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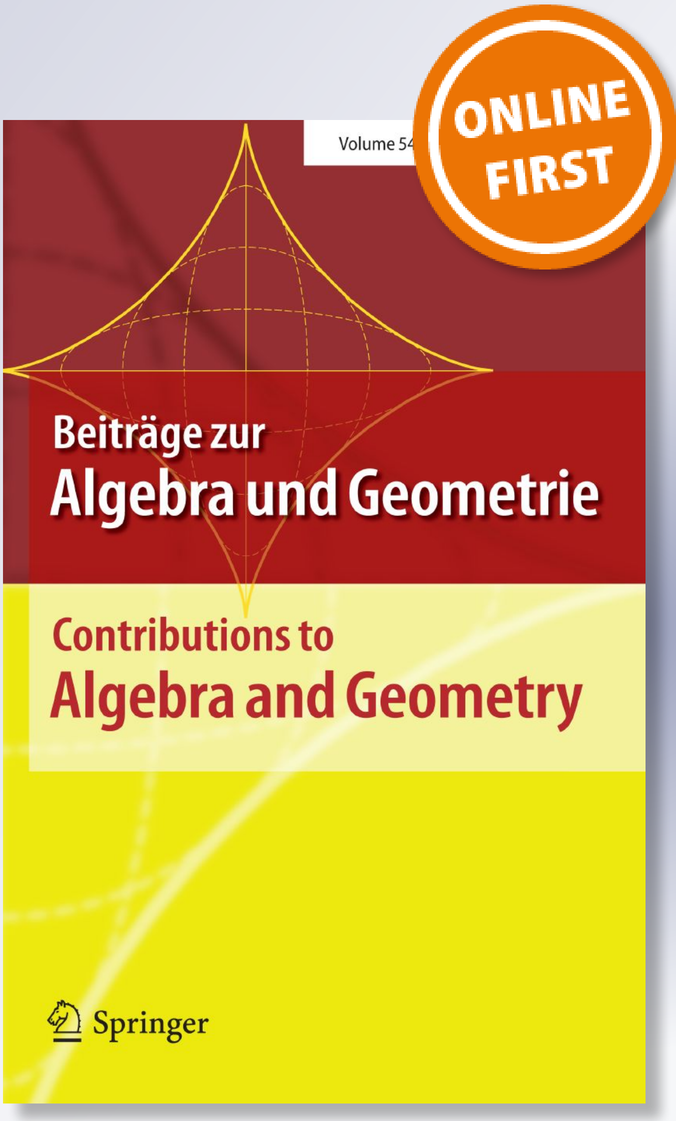

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ORIGINAL PAPER

A formula for the g-angle between two subspaces of a normed space

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Abstract We develop the notion of g-angle between two subspaces of a normed space. In particular, we discuss the g-angle between a 1-dimensional subspace and a t-dimensional subspace for $t \geq 1$ and the g-angle between a 2-dimensional subspace and a t-dimensional subspace for $t \geq 2$. Moreover, we present an explicit formula for the g-angle between two subspaces of ℓ^p spaces.

Keywords g-Angles · Subspaces · Normed spaces · ℓ^p Spaces

Mathematics Subject Classification 15A03 · 46B20 · 51N15 · 52A21

1 Introduction

In an inner product space $(X, \langle \cdot, \cdot \rangle)$, we can calculate the angles between two vectors and two subspaces. In particular, the angle $\theta = \theta(x, y)$ between two nonzero vectors x and y in X is defined by $\cos\theta := \frac{\langle x,y \rangle}{\|x\|\|y\|}$ where $\|x\| := \langle x, x \rangle^{\frac{1}{2}}$ denotes the induced norm on X. One may observe that the angle θ in X satisfies the following basic properties (see Diminnie et al. 1988);

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(a) *Parallelism* $\theta(x, y) = 0$ if and only if x and y are of the same direction; $\theta(x, y) = \pi$ if and only if x and y are of opposite direction.
(b) *Symmetry* $\theta(x, y) = \theta(y, x)$ for every $x, y \in X$.
(c) *Homogeneity*

$$\theta(ax, by) = \begin{cases} \theta(x, y), & ab > 0 \\ \pi - \theta(x, y), & ab < 0. \end{cases}$$

(d) *Continuity* If $x_n \to x$ and $y_n \to y$ (in the norm), then $\theta(x_n, y_n) \to \theta(x, y)$.

In a normed space, the concept of angles between two vectors has been studied intensively (see, for instance, Balestro et al. 2017; Diminnie et al. 1986; Gunawan et al. 2008; Horváth 2010; Milicic 2007; Thürey 2009; Zhi-Zhi et al. 2011). Here we shall be interested in the notion of angles between two subspaces of a normed space using a semi-inner product.

Let $(X, \|\cdot\|)$ be a real normed space. The functional $g : X^2 \to \mathbb{R}$ defined by the formula

$$g(x, y) := \frac{1}{2}\|x\|\left[\tau_+(x, y) + \tau_-(x, y)\right],$$

with

$$\tau_\pm(x, y) := \lim_{t\to\pm 0} \frac{\|x + ty\| - \|x\|}{t},$$

satisfies the following properties:

(1) $g(x, x) = \|x\|^2$ for every $x \in X$;
(2) $g(ax, by) = ab \cdot g(x, y)$ for every $x, y \in X$ and $a, b \in \mathbb{R}$;
(3) $g(x, x + y) = \|x\|^2 + g(x, y)$ for every $x, y \in X$;
(4) $|g(x, y)| \le \|x\| \cdot \|y\|$ for every $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear in y, then g is called a *semi-inner product* on X. For example, consider the space ℓ^p $(1 \le p < \infty)$ with the norm $\|x\|_p := \left[\sum_{k=1}^{\infty} |\xi_k|^p\right]^{\frac{1}{p}}$, $x = (\xi_k)$. Then the functional

$$g(x, y) := \|x\|_p^{2-p} \sum_{k=1}^{\infty} |\xi_k|^{p-1}\mathrm{sgn}\,(\xi_k)\,\eta_k, \quad x := (\xi_k)\,, \; y := (\eta_k) \in \ell^p,$$

is a semi-inner product on ℓ^p $(1 \le p < \infty)$ (Giles 1967; Gunawan et al. 2008). Note that, in general, g is not commutative.

Using a semi-inner product g, Milicic (1993b) introduced the notion of g-*orthogonality* on X, namely x is said to be g-orthogonal to y, denoted by $x \perp_g y$, if $g(x, y) = 0$. Note that in an inner product space, the functional $g(x, y)$ is identical with the inner product $\langle\cdot,\cdot\rangle$, and so the g-orthogonality coincides with the usual orthogonality. In this article, we will develop the notion of g-angles between two subspaces of a normed space and discuss its properties. We will begin our discussion by studying the g-angle between two vectors in a normed space.

2 Main results

2.1 The g-angle between two vectors

From now on, let $(X, \|\cdot\|)$ be a real normed space, unless otherwise stated. In connection with the notion of g-ortogonality, we define the *g-angle* between two nonzero vectors x and y in X, denoted by $A_g(x, y)$, by the formula

$$A_g(x, y) := \arccos \frac{g(y, x)}{\|x\| \cdot \|y\|}.$$

Note that $A_g(x, y) = \frac{1}{2}\pi$ if and only if $g(y, x) = 0$ or $y \perp_g x$. If X is an inner product space, the g-angle in X is identical with the usual angle.

Proposition 2.1 *The g-angle $A_g(\cdot, \cdot)$ satisfies the following properties:*

(a) *If x and y are of the same direction, then $A_g(x, y) = 0$; if x and y are of opposite direction, then $A_g(x, y) = \pi$ (part of the parallelism property).*
(b) *$A_g(ax, by) = A_g(x, y)$ if $ab > 0$; $A_g(ax, by) = \pi - A_g(x, y)$ if $ab < 0$ (the homogeneity property).*
(c) *If $x_n \to x$ (in the norm), then $A_g(x_n, y) \to A_g(x, y)$ (part of the continuity property).*

Proof (a) Let $y = kx$ for an arbitrary nonzero vector x in X and $k \in \mathbb{R} - \{0\}$. We have

$$A_g(x, y) = \arccos \frac{g(kx, x)}{\|x\| \cdot \|kx\|} = \arccos \frac{k \cdot g(x, x)}{|k|\|x\|^2} = \arccos \frac{k\|x\|^2}{|k|\|x\|^2}.$$

If $y = kx$ with $k > 0$, then $A_g(x, y) = \arccos(1) = 0$. If $y = kx$ with $k < 0$, then $A_g(x, y) = \arccos(-1) = \pi$.

(b) Let a and $b \in \mathbb{R} - \{0\}$. Observe that

$$A_g(ax, by) = \arccos \frac{ab \cdot g(y, x)}{|ab|(\|x\| \cdot \|y\|)}.$$

If $ab > 0$, then $A_g(ax, by) = A_g(x, y)$. Likewise, if $ab < 0$, then $A_g(ax, by) = \arccos\left(-\frac{g(y,x)}{\|x\|\cdot\|y\|}\right)$. Hence $A_g(ax, by) = \pi - A_g(x, y)$.

(c) If $x_n \to x$ (in the norm), then

$$|g(y, x_n - x)| \leq \|y\| \cdot \|x_n - x\| \longrightarrow 0.$$

Observe that $g(y, x_n - x) = g(y, x_n) - g(y, x)$. We have $g(y, x_n) \longrightarrow g(y, x)$. Hence

$$A_g(x_n, y) \to A_g(x, y),$$

as desired.

□

Remark 2.2 Since g, in general, is not commutative, the g-angle does not satisfy the symmetry property. For instance, in ℓ^1 with $g(y, x) := \|y\|_1 \sum_{k=1}^{\infty} \text{sgn}(\eta_k)\xi_k$, take $x := (1, 1, 0, \dots)$ and $y := (-1, 2, 0, \dots)$, so that we have $g(y, x) = 0 \neq g(x, y) = 2$. Likewise, the g-angle does not satisfy the continuity property. For instance, in ℓ^1 with the above functional g, take $y_n := (\frac{1}{n}, 1, 0, \dots)$, $x_n := (1 + \frac{1}{n}, 1, 0, \dots)$, $y := (0, 1, 0, \dots)$ and $x := (1, 1, 0, \dots)$, so that we obtain $g(y_n, x_n) \nrightarrow g(y, x)$.

2.2 The g-angle between a 1-dimensional subspace and a t-dimensional subspace

Here, using a semi-inner product g, we will discuss the notion of g-angles between two subspaces of a normed space. We first state the connection between the Gram determinant $\Gamma(x_1, \dots, x_n) := \det[g(x_i, x_k)]$, where $g(x_i, x_k)$ is the kth element of the ith row, and the linearly independence of $\{x_1, \dots, x_n\}$ as in the following theorem.

Theorem 2.3 *Let g be a semi-inner product in X and $\{x_1, \dots, x_n\} \subset X$. If $\Gamma(x_1, \dots, x_n) \neq 0$, then $\{x_1, \dots, x_n\}$ is linearly independent.*

Proof Suppose, on the contrary, that $\{x_1, \dots x_n\}$ is linearly dependent. Then there is an index j with $1 \leq j \leq n$ so that x_j is a linear combination of $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n$. Here the jth column of Γ is a linear combination of the other columns. As a consequence, we have $\Gamma(x_1, \dots, x_n) = 0$. Hence, $\{x_1, \dots, x_n\}$ must be a linearly independent set. □

Remark 2.4 The converse of this theorem is not true. For example, take $x_1 := (1, 2, 0, \dots)$ and $x_2 := (2, 1, 0, \dots)$ in ℓ^1 with the usual semi-inner product g. Clearly x_1 and x_2 are linearly independent. But one may check that

$$g(x_i, x_j) = \|x_i\|_1 \sum_{k=1}^{\infty} \text{sgn}(x_{ik}) x_{jk} = 9$$

for $i, j = 1, 2$, and hence $\Gamma(x_1, x_2) = \begin{vmatrix} 9 & 9 \\ 9 & 9 \end{vmatrix} = 0.$

We shall now define the g-orthogonal projection of y on subspace S as follows.

Definition 2.5 (Milicic 1993a) Let y be a vector of X and $S = \text{span}\{x_1, \dots, x_n\}$ be subspace of X with $\Gamma(x_1, \dots, x_n) = \det[g(x_i, x_k)] \neq 0$. The *$g$-orthogonal projection* of y on S, denoted by y_S, is defined by

$$y_S := -\frac{1}{\Gamma(x_1, \dots, x_n)} \begin{vmatrix} 0 & x_1 & \dots & x_n \\ g(x_1, y) & g(x_1, x_1) & \dots & g(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_n, y) & g(x_n, x_1) & \dots & g(x_n, x_n) \end{vmatrix},$$

and its *g-orthogonal complement* $y - y_S$ is given by

$$y - y_S = \frac{1}{\Gamma(x_1, \ldots, x_n)} \begin{vmatrix} y & x_1 & \ldots & x_n \\ g(x_1, y) & g(x_1, x_1) & \ldots & g(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_n, y) & g(x_n, x_1) & \ldots & g(x_n, x_n) \end{vmatrix}.$$

Note that the notation of the determinant $|\cdot|$ here has a special meaning because the elements of the matrix are not in the same field. Since

$$g(x_i, y - y_S) = \frac{1}{\Gamma(x_1, \ldots, x_n)} \begin{vmatrix} g\,(x_i, y) & g\,(x_i, x_1) & \ldots & g\,(x_i, x_n) \\ g\,(x_1, y) & g\,(x_1, x_1) & \ldots & g\,(x_1, x_n) \\ \vdots & \vdots & \ddots & \vdots \\ g\,(x_n, y) & g\,(x_n, x_1) & \ldots & g\,(x_n, x_n) \end{vmatrix},$$

we obtain $x_i \perp_g y - y_S$ for every $i = 1, \ldots, n$. For example, if $S = \text{span}\,\{x\}$, then the g-orthogonal projection of y on x is

$$y_x = \frac{g(x, y)}{\|x\|^2} x,$$

and $y - y_x$ is the g-orthogonal complement y on x. Notice here that $x \perp_g y - y_x$. Note that the g-angle between two vectors in a normed space is also the g-angle between these two vectors in the subspace spanned by them.

Next, let $x_1, \ldots, x_n \in X$ be a finite sequence of linearly independent vectors. We can construct a *left g-orthonormal sequence* $x_1^*, \ldots, x_n^*$ with $x_1^* := \frac{x_1}{\|x_1\|}$ and

$$x_k^* := \frac{x_k - (x_k)_{s_{k-1}}}{\|x_k - (x_k)_{s_{k-1}}\|}, \tag{1}$$

where $S_{k-1} = \text{span}\left\{x_1^*, \ldots, x_{k-1}^*\right\}$, $k = 2, \ldots, n$. We observe that $x_k^* \perp_g x_l^*$ for $k, l = 1, \ldots, n$ with $k < l$) (see Gunawan et al. 2005; Milicic 1993a).

Using the g-orthogonal projection, we define the g-angle between a 1-dimensional subspace $U = \text{span}\{u\}$ and a t-dimensional subspace $V = \text{span}\{v_1, \ldots, v_t\}$ of X with $\Gamma(v_1, \ldots, v_t) \neq 0$ and $t \geq 1$ by

$$\cos^2 A_g(U, V) := \frac{(g(u_V, u))^2}{\|u\|^2 \|u_V\|^2}, \tag{2}$$

where u_V denote the g-ortogonal projection of u on V. Note that if $U \subseteq V$, then $A_g(U, V) = 0$. One may observe that if X is an inner product space, then the definition of the g-angle in (2) and the usual definition of angle between two subspaces of X are equivalent.

If we now write $u = u_V + u_V^{\perp}$ with $u_V^{\perp}$ is the g-orthogonal complement of u on V, then (2) becomes

$$\cos^2 A_g(U, V) = \frac{\|u_V\|^2}{\|u\|^2},$$

which tells us that the value of $\cos A_g(U, V)$ is equal to the ratio between the 'length' of the g-orthogonal projection of u on V and the 'length' of u. If $X = \ell^p$ with the semi-inner product g, then

$$\cos^2 A_g(U, V) = \frac{\|u_V\|_p^2}{\|u\|_p^2}.$$

Therefore, an explicit formula for the cosine of the g-angle between a 1-dimensional subspace $U = \mathrm{span}\{u\}$ and t-dimensional subspace $V = \mathrm{span}\{v_1, \ldots, v_t\}$ of ℓ^p can be presented as follows.

Fact 2.6 *If $U = span\{u\}$ is a 1-dimensional subspace and $V = span\{v_1, \ldots, v_t\}$ is a t-dimensional subspace of ℓ^p with $\Gamma(v_1, \ldots, v_t) \neq 0$, then*

$$\cos^2 A_g(U, V) = \left[\sum_{j_{t+1}} \left|\sum_{j_t} \cdots \sum_{j_1} \left(\frac{1}{\|u\|_p} \prod_{i=1}^{t} |v_{ij_i}^*|^{p-1} sgn(v_{ij_i}^*)\right) \begin{vmatrix} v_{1j_1} & \cdots & v_{1j_t} & v_{1j_{t+1}} \\ \vdots & \ddots & \vdots & \vdots \\ v_{tj_1} & \cdots & v_{tj_t} & v_{tj_{t+1}} \\ u_{j_1} & \cdots & u_{j_t} & 0 \end{vmatrix}\right|^p\right]^{\frac{2}{p}},$$

where in each summation the index ranges from 1 to ∞.

Proof Suppose that $V = \mathrm{span}\,\{v_1, \ldots, v_t\}$ with $\Gamma(v_1, \ldots, v_t) \neq 0$. According to Theorem 2.3, $\{v_1, \ldots, v_t\}$ is linearly independent. Using $v_1^* = \frac{v_1}{\|v_1\|}$ and so forth as in (1), we obtain left g-orthonormal set $\left\{v_1^*, \ldots, v_t^*\right\}$. Notice that $\mathrm{span}\{v_1, \ldots, v_t\} = \mathrm{span}\{v_1^*, \ldots, v_t^*\}$. Hence

$$u_V = -\frac{1}{\Gamma\left(v_1^*, \ldots, v_t^*\right)} \begin{vmatrix} 0 & v_1^* & \cdots & v_t^* \\ g(v_1^*, u) & g(v_1^*, v_1^*) & \cdots & g(v_1^*, v_t^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(v_t^*, u) & g(v_t^*, v_1^*) & \cdots & g(v_t^*, v_t^*) \end{vmatrix}.$$

Observe that $\Gamma(v_1^*, \ldots, v_t^*) = 1$, and so

$$\|u_V\|_p = \left(\sum_{j_{t+1}} \left\| \begin{matrix} 0 & v_{1j_{t+1}}^* & \cdots & v_{tj_{t+1}}^* \\ g(v_1^*, u) & g(v_1^*, v_1^*) & \cdots & g(v_1^*, v_t^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(v_t^*, u) & g(v_t^*, v_1^*) & \cdots & g(v_t^*, v_t^*) \end{matrix} \right\|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{j_{t+1}} \left\| \begin{matrix} g(v_1^*, v_1^*) & \cdots & g\left(v_1^*, v_t^*\right) & g(v_1^*, u) \\ \vdots & \ddots & \vdots & \vdots \\ g(v_t^*, v_1^*) & \cdots & g(v_t^*, v_t^*) & g(v_t^*, u) \\ v_{1j_{t+1}}^* & \cdots & v_{tj_{t+1}}^* & 0 \end{matrix} \right\|^p \right)^{\frac{1}{p}}$$

$$= \left(\sum_{j_{t+1}} \left\| \begin{matrix} g(v_1^*, v_1^*) & \cdots & g(v_t^*, v_1^*) & v_{1j_{t+1}}^* \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_t^*) & \cdots & g(v_t^*, v_t^*) & v_{tj_{t+1}}^* \\ g(v_1^*, u) & \cdots & g(v_t^*, u) & 0 \end{matrix} \right\|^p \right)^{\frac{1}{p}}.$$

Hence, we have

$$\cos^2 A_g(U, V) = \frac{\|u_V\|_p^2}{\|u\|_p^2} = \left(\sum_{j_{t+1}} \left| \frac{1}{\|u\|_p} \begin{vmatrix} g\left(v_1^*, v_1^*\right) & \cdots & g\left(v_t^*, v_1^*\right) & v_{1j_{t+1}}^* \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_t^*) & \cdots & g(v_t^*, v_t^*) & v_{tj_{t+1}}^* \\ g(v_1^*, u) & \cdots & g(v_t^*, u) & 0 \end{vmatrix} \right|^p \right)^{\frac{2}{p}}.$$

Since $v_1^* = v_1$ and $g(x, y)$ is linear in y, we have

$$\cos^2 A_g(U, V) = \left(\sum_{j_{t+1}} \left| \frac{1}{\|u\|_p} \begin{vmatrix} g(v_1^*, v_1) & \cdots & g(v_t^*, v_1) & v_{1j_{t+1}} \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_t) & \cdots & g(v_t^*, v_t) & v_{tj_{t+1}} \\ g(v_1^*, u) & \cdots & g(v_t^*, u) & 0 \end{vmatrix} \right|^p \right)^{\frac{2}{p}}.$$

Next, substituting $g(v_i^*, v_k) = \|v_i^*\|_p^{2-p} \sum_{j_i} |v_{ij_i}^*|^{p-1} \mathrm{sgn}(v_{ij_i}^*) v_{kj_i}$ and using properties of determinants, we obtain

$$\cos^2 A_g(U, V) = \left[\sum_{j_{t+1}} \left| \sum_{j_t} \cdots \sum_{j_1} \left(\frac{1}{\|u\|_p} \prod_{i=1}^{t} |v_{ij_i}^*|^{p-1} \mathrm{sgn}(v_{ij_i}^*) \right) \begin{vmatrix} v_{1j_1} & \cdots & v_{1j_t} & v_{1j_{t+1}} \\ \vdots & \ddots & \vdots & \vdots \\ v_{tj_1} & \cdots & v_{tj_t} & v_{tj_{t+1}} \\ u_{j_1} & \cdots & u_{j_t} & 0 \end{vmatrix} \right|^p \right]^{\frac{2}{p}}.$$

This proves the fact. □

Example 2.7 Consider ℓ^1 with the usual semi-inner product g. Take $U = \text{span}\{u\}$ and $V = \text{span}\{v_1, v_2\}$, with $u := (1, 2, 1, 0, \ldots)$, $v_1 := (1, 0, 0, 0, \ldots)$, and $v_2 := (0, 1, 0, 0, \ldots)$. We obtain

$$\begin{aligned}\cos^2 A_g(U, V) &= \frac{1}{16}\left[\sum_{j_3}\left|\sum_{j_2}\sum_{j_1}\text{sgn}(v_{1j_1})\text{sgn}(v_{2j_2})\begin{vmatrix} v_{1j_1} & v_{1j_2} & v_{1j_3} \\ v_{2j_1} & v_{2j_2} & v_{2j_3} \\ u_{j_1} & u_{j_2} & 0 \end{vmatrix}\right|\right]^2 \\ &= \frac{1}{16}\left[\sum_{j_3}\left|\sum_{j_2}\text{sgn}(v_{2j_2})\begin{vmatrix} 1 & v_{1j_2} & v_{1j_3} \\ 0 & v_{2j_2} & v_{2j_3} \\ 1 & u_{j_2} & 0 \end{vmatrix}\right|\right]^2 \\ &= \frac{1}{16}\left[\left|\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{vmatrix}\right| + \left|\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{vmatrix}\right|\right]^2 = \frac{9}{16}.\end{aligned}$$

Hence, the g-angle between the two subspaces U and V is $\arccos(\frac{3}{4})$.

2.3 The g-angle between a 2-dimensional subspace and a t-dimensional subspace

In this section, we discuss the g-angle between subspaces U and V where U is a 2-dimensional subspace and V is a t-dimensional subspace with $t \geq 2$. First, we define the function $\Lambda(\cdot, \cdot)$ on $X \times X$ by

$$\Lambda(x, y) := \begin{vmatrix} |g(x, x)| & |g(x, y)| \\ |g(y, x)| & |g(y, y)| \end{vmatrix}^{\frac{1}{2}}. \tag{3}$$

Note that in a real inner product space $(X, \langle\cdot, \cdot\rangle)$, $\Lambda(\cdot, \cdot)$ is identical with the *standard* 2-*norm* $\|\cdot, \cdot\|$ which is given by

$$\|x, y\| := \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}^{\frac{1}{2}}.$$

The following proposition lists some properties of the function $\Lambda(\cdot, \cdot)$.

Proposition 2.8 *The function $\Lambda(\cdot, \cdot)$ defined by (3) satisfies the following properties:*

(a) $\Lambda(x, y) \geq 0$ *for every* $x, y \in X$*; If* x *and* y *are linearly dependent, then* $\Lambda(x, y) = 0$*;*
(b) $\Lambda(x, y) = \Lambda(y, x)$ *for every* $x, y \in X$*;*
(c) $\Lambda(\alpha x, y) = \alpha\Lambda(x, y)$ *for every* $x, y \in X$ *and* $\alpha \in \mathbb{R}$*;*
(d) $\Lambda(x, y) \leq \|x\| \cdot \|y\|$ *for every* $x, y \in X$.

Proof (a) Using properties of determinants and the functional g, we have

$$(\Lambda(x, y))^2 = \|x\|^2\|y\|^2 - |g(x, y)||g(y, x)| \geq \|x\|^2\|y\|^2 - \|x\|^2\|y\|^2 = 0.$$

Let $y = kx$ with $k \in \mathbb{R}$. Then we have $\Lambda(ky, y) = \sqrt{\|ky\|^2\|y\|^2 - |g(ky, y)||g(y, ky)|} = 0$.

(b) Observe that

$$\begin{aligned}\Lambda(x, y) &= \sqrt{\|x\|^2\|y\|^2 - |g(x, y)||g(y, x)|} \\ &= \sqrt{\|y\|^2\|x\|^2 - |g(y, x)||g(x, y)|} \\ &= \Lambda(y, x).\end{aligned}$$

(c) Observe that

$$\Lambda(\alpha x, y) = \sqrt{\|\alpha x\|^2\|y\|^2 - |g(\alpha x, y)||g(y, \alpha x)|} = |\alpha|\Lambda(x, y).$$

(d) Observe that

$$\Lambda(x, y) = \sqrt{\|x\|^2\|y\|^2 - |g(x, y)||g(y, x)|} \leq \|x\| \cdot \|y\|,$$

as desired.

□

Remark 2.9 The converse of part (a) is not true. For instance, take $x := (1, 2, 0, \dots)$ and $y := (2, 1, 0, \dots)$ in ℓ^1 (with the semi-inner product g, as usual). Clearly x and y are linearly independent. But one may check that $\Lambda(x, y) = 0$. Likewise, Λ does not satisfy the triangle inequality. For example, take $x := (3, 1, 0, \dots)$, $y := (-2, 0, 0, \dots)$, $z := (0, 2, 0, \dots)$ in ℓ^1. Then we have $\|x\| = 4$, $\|y\| = 2$, $\|z\| = 2$, $\|y + z\| = 4$, $g(x, y) = -8$, $g(y, z) = -6$, $g(x, z) = 8$, $g(z, x) = 2$, $g(x, y + z) = 0$, and $g(y + z, x) = -8$. Hence $\Lambda(x, y) = 4$, $\Lambda(x, z) = 4\sqrt{3}$, and $\Lambda(x, y + z) = 16$, so that $\Lambda(x, y + z) > \Lambda(x, y) + \Lambda(x, z)$.

Using the function $\Lambda(\cdot, \cdot)$, we now define the g-angle between a 2-dimensional subspace $U := \text{span}\{u_1, u_2\}$ of X with $\Lambda(u_1, u_2) \neq 0$ and a t-dimensional subspace $V := \text{span}\{v_1, \dots, v_t\}$ of X with $\Gamma(v_1, \dots, v_t) \neq 0$ ($t \geq 2$) by

$$\cos^2 A_g(U, V) := \frac{(\Lambda(u_{1V}, u_{2V}))^2}{(\Lambda(u_1, u_2))^2} \tag{4}$$

where u_{iV} denote the g-orthogonal projection of u_i's on V with $i = 1, 2$. Note that in a standard 2-normed space, the definition of g-angle in (4) is identical with the angle defined in Gunawan et al. (2005).

According to the following proposition, the definition of g-angle in (4) makes sense.

Proposition 2.10 *The ratio on the right hand side of* (4) *is a number in* $[0, 1]$, *and it is independent of the choice of basis for* U *and* V.

Proof Assuming particularly that $\{u_1, u_2\}$ is left orthonormal, we have $\Lambda(u_1, u_2) = 1$ and

$$(\Lambda(u_{1V}, u_{2V}))^2 = \|u_{1V}\|^2\|u_{2V}\|^2 - |g(u_{1V}, u_{2V})||g(u_{2V}, u_{1V}|.$$

According to Milicic (1993a), we have $\|u_{iV}\| \leq \|u_i\|$ for $i = 1, 2$. Hence $(\Lambda(u_{1V}, u_{2V}))^2 \leq 1$. Therefore, the ratio is a number in $[0, 1]$.
Secondly, note that the g-orthogonal projection of u_i's on V is independent of the choice of basis for V (Milicic 1993a). Moreover, since g-orthogonal projections are linear transformations, the ratio of (4) is also invariant under any change of basis for U. Indeed, the ratio is unchanged if swap u_1 and u_2, replace u_1 with $u_1 + \alpha u_2$, or replace u_1 with αu_2 where $\alpha \neq 0$. □

3 Concluding remarks

The formula (4) can be used to compute the g-angle between two subspaces of ℓ^p as follows. Let $V = \text{span}\{v_1, \ldots, v_t\}$ with $\Gamma(v_1, \ldots, v_t) \neq 0$. According to Theorem 2.3, $\{v_1, \ldots, v_t\}$ is linearly independent. Using $v_1^* = \frac{v_1}{\|v_1\|}$ and so forth as in (1), we obtain the left g-orthonormal set $\{v_1^*, \ldots, v_t^*\}$. Here span $\{v_1, \ldots, v_t\} =$ span $\{v_1^*, \ldots, v_t^*\}$. Hence, for $i = 1, 2$, we have

$$u_{iV} = -\begin{vmatrix} 0 & v_1^* & \cdots & v_t^* \\ g(v_1^*, u_i) & g(v_1^*, v_1^*) & \cdots & g(v_1^*, v_t^*) \\ \vdots & \vdots & \ddots & \vdots \\ g(v_t^*, u_i) & g(v_t^*, v_1^*) & \cdots & g(v_t^*, v_t^*) \end{vmatrix} = -\begin{vmatrix} g(v_1^*, v_1^*) & \cdots & g(v_t^*, v_1^*) & v_1^* \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_t^*) & \cdots & g(v_t^*, v_t^*) & v_t^* \\ g(v_1^*, u_i) & \cdots & g(v_t^*, u_i) & 0 \end{vmatrix}.$$

Since $v_1^* = v_1$ and $g(x, y)$ is linear in y, we obtain

$$u_{iV} = -\begin{vmatrix} g(v_1^*, v_1) & \cdots & g(v_t^*, v_1) & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ g(v_1^*, v_t) & \cdots & g(v_t^*, v_t) & v_t \\ g(v_1^*, u_i) & \cdots & g(v_t^*, u_i) & 0 \end{vmatrix}.$$

Substituting $g(v_k^*, v_i) = \|v_k^*\|_p^{2-p} \sum_{j_k} |v_{kj_k}^*|^{p-1} \text{sgn}(v_{kj_k}^*) v_{ij_k}$, we get

$$u_{iV} = -\sum_{j_t} \cdots \sum_{j_1} |v_{1j_t}^*|^{p-1} \text{sgn}(v_{1j_t}^*) \cdots |v_{tj_1}^*|^{p-1} \text{sgn}(v_{tj_1}^*) \begin{vmatrix} v_{1j_1} & \cdots & v_{1j_t} & v_1 \\ \vdots & \ddots & \vdots & \vdots \\ v_{tj_1} & \cdots & v_{tj_t} & v_t \\ u_{ij_1} & \cdots & u_{ij_t} & 0 \end{vmatrix}.$$

Using this formula, the value of $\cos^2 A_g(U, V)$ (and hence $A_g(U, V)$) can be computed. For instance, in ℓ^1 (with the usual semi-inner product g), let $U = \text{span}\{u_1, u_2\}$ and $V = \text{span}\{v_1, v_2, v_3\}$ with $u_1 := (1, 1, 2, 3, 0, \ldots)$, $u_2 := (2, 1, -3, 2, 0, \ldots)$, $v_1 := (1, 0, 0, 0, 0, \ldots)$, $v_2 := (0, 1, 0, 0, 0, \ldots)$, and $v_3 := (0, 0, 1, 0, 0, \ldots)$. We obtain $u_{1V} = (1, 1, 2, 0, 0, \ldots)$ and $u_{2V} = (2, 1, -3, 0, 0, \ldots)$. Moreover, $\|u_1\| = 7$, $\|u_2\| = 8$, $g(u_1, u_2) = 14$, $g(u_2, u_1) = 24$, $\|u_{1V}\| = 4$, $\|u_{2V}\| = 6$, $g(u_{1V}, u_{2V}) = 0$, and $g(u_{2V}, u_{1V}) = 0$. Thus $\cos^2 A_g(U, V) = \frac{36}{167}$, so that $A_g(U, V) = \arccos(\frac{6}{167}\sqrt{167})$.

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