_{j+1} \subset \mathcal{O}_j. \tag{61}$$

If we invoke Lemma 5.5, then f can be decomposed;

$$f = g_j + b_j, \quad b_j = \sum_k b_{j,k}, \quad b_{j,k} = (f - c_{j,k})\eta_{j,k}$$

where each $b_{j,k}$ is supported in a cube $Q^*_{j,k}$ as is described in Lemma 5.5.

We know that

$$f = \sum_{j=-\infty}^{\infty} (g_{j+1} - g_j), \tag{62}$$

with the sum converging in the sense of distributions from Lemma 5.7. Here, going through the same argument as the one in [48, pp.108–109], we have an expression;

$$f = \sum_{j,k} A_{j,k}, \quad g_{j+1} - g_j = \sum_k A_{j,k} \quad (j \in \mathbb{Z}) \tag{63}$$

in the sense of distributions, where each $A_{j,k}$, supported in $Q^*_{j,k}$, satisfies the pointwise estimate $|A_{j,k}(x)| \leq C_0 2^j$ for some universal constant C_0 and the moment condition $\displaystyle\int_{\mathbb{R}^n} A_{j,k}(x) q(x)\, dx = 0$ for every $q \in \mathcal{P}_d(\mathbb{R}^n)$. With these observations in mind, let us set

$$a_{j,k} \equiv \frac{A_{j,k}}{C_0 2^j}, \quad \kappa_{j,k} \equiv C_0 2^j.$$

Then we automatically obtain that each $a_{j,k}$ satisfies

$$|a_{j,k}| \leq \chi_{Q^*_{j,k}}, \int_{\mathbb{R}^n} x^\beta a_{j,k}(x)\, dx = 0 \quad (|\beta| \leq L)$$

and that $f = \sum_{j,k} \kappa_{j,k} a_{j,k}$ in the topology of $H\mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$ thanks to Proposition 1.4, once we prove the estimate of coefficients. Rearrange $\{a_{j,k}\}$ and so on to obtain $\{a_j\}$ and so on. Thus, we are given a collection;

$$\{(\kappa_{j,k}; Q^*_{j,k})\}_{j,k} = \{(\lambda_j; Q_j)\}_j \in (0,\infty) \times \mathcal{Q}. \tag{64}$$

To establish (4) we need to estimate

$$\mu \equiv \left\| \left(\sum_{j=-\infty}^{\infty} (\lambda_j \chi_{Q_j})^v \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

So, by (64), we have

$$\mu = \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k \in K_j} (\kappa_{j,k} \chi_{Q^*_{j,k}})^v \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

If we insert the definition of κ_j, then we have

$$\mu = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} \sum_{k \in K_j} (2^j \chi_{Q^*_{j,k}})^v \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}} = C_0 \left\| \left(\sum_{j=-\infty}^{\infty} 2^{jv} \sum_{k \in K_j} \chi_{Q^*_{j,k}} \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Observe that (56) together with the bounded overlapping property yields

$$\chi_{\mathcal{O}_j}(x) \le \sum_{k \in K_j} \chi_{Q^*_{j,k}}(x) \le \chi_{200 Q^*_{j,k}}(x) \le C \chi_{\mathcal{O}_j}(x) \quad (x \in \mathbb{R}^n).$$

Thus, we have

$$\mu \le C \left\| \left(\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}\right)^v \right)^{1/v} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

Recall that $\mathcal{O}_j \supset \mathcal{O}_{j+1}$ for each $j \in \mathbb{Z}$. Consequently we have

$$\sum_{j=-\infty}^{\infty} \left(2^j \chi_{\mathcal{O}_j}(x)\right)^v \sim \left(\sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j}(x) \right)^v \sim \left(\sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}}(x) \right)^v.$$

Thus, we obtain

$$\mu \le C \left\| \sum_{j=-\infty}^{\infty} 2^j \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} \right\|_{\mathcal{M}_{\phi,\Phi}}.$$

It follows from the definition of $\mathcal{O}_j$ that we have $2^j < \mathcal{M}f(x)$ for all $x \in \mathcal{O}_j$. Hence, we have

$$\mu \le C \left\| \sum_{j=-\infty}^{\infty} \chi_{\mathcal{O}_j \setminus \mathcal{O}_{j+1}} \mathcal{M}f \right\|_{\mathcal{M}_{\phi,\Phi}} \le C \|\mathcal{M}f\|_{\mathcal{M}_{\phi,\Phi}}.$$

This is the desired result. □

5.3 Decompositions of Morrey spaces

According to the best knowledge of the authors, it seems that there are three decompositions for Morrey spaces. In 2005, Kruglyak and Kuznetsov considered the Calderón-Zygmund decomposition [20]. In 2007, the "so called" smooth decomposition is obtained [44]. The key idea is to develop the one obtained in [1, 13, 50, 51]. This decomposition is investigated very intensively in [14, 21, 22, 31, 32, 34, 35]. Later, in [45], by using this atomic decomposition, the above scale turned out to be the one defined by Yang and Yuan [53, 54]. See [55, 56, 57, 58] for more recent advance for this new scale. See [12, 35] for the extension of this scale on domains. In particular, Haroske and Skrzypczak considered the embedding properties in [11, 12]. See [15] for the boundedness of operators. We refer to [59] for an exhaustive account of these function spaces as well as the results on these decomposition results.

6 Olsen inequalities

As an application of the decomposition of generalized Orlicz-Morrey spaces, we shall prove the Olsen inequalities on generalized Orlicz-Morrey spaces. Here, we recall a simple case with a detailed proof.

Theorem 6.1. *Let $0 < \alpha < n, 1 < p \le p_0 < \infty, 1 < q \le q_0 < \infty$, and $1 < r \le r_0 < \infty$. If p, p_0, q, q_0, r and r_0 satisfy*

$$r < q, \tag{65}$$

$$\frac{1}{p_0} > \frac{\alpha}{n}, \tag{66}$$

$$\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}, \tag{67}$$

$$\frac{p}{p_0} = \frac{\frac{1}{r_0} - \frac{1}{q_0}}{\frac{1}{r} - \frac{1}{q}}. \tag{68}$$

Then there exists a constant $C > 0$ which is independent of f and g such that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \le C \|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q}$$

for all $f \in \mathcal{M}^{p_0}_p(\mathbb{R}^n)$ and $g \in \mathcal{M}^{q_0}_q(\mathbb{R}^n)$.

Proof. Define s and s_0 by $\frac{1}{s_0} \equiv \frac{1}{r_0} - \frac{1}{q_0}$ and $\frac{1}{s} \equiv \frac{1}{r} - \frac{1}{q}$. By condition (65), we get $\frac{1}{s} = \frac{q-r}{qr} > 0$. Since $0 < \frac{1}{s} = \frac{1}{r} - \frac{1}{q} < \frac{1}{r} < 1$ by (67), we have $s > 1$. From (66) and (67), we obtain

$$\frac{1}{s_0} = \frac{1}{r_0} - \frac{1}{q_0} = \frac{1}{p_0} - \frac{\alpha}{n} > 0.$$

On the other hand,

$$\frac{1}{s_0} = \frac{1}{p_0} - \frac{\alpha}{n} < \frac{1}{p_0} < 1. \tag{69}$$

Thus, $s_0 > 1$. Since $1 < p \leq p_0$, we have

$$\frac{s}{s_0} = \frac{\frac{1}{r_0} - \frac{1}{q_0}}{\frac{1}{r} - \frac{1}{q}} = \frac{p}{p_0} \leq 1 \tag{70}$$

thanks to (68). Thus, $1 < s \leq s_0$. By using the Hölder inequality and the definition of s and s_0, we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \leq \|I_\alpha f\|_{\mathcal{M}^{s_0}_s} \|g\|_{\mathcal{M}^{q_0}_q}.$$

By using (69) and (70) and the Adams theorem on the boundedness of I_α on Morrey spaces, we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}^{r_0}_r} \leq C\|f\|_{\mathcal{M}^{p_0}_p} \|g\|_{\mathcal{M}^{q_0}_q}.$$

So, we are done. □

To prove the result for general cases, we invoke the boundedness of the generalized fractional integral operator T_ρ which is defined by

$$T_\rho f(x) \equiv \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy$$

for a measurable function $\rho : (0, \infty) \to (0, \infty)$ and for all suitable functions f on $\mathbb{R}^n$.

Lemma 6.2. [39, Corollary 2.11] *Let $\phi \in \mathcal{G}_1$, $\Phi \in \Delta_2 \cap \nabla_2$ and $0 < b \leq 1$. Let $\rho : (0, \infty) \to (0, \infty)$ be a measurable function satisfying*

$$\int_0^t \frac{\rho(s)}{s} ds \leq C\rho(t) \tag{71}$$

and

$$\frac{1}{C_0} \leq \frac{\rho(s)}{\rho(t)} \leq C_0, \ \text{if } \frac{1}{2} \leq \frac{s}{t} \leq 2. \tag{72}$$

Set $\eta(t) \equiv \phi(t)^b$ for $t > 0$ and and $\Psi(t) \equiv \Phi(t^{1/b})$ for $t \geq 0$. Assume in addition that there exists a constant $C > 0$ such that

$$\frac{\rho(t)}{\phi(t)} + \int_t^\infty \frac{\rho(s)}{\phi(s)s} ds \leq \frac{C}{\phi(t)^b} \tag{73}$$

for all $t > 0$. Then

$$\|T_\rho f\|_{\mathcal{M}_{\eta,\Psi}} \leq C\|f\|_{\mathcal{M}_{\phi,\Phi}}$$

for every $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

Remark that (73) is necessary for the boundedness of T_ρ, when $\Phi(t) = t^p$ with $p \in (1,\infty)$ according to [4]. If $\rho(t) = t^\alpha$ where $0 < \alpha < n$, then we obtain the boundedness of fractional integral operator I_α on generalized Orlicz-Morrey spaces; See [10, Theorem 3.1] for the weak type inequalities for I_α.

Corollary 6.3. *Let $\phi \in \mathcal{G}_1$ and $\Phi \in \Delta_2 \cap \nabla_2$. Assume that ϕ satisfies*

$$\frac{t^\alpha}{\phi(t)} + \int_t^\infty \frac{s^{\alpha-1}}{\phi(s)}\, ds \le \frac{C}{\phi(t)^b} \tag{74}$$

for some $b \in (0,1]$ and for every $t > 0$. Define $\eta(t) \equiv \phi(t)^b$ for $t > 0$ and $\Psi(t) \equiv \Phi\left(t^{1/b}\right)$ for $t \ge 0$. Then, there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{M}_{\eta,\Psi}} \le C\|f\|_{\mathcal{M}_{\phi,\Phi}} \tag{75}$$

for every $f \in \mathcal{M}_{\phi,\Phi}(\mathbb{R}^n)$.

We present one of the formulations of the Olsen inequality in the following theorem.

Theorem 6.4. *Let $b \in (0,1]$. Let $\phi_1, \phi_2, \phi_3 \in \mathcal{G}_1$ and $\Phi_1, \Phi_2, \Phi_3 \in \Delta_2 \cap \nabla_2$. Assume that*

$$\frac{t^\alpha}{\phi_1(t)} + \int_t^\infty \frac{s^{\alpha-1}}{\phi_1(s)}\, ds \le \frac{C}{\phi_1(t)^b} \tag{76}$$

for every $t > 0$. If $\phi_1, \phi_2, \phi_3, \Phi_1, \Phi_2$ and Φ_3 satisfy

$$\left(\Phi_1^{-1}(t)\right)^b \Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \text{ and } \phi_3(t) \le \phi_1(t)^b \phi_2(t), \tag{77}$$

then there exists a constant $C > 0$ such that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le C\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}} \|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \tag{78}$$

for every $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Proof. Define $\eta_1(t) \equiv \phi_1(t)^b$ for $t > 0$ and define $\Psi_1(t) \equiv \Phi_1(t^{1/b})$ and $\Theta(t) \equiv \left(\Phi_1^{-1}(t)\right)^b$ for $t \ge 0$. The latter half of the condition (77) reads

$$\phi_3(t) \le \eta_1(t)\phi_2(t) \quad (t > 0).$$

Then

$$(\Psi_1 \circ \Theta)(t) = \Psi_1\left(\left(\Phi_1^{-1}(t)\right)^b\right) = \Phi_1(\Phi_1^{-1}(t)) = t$$

and

$$(\Theta \circ \Psi_1)(t) = \Theta\left(\Phi_1(t^{1/b})\right) = \left(\Phi_1^{-1}\left(\Phi_1(t^{1/b})\right)\right)^b = (t^{1/b})^b = t$$

for all $t \ge 0$. Thus, we conclude $\Psi_1^{-1}(t) = \Theta(t) = \left(\Phi_1^{-1}(t)\right)^b$. By using the assumption on Φ_1, Φ_2 and Φ_3, we have

$$\Psi_1^{-1}(t)\Phi_2^{-1}(t) \le \Phi_3^{-1}(t) \quad (t \ge 0).$$

We use Theorem 3.4 to get

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le 2\|I_\alpha f\|_{\mathcal{M}_{\eta_1,\Psi_1}} \|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}.$$

By using Corollary 6.3, we obtain the inequality (78). □

Remark 6.5. Let $1 < p \le p_0 < \infty, 1 < q \le q_0 < \infty$ and $1 < r \le r_0 < \infty$. We assume that p, p_0, q, q_0, r and r_0 satisfy

(6.a) $\frac{1}{p_0} > \frac{\alpha}{n}$

(6.b) $\frac{1}{r_0} = \frac{1}{q_0} + \frac{1}{p_0} - \frac{\alpha}{n}$

(6.c) $\frac{p}{p_0} = \frac{\frac{1}{r_0} - \frac{1}{q_0}}{\frac{1}{r} - \frac{1}{q}}$

We define $b \equiv 1 - \frac{\alpha}{n}p_0$, $\Phi_1(t) \equiv t^p$, $\phi_1(t) \equiv t^{\frac{n}{p_0}}$, $\Phi_2(t) \equiv t^q$, $\phi_2(t) \equiv t^{\frac{n}{q_0}}$, $\Phi_3(t) \equiv t^r$, and $\phi_3(t) \equiv t^{\frac{n}{r_0}}$. Note that, b and the functions ϕ_i and Φ_i $(i = 1, 2, 3)$ satisfy the assumption of Theorem 6.4 and we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_r^{r_0}} \le C\|f\|_{\mathcal{M}_p^{p_0}}\|g\|_{\mathcal{M}_q^{q_0}}. \tag{79}$$

The inequality (79) and the related results on Morrey spaces can be found in [17].

By using the decomposition of generalized Orlicz-Morrey spaces, we can obtain another Olsen inequality for fractional integral operators on generalized Orlicz-Morrey spaces. Before giving the main result, we present this preliminary.

Lemma 6.6. [17, Lemma 4.2] *Let $L \in \mathbb{N} \cup \{0\}$, $a \in L^\infty(\mathbb{R}^n)$, $Q \in \mathcal{Q}$, and* $\mathrm{supp}(a) \subset Q$. *Assume that $\int_{\mathbb{R}^n} x^\beta a(x)\, dx = 0$ for all multi-indices β with $|\beta| \le L$. Then,*

$$|I_\alpha a(x)| \le C\|a\|_{L^\infty}\ell(Q)^\alpha \sum_{k=1}^{\infty} \frac{1}{2^{k(n+L+1-\alpha)}}\chi_{2^kQ}(x).$$

As applications of Theorems 1.1 and 1.3, we obtain the following estimate.

Theorem 6.7. *Let $b \in (0, 1]$. Let also $\phi_1, \phi_2, \phi_3 \in \mathcal{G}_1$ and $\Phi_1, \Phi_2, \Phi_3 \in \Delta_2 \cap \nabla_2$. Assume that ϕ_1 satisfies the inequality* (76). *If $\phi_1, \phi_2, \phi_3, \Phi_1, \Phi_2$ and Φ_3 fulfill the following conditions*:

1. *$\phi_1, \phi_3, \Phi_1, \Phi_3$ are related by:*

$$\phi_3(t) = \phi_1(t)^b \quad (t > 0), \tag{80}$$

and

$$\Phi_3(t) = \Phi_1(t^{1/b}) \quad (t \ge 0). \tag{81}$$

2. *The functions $1/\phi_1$ and ϕ_2/ϕ_3 satisfy the "so-called" $\mathbb{Z}_1$-condition;*

$$\int_t^\infty \frac{ds}{\phi_1(s)s} \le \frac{C}{\phi_1(t)}, \tag{82}$$

and

$$\int_t^\infty \frac{\phi_2(s)}{\phi_3(s)s}\, ds \le C\frac{\phi_2(t)}{\phi_3(t)} \tag{83}$$

for every $t > 0$.

3. Denote by $\tilde{\Phi}_2$ and $\tilde{\Phi}_3$ the conjugate of Φ_2 and Φ_3, respectively. Then

$$\|M^{\tilde{\Phi}_2} g\|_{\tilde{\Phi}_3;Q} \leq C\|g\|_{\tilde{\Phi}_3;Q}. \tag{84}$$

4. The functions ϕ_1, ϕ_2 satisfy

$$\frac{t^\alpha}{\phi_1(t)\phi_2(t)} + \int_t^\infty \frac{s^{\alpha-1}}{\phi_1(s)\phi_2(s)}\,ds \leq \frac{C}{\phi_1(t)^b} \quad (t>0). \tag{85}$$

Then, there exists a constant $C>0$ such that

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \leq C\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}} \|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}.$$

for all $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Proof. Let $f \in \mathcal{M}_{\phi_1,\Phi_1}(\mathbb{R}^n)$ and $g \in \mathcal{M}_{\phi_2,\Phi_2}(\mathbb{R}^n) \setminus \{0\}$. We invoke Theorem 1.1 with $v=1$. Then, thanks to (82), there exists a triplet $\{\lambda_j\}_{j=1}^\infty \subset [0,\infty)$, $\{Q_j\}_{j=1}^\infty \subset \mathcal{Q}$ and $\{a_j\}_{j=1}^\infty \subset L^\infty(\mathbb{R}^n)$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $\mathcal{S}'(\mathbb{R}^n)$ and that

$$|a_j| \leq \chi_{Q_j}, \quad \int_{\mathbb{R}^n} x^\beta a_j(x)\,dx = 0, \quad \left\|\sum_{j=1}^\infty \lambda_j \chi_{Q_j}\right\|_{\mathcal{M}_{\phi_1,\Phi_1}} \leq C\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}} \tag{86}$$

for all multi-indices β with $|\beta| \leq L$. Here the constant $C>0$ is independent of f. According to Lemma 6.6, we have

$$|g(x) I_\alpha a_j(x)| \leq C\ell(Q_j)^\alpha \sum_{k=1}^\infty \frac{1}{2^{k(n+L+1-\alpha)}} |g(x)|\chi_{2^k Q_j}(x).$$

With this in mind, we set

$$H_k(x) \equiv \sum_{j=1}^\infty \frac{\lambda_j}{2^{k(n+L+1-\alpha)}} \ell(Q_j)^\alpha |g(x)|\chi_{2^k Q_j}(x).$$

Note that, the conditions (83) and (84) play the role of (6) and (7) in Theorem 1.3. Let us set

$$\Lambda_{j,k} \equiv \frac{\lambda_j \|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \ell(Q_j)^\alpha}{2^{k(n+L+1-\alpha)}\phi_2(\ell(2^k Q_j))}, \quad G_{j,k} \equiv \frac{\phi_2(\ell(2^k Q_j))}{\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}} g\chi_{2^k Q_j}.$$

Then, $\mathrm{supp}(G_{j,k}) \subset 2^k Q_j$ and $\|G_{j,k}\|_{\mathcal{M}_{\phi_2,\Phi_2}} \leq \phi_2(\ell(2^k Q_j))$. By using Theorem 1.3, we have

$$\begin{aligned}
\|H_k\|_{\mathcal{M}_{\phi_3,\Phi_3}} &\leq C\left\|\sum_{j=1}^\infty \Lambda_{j,k}\chi_{2^k Q_j}\right\|_{\mathcal{M}_{\phi_3,\Phi_3}} \\
&\leq C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\|\sum_{j=1}^\infty \frac{\lambda_j \ell(Q_j)^\alpha}{2^{k(n+L+1-\alpha)}\phi_2(\ell(2^k Q_j))}\chi_{2^k Q_j}\right\|_{\mathcal{M}_{\phi_3,\Phi_3}}.
\end{aligned}$$

An arithmetic shows that $\chi_{2^kQ_j} \le 2^{kn} M_{\chi_{Q_j}}$. Thus,

$$\begin{aligned}
&\|H_k\|_{\mathcal{M}_{\phi_3,\Phi_3}} \\
&\le C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\| \sum_{j=1}^{\infty} \frac{\lambda_j \ell(Q_j)^\alpha}{2^{k(-n+L+1-\alpha)}\phi_2(\ell(2^kQ_j))} (M\chi_{2^kQ_j})^2 \right\|_{\mathcal{M}_{\phi_3,\Phi_3}} \\
&= C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left(\left\| \left(\sum_{j=1}^{\infty} \frac{\lambda_j \ell(Q_j)^\alpha}{2^{k(-n+L+1-\alpha)}\phi_2(\ell(2^kQ_j))} (M\chi_{2^kQ_j})^2 \right)^{\frac{1}{2}} \right\|_{\mathcal{M}_{\sqrt{\phi_3},\Phi_3(\cdot^2)}} \right)^2.
\end{aligned}$$

By Theorem 4.1, we have

$$\|H_k\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\| \sum_{j=1}^{\infty} \frac{\lambda_j \ell(Q_j)^\alpha}{2^{k(-n+L+1-\alpha)}\phi_2(\ell(Q_j))} \chi_{Q_j} \right\|_{\mathcal{M}_{\phi_3,\Phi_3}}. \tag{87}$$

Choosing L large enough, we can add (87) over k to conclude

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\| \sum_{j=1}^{\infty} \frac{\lambda_j \ell(Q_j)^\alpha}{\phi_2(\ell(Q_j))} \chi_{Q_j} \right\|_{\mathcal{M}_{\phi_3,\Phi_3}}.$$

Let us set

$$\rho(t) \equiv \frac{t^\alpha}{\phi_2(t)} \quad (t > 0).$$

Then we have

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\| T_\rho \left[\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right] \right\|_{\mathcal{M}_{\phi_3,\Phi_3}}.$$

If we invoke Lemma 6.2 and use assumptions (80), (81) and (85), then we obtain

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_{\phi_3,\Phi_3}} \le C\|g\|_{\mathcal{M}_{\phi_2,\Phi_2}} \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{\mathcal{M}_{\phi_1,\Phi_1}} \le C\|f\|_{\mathcal{M}_{\phi_1,\Phi_1}} \|g\|_{\mathcal{M}_{\phi_2,\Phi_2}}.$$

This is the desired result. □

Remark 6.8.

1. Let $1 < p \le p_0 < \infty, 1 < q \le q_0 < \infty$ and $1 < r \le r_0 < \infty$. We assume that p, p_0, q, q_0, r and r_0 satisfy the conditions $(6.a)$ and $(6.b)$ and also the conditions:

   $(6.d)$ $r < q$

   $(6.e)$ $\frac{1}{q_0} \le \frac{\alpha}{n}$

(6.f) $\frac{r}{r_0} = \frac{p}{p_0}$

We define $b \equiv \frac{p_0}{r_0}$, $\Phi_1(t) \equiv t^p$, $\phi_1(t) \equiv t^{\frac{n}{p_0}}$, $\Phi_2(t) \equiv t^q$, $\phi_2(t) \equiv t^{\frac{n}{q_0}}$, $\Phi_3(t) \equiv t^r$, and $\phi_3(t) \equiv t^{\frac{n}{r_0}}$. We can verify that b and the functions ϕ_i and Φ_i $(i = 1, 2, 3)$ satisfy the assumption of Theorem 6.4 and we can recover the Olsen inequality on Morrey spaces in [17, Theorem 1.7].

2. Once we have a counterpart to Lemma 6.6, we will have obtained the Olsen inequality for T_ρ on generalized Orlicz-Morrey spaces. This is left as our future work.

As a corollary of Theorem 6.7, we can recapture the earlier result.

Theorem 6.9. [37] *Let* $0 < \alpha < n, 1 < p \le p_0 < \infty, 1 < q \le q_0 < \infty$, *and* $1 < r \le r_0 < \infty$. *If* p, p_0, q, q_0, r *dan* r_0 *satisfy* (65), (66), (67) *as well as*

$$\frac{1}{q_0} \le \frac{\alpha}{n} \tag{88}$$

$$\frac{r}{r_0} = \frac{p}{p_0}, \tag{89}$$

then there exists a constant $C > 0$ *which is independent of* f *and* g *such that*

$$\|g \cdot I_\alpha f\|_{\mathcal{M}_r^{r_0}} \le C \|f\|_{\mathcal{M}_p^{p_0}} \|g\|_{\mathcal{M}_q^{q_0}}$$

for all $f \in \mathcal{M}_p^{p_0}(\mathbb{R}^n)$ *and* $g \in \mathcal{M}_q^{q_0}(\mathbb{R}^n)$.

To conclude this paper, we give some examples.

Example 6.10. Note that, $\alpha = 1, n = 5, p = 2, p_0 = 3, q = 5, q_0 = 15, r = \frac{5}{2}$, and $r_0 = 5$ satisfy the assumption of Theorem 6.1 but these parameters fail the hypothesis (89). In fact

$$\frac{r}{r_0} = \frac{1}{2} \neq \frac{2}{3} = \frac{p}{p_0}.$$

On the other hand, $q = q_0 = \frac{n}{\alpha}$, $p = r = \frac{n+2}{\alpha+2}$, and $p_0 = r_0 = \frac{n+1}{\alpha+1}$ fail the assumption (68). In fact,

$$\frac{p}{p_0} \neq \frac{\frac{1}{r_0} - \frac{1}{q_0}}{\frac{1}{r} - \frac{1}{q}},$$

but these parameters satisfy the assumption of Theorem 6.9. These two examples show that Theorem 6.1 and Theorem 6.9 are independent.

Example 6.11. Let $q = q_0 = \frac{n}{\alpha}$, $p = r = \frac{n+2}{\alpha+2}$, $p_0 = r_0 = \frac{n+1}{\alpha+1}$. Define $\phi_1(t) = t^{\frac{n}{p_0}}$, $\Phi_1(t) = t^p$, $\phi_2(t) = t^{\frac{n}{q_0}}$, $\Phi_2(t) = t^q$, $\phi_3(t) = t^{\frac{n}{r_0}}$ and $\Phi_3(t) = t^r$. If $b = 1 - \frac{\alpha}{n} p_0$, then

$$\frac{\Phi_1^{-1}(t)^b \Phi_2^{-1}(t)}{\Phi_3^{-1}(t)} = t^{\frac{\alpha(\alpha-n)}{n(n+2)(\alpha+1)}}.$$

Consequently, $\frac{\Phi_1^{-1}(t_0)^b \Phi_2^{-1}(t_0)}{\Phi_3^{-1}(t_0)} > 1$, for some $t_0 > 0$. Thus, these functions fails the assumption of Theorem 6.4. However, we can check that these functions satisfy the hypothesis of Theorem 6.7.

Meanwhile, $\alpha = 1, n = 5, p = 2, p_0 = 3, q = 5, q_0 = 15, r = \frac{5}{2}$ and $r_0 = 5$ satisfy the conditions in Remark 6.5 but these parameters fail the conditions in Remark 6.8.

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References

[1] A. Mazzucato, Decomposition of Besov-Morrey spaces. Harmonic analysis at Mount Holyoke (South Hadley, MA, 2001), 279–294, Contemp. Math., **320**, Amer. Math. Soc., Providence, RI, 2003.

[2] J. Duoandikoetxea, Fourier Analysis. Translated and revised from the 1995 Spanish original by David Cruz-Uribe. Graduate Studies in Mathematics, 29. American Mathematical Society, Providence, RI, 2001.

[3] Eridani, H. Gunawan and M.I. Utoyo, A characterization for fractional integral operators on generalized Morrey spaces, Anal. Theory Appl. **28** (2012), No. 3, 263–267.

[4] Eridani, H. Gunawan, E. Nakai and Y. Sawano, Characterizations for the generalized fractional integral operators on Morrey spaces, Math. Ineq. and Appl. **17**, No. 2 (2014), 761–777.

[5] C. Fefferman and E.M. Stein, Some maximal inequalities, Amer. J. Math, **93** (1971), 107–115.

[6] S. Gala, A.M. Ragusa, Y. Sawano, and H. Tanaka, Uniqueness criterion of weak solutions for the dissipative quasi-geostrophic equations in Orlicz–Morrey spaces, Appl. Anal. **93** (2014), no. 2, 356–368.

[7] S. Gala, Y. Sawano, and H. Tanaka, A new Beale-Kato-Majda criteria for the 3D magneto-micropolar fluid equations in the Orlicz-Morrey space, Math. Methods Appl. Sci. **35** (2012), no. 11, 1321–1334.

[8] S. Gala, Y. Sawano and H. Tanaka, On the uniqueness of weak solutions of the 3D MHD equations in the Orlicz-Morrey space, Appl. Anal. **92** (2013), no. 4, 776–783.

[9] A. Gogatishvili and R. Mustafayev, New pre-dual space of Morrey space, J. Math. Anal. Appl. **397** (2013), no. 2, 678–692.

[10] H. Gunawan, D.I. Hakim, Y. Sawano, and I. Sihwaningrum, Weak type inequalities for some integral operator on generalized nonhomogeneous Morrey spaces, J. Funct. Spaces Appl., vol.2013, 2013.

[11] D.D. Haroske and L. Skrzypczak, Continuous embeddings of Besov-Morrey function spaces, Acta Math. Sin. **28** (2012), no. 7, 1307–1328.

[12] D.D. Haroske and L. Skrzypczak, Embeddings of Besov-Morrey spaces on bounded domains, Studia Math. **218** (2013), no. 2, 119–144.

[13] L. Hedberg and Y. Netrusov, An axiomatic approach to function spaces, spectral synthesis, and Luzin approximation, Mem. Amer. Math. Soc. **188** (2007), vi+97 pp.

[14] K.P. Ho, Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, Anal. Math. **38** (2012), no. 3, 173–185.

[15] K.P. Ho, Vector-valued singular integral operators on Morrey type spaces and variable Triebel-Lizorkin-Morrey space, Ann. Acad. Sci. Fenn. Math. **37** (2012), no. 2, 375–406.

[16] T. Iida, Weighted inequalities on Morrey spaces for linear and multilinear fractional integrals with homogeneous kernels, Taiwanese Math. J. Vol. 18, No. 1 147–185.

[17] T. Iida, Y. Sawano, and H. Tanaka, Atomic decomposition for Morrey spaces, Z. Anal. Anwendungen **33** (2014) 2, 149–170.

[18] H. Jia and H. Wang, Decomposition of Hardy-Morrey spaces, J. Math. Anal. Appl. **354** (2009), no. 1, 99–110.

[19] E.A. Kalita, Dual Morrey spaces, Dokl. Akad. Nauk **361** (1998), no. 4, 447–449.

[20] N. Kruglyak and E.A. Kuznetsov, Smooth and nonsmooth Calderón-Zygmund type decompositions for Morrey spaces, J. Fourier Anal. Appl. **11** (2005), no. 6, 697–714.

[21] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, New characterizations of Besov-Triebel-Lizorkin-Hausdorff spaces including coorbits and wavelets, J. Fourier Anal. Appl. **18** (2012), no. 5, 1067–1111.

[22] Y. Liang, Y. Sawano, T. Ullrich, D. Yang and W. Yuan, A new framework for generalized Besov-type and Triebel-Lizorkin-type spaces, Dissertationes Math. **489** (2013), 114 pp.

[23] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[24] E. Nakai, Hardy–Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces, Math. Nachr. **166**, 1994, 95–103.

[25] E. Nakai, Generalized fractional integral operator on Orlicz-Morrey spaces, in Banach and Function spaces (Kitakyushu, 2003), Yokohama Publ., Yokohama, 2004, 323–333.

[26] E. Nakai, Orlicz-Morrey spaces and Hardy–Littlewood maximal function, Studia Math. **188**, 2008, 193–221.

[27] E. Nakai, Orlicz-Morrey spaces and their preduals, in Banach and Function spaces (Kitakyushu, 2009), Yokohama Publ., Yokohama, 2011, 187–205.

[28] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal., (2012) **262**, 3665–3748.

[29] E. Nakai and Y. Sawano, Orlicz-Hardy spaces and their duals, Science China Mathematics, online.

[30] P. Olsen, Fractional integration, Morrey spaces and Schrödinger equation, Comm. Partial Differential Equations, **20** (1995), 2005–2055.

[31] H. Rauhut and T. Ullrich, Generalized coorbit space theory and inhomogeneous function spaces of Besov-Lizorkin-Triebel type, J. Funct. Anal. **260** (2011), 3299–3362.

[32] M. Rosenthal, Local means, wavelet bases and wavelet isomorphisms in Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Math. Nachr. **286** (2013), no. 1, 59–87.

[33] Y. Sawano, Wavelet characterization of Besov-Morrey and Triebel-Lizorkin-Morrey spaces, Funct. Approx. Comment. Math. **38** (2008), part 1, 93–107.

[34] Y. Sawano, A note on Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, Acta Math. Sin. (Engl. Ser.) **25** (2009), 1223–1242.

[35] Y. Sawano, Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces on domains in $\mathbb{R}^n$, Math. Nachr. **283** (2010), no. 10, 1–32.

[36] Y. Sawano, Atomic decompositions of Hardy spaces with variable exponents and its application to bounded linear operators, Integral Equations Operator Theory **77** (2013), no. 1, 123–148.

[37] Y. Sawano, S. Sugano and H. Tanaka, Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces, Trans. Amer. Math. Soc. **363** (2011), no. 12, 6481–6503.

[38] Y. Sawano, S. Sugano and H. Tanaka, Olsen's inequality and its applications to Schrödinger equations, Suurikaiseki Kôkyûroku Bessatsu **B26**, (2011) 51–80.

[39] Y. Sawano, S. Sugano and H. Tanaka, Orlicz-Morrey spaces and fractional operators, Potential Anal., **36**, No. 4 (2012), 517–556.

[40] I. Sihwaningrum, H.P. Suryawan and H. Gunawan, Fractional integral operators and Olsen inequalities on non-homogeneous spaces, Aust. J. Math. Anal. Appl. **7** (2010), no. 1, Art. 14, 6 pp.

[41] S. Sugano, Some inequalities for generalized fractional integral operators on generalized Morrey spaces, Math. Inequal. Appl., **14** (2011), no.4, 849–865.

[42] S. Sugano and H. Tanaka, Boundedness of fractional integral operators on generalized Morrey spaces, Sci. Math. Jpn., **58** (2003), no. 3, 531–540 (Sci. Math. Jpn. Online **8** (2003), 233–242).

[43] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, Acta Math. Sinica, **21** (2005), no. 6, 1535–1544.

[44] Y. Sawano and H. Tanaka, Decompositions of Besov-Morrey spaces and Triebel-Lizorkin-Morrey spaces, Math. Z. **257** (2007), no. 4, 871-905.

[45] Y. Sawano, D. Yang and W. Yuan, New applications of Besov-type and Triebel-Lizorkin-type tpaces, J. Math. Anal. Appl. **363** (2010), 73–85.

[46] Y. Sawano and H. Wadade, On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space, J. Fourier Anal. Appl. **19** (2013), no. 1, 20–47.

[47] S. Sugano, Some inequalities for generalized fractional integral operators on generalized Morrey spaces, Math. Inequal. Appl. **14** (2011), no. 4, 849–865.

[48] E.M. Stein, Harmonic Analysis, real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[49] H. Tanaka, Morrey spaces and fractional operators, J. Aust. Math. Soc. **88** (2010), no. 2, 247–259.

[50] L. Tang and J. Xu, Some properties of Morrey type Besov-Triebel spaces, Math. Nachr **278** (2005), no 7–8, 904–917.

[51] T. Ullrich, Continuous characterizations of Besov-Lizorkin-Triebel spaces and new interpretations as coorbits, J. Funct. Space Appl., Article ID 163213, 47 pages.

[52] M.I. Utoyo, T. Nusantara and B.S. Widodo, Fractional integral operator and Olsen inequality in the non-homogeneous classic Morrey space, Int. J. Math. Anal. (Ruse) **6** (2012), no. 29–32, 1501–1511.

[53] D. Yang and W. Yuan, A new class of function spaces connecting Triebel-Lizorkin spaces and Q spaces, J. Funct. Anal. **255** (2008), 2760–2809.

[54] D. Yang and W. Yuan, New Besov-type spaces and Triebel-Lizorkin-type spaces including Q spaces, Math. Z. **265** (2010), 451–480.

[55] D. Yang and W. Yuan, Characterizations of Besov-type and Triebel-Lizorkin-type spaces via maximal functions and local means, Nonlinear Analysis **73** (2010), 3805–3820.

[56] D. Yang and W. Yuan, Dual properties of Triebel-Lizorkin-type spaces and their applications, Z. Anal. Anwend. **30** (2011), 29–58.

[57] D. Yang and W. Yuan, Relations among Besov-type spaces, Triebel-Lizorkin-type spaces and generalized Carleson measure spaces, Appl. Anal. **92** (2013), no. 3, 549–561.

[58] W. Yuan, Y. Sawano and D. Yang, Decompositions of Besov-Hausdorff and Triebel-Lizorkin-Hausdorff spaces and their applications, J. Math. Anal. Appl. **369** (2010), 736–757.

[59] W. Yuan, W. Sickel and D. Yang, Morrey and Campanato Meet Besov, Lizorkin and Triebel, Lecture Notes in Mathematics, 2005, Springer-Verlag, Berlin, 2010, xi+281 pp.

[60] C. Zorko, Morrey space, Proc. Amer. Math. Soc. **98** (1986), no. 4, 586–592.