Generalized Hölder's Inequality on Morrey Spaces

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Abstract

The aim of this paper is to present necessary and sufficient conditions for generalized Hölder's inequality on generalized Morrey spaces. We also obtain similar results on weak Morrey spaces and on generalized weak Morrey spaces. The necessary and sufficient conditions for the generalized Hölder's inequality on these spaces are obtained through estimates for characteristic functions of balls in $\mathbb{R}^d$.

Keywords: Hölder's inequality, generalized Hölder's inequality, Morrey spaces, weak Morrey spaces, generalized Morrey spaces, generalized weak Morrey spaces.

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1 Introduction and Preliminaries

Several authors have made important observations about Hölder's inequality in the last three decades (see [1, 2, 9, 12]). Recently, Masta *et al.* [8] obtained sufficient and necessary conditions for the generalized Hölder's inequality on Lebesgue spaces. In this paper, we are interested in studying the generalized Hölder's inequality on Morrey spaces and on generalized Morrey spaces. In particular, we shall prove sufficient and necessary conditions for generalized Hölder's inequality on those spaces. In addition, we also prove similar result on weak Morrey spaces and on generalized weak Morrey spaces.

Let us first recall the definition of Morrey spaces. For $1 \le p \le q < \infty$, the *Morrey space*

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$\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all p-locally integrable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d, r>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left(\int_{B(a,r)} |f(y)|^p \, dy\right)^{\frac{1}{p}} < \infty.$$

Here, $B(a,r)$ denotes the open ball in $\mathbb{R}^d$ centered at a with radius r, and $|B(a,r)|$ denotes its Lebesgue measure. One might observe that $\|\cdot\|_{\mathcal{M}^p_q}$ defines a norm on $\mathcal{M}^p_q$, and makes the space complete [11]. Also note that if $q = p$, then $\mathcal{M}^p_q(\mathbb{R}^d) = L^p(\mathbb{R}^d)$. Thus, $\mathcal{M}^p_q(\mathbb{R}^d)$ can be viewed as a generalization of the Lebesgue space $L^p(\mathbb{R}^d)$.

The following theorem presents sufficient and necessary conditions for Hölder's inequality on Morrey spaces.

Theorem 1.1. *Let $1 \le p \le q < \infty$, $1 \le p_1 \le q_1 < \infty$, and $1 \le p_2 \le q_2 < \infty$. Then the following statements are equivalent:*

(1) $\frac{1}{p_1} + \frac{1}{p_2} \le \frac{1}{p}$ *and* $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$.

(2) $\|fg\|_{\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^{p_1}_{q_1}} \|g\|_{\mathcal{M}^{p_2}_{q_2}}$, *for every* $f \in \mathcal{M}^{p_1}_{q_1}(\mathbb{R}^d)$ *and* $g \in \mathcal{M}^{p_2}_{q_2}(\mathbb{R}^d)$.

Let us now move to the weak Morrey spaces. For $1 \le p \le q < \infty$, the *weak Morrey space* $w\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all measurable functions f on $\mathbb{R}^d$ for which $\|f\|_{w\mathcal{M}^p_q} < \infty$, where

$$\|f\|_{w\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \gamma \left|\{x \in B(a,r) \colon |f(x)| > \gamma\}\right|^{\frac{1}{p}}.$$

Note that $\|\cdot\|_{w\mathcal{M}^p_q}$ defines a quasi-norm on $w\mathcal{M}^p_q(\mathbb{R}^d)$. If $q = p$, then $w\mathcal{M}^p_q(\mathbb{R}^d) = wL^p(\mathbb{R}^d)$. Here, $w\mathcal{M}^p_q(\mathbb{R}^d)$ can be viewed as a generalization of weak Lebesgue spaces $wL^p(\mathbb{R}^d)$. The relation between $w\mathcal{M}^p_q(\mathbb{R}^d)$ and $\mathcal{M}^p_q(\mathbb{R}^d)$ is shown in the following lemma.

Lemma 1.2. *[6] Let $1 \le p \le q < \infty$. Then $\mathcal{M}^p_q(\mathbb{R}^d) \subseteq w\mathcal{M}^p_q(\mathbb{R}^d)$ with*

$$\|f\|_{w\mathcal{M}^p_q} \le \|f\|_{\mathcal{M}^p_q},$$

for every $f \in \mathcal{M}^p_q(\mathbb{R}^d)$.

This lemma will be useful for us to study sufficient and necessary conditions for generalized Hölder's inequality on weak Morrey spaces.

Next we present the definition of generalized Morrey spaces and generalized weak Morrey spaces. For $1 \le p \le q < \infty$, let $\mathcal{G}_p$ be the set of all functions $\phi : (0,\infty) \to (0,\infty)$ which satisfy $\phi(r) > \phi(s)$ and $r^{\frac{d}{p}}\phi(r) < s^{\frac{d}{p}}\phi(s)$ for every $0 < r < s < \infty$. (Note that if $\phi \in \mathcal{G}_p$, then ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that $\frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C$ for $1 \le \frac{r}{s} \le 2$.) For $\phi \in \mathcal{G}_p$, the *generalized Morrey space* $\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined as the set of measurable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d, r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty.$$

Note that $\mathcal{M}^p_\phi(\mathbb{R}^d) = \mathcal{M}^p_q(\mathbb{R}^d)$ for $\phi(r) := r^{-\frac{d}{q}}$, $1 \le p \le q < \infty$. Meanwhile, for $\phi \in \mathcal{G}_p$, the *generalized weak Morrey space* $w\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{w\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} \frac{\gamma\,|\{x \in B(a,r)\colon |f(x)| > \gamma\}|^{\frac{1}{p}}}{\phi(r)|B(a,r)|^{\frac{1}{p}}} < \infty.$$

Here $\|\cdot\|_{w\mathcal{M}^p_\phi}$ is a quasi norm on $w\mathcal{M}^p_\phi(\mathbb{R}^d)$. Furthermore, $w\mathcal{M}^p_\phi(\mathbb{R}^d) = w\mathcal{M}^p_q(\mathbb{R}^d)$ for $\phi(r) = r^{-\frac{d}{q}}$. The relation between the generalized Morrey spaces and their weak type is given in the following lemma.

Lemma 1.3. *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Then $\mathcal{M}^p_\phi(\mathbb{R}^d) \subseteq w\mathcal{M}^p_\phi(\mathbb{R}^d)$ with*

$$\|f\|_{w\mathcal{M}^p_\phi} \le \|f\|_{\mathcal{M}^p_\phi},$$

for every $f \in \mathcal{M}^p_\phi(\mathbb{R}^d)$.

In Section 2 we state our main results, and in Section 3 we present the proofs.

2 Statement of The Results

Our main results are presented in the following theorems. The first theorem is more general than Theorem 1.1.

Theorem 2.1. *Let $m \ge 2$. If $1 \le p \le q < \infty$ and $1 \le p_i \le q_i < \infty$, $i = 1, 2, \ldots, m$, then the following statements are equivalent:*

(1) $\sum\limits_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ *and* $\sum\limits_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

(2) $\left\|\prod\limits_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod\limits_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$, *for every $f_i \in \mathcal{M}^{p_i}_{q_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.*

Theorem 2.2. *Let $m \ge 2$. If $1 \le p \le q < \infty$ and $1 \le p_i \le q_i < \infty$, $i = 1, 2, \ldots, m$, then the following statements are equivalent:*

(1) $\sum\limits_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ *and* $\sum\limits_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

(2) $\left\|\prod\limits_{i=1}^m f_i\right\|_{w\mathcal{M}^p_q} \le m \prod\limits_{i=1}^m \|f_i\|_{w\mathcal{M}^{p_i}_{q_i}}$, *for every $f_i \in w\mathcal{M}^{p_i}_{q_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.*

On generalized Morrey spaces and on generalized weak Morrey spaces, we have the following theorems.

Theorem 2.3. *Let $m \ge 2$. If $1 \le p, p_i < \infty$ and $\phi, \phi_i \in \mathcal{G}_p$, for all $i = 1, 2, \ldots, m$, then the following statements are equivalent:*

(1) $\sum\limits_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ *and* $\prod\limits_{i=1}^m \phi_i(r) \le \phi(r)$, *for every $r > 0$.*

(2) $\left\|\prod\limits_{i=1}^m f_i\right\|_{\mathcal{M}^p_\phi} \le \prod\limits_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$, *for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, $i = 1, \ldots, m$.*

Theorem 2.4. *Let $m \geq 2$. If $1 \leq p, p_i < \infty$ and $\phi, \phi_i \in \mathcal{G}_p$, for all $i = 1, 2, \dots, m$, then the following statements are equivalent:*

(1) $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$ *and* $\prod_{i=1}^{m} \phi_i(r) \leq \phi(r)$, *for every* $r > 0$.

(2) $\left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_{\phi}^{p}} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}_{\phi_i}^{p_i}}$, *for every* $f_i \in w\mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$, $i = 1, \dots, m$.

3 Proof of Theorems

Now we come to the proof of theorems in Section 2. Here, the letter C denotes a constant that may change from line to line. To prove our results, we shall use Lemma 1.2, Lemma 1.3, and the following lemma.

Lemma 3.1. *[3, 4, 6] Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. For $a_0 \in \mathbb{R}^d$ and $R > 0$, we have*

$$\|\chi_{B(a_0,R)}\|_{\mathcal{M}_{\phi}^{p}} = \|\chi_{B(a_0,R)}\|_{w\mathcal{M}_{\phi}^{p}} = \frac{1}{\phi(R)}. \tag{1}$$

Proof. This fact is proved in [3, 4, 6]; we rewrite the proof here for convenience. Let $B_0 = B(a_0, R) \subseteq \mathbb{R}^d$ for some $a_0 \in \mathbb{R}^d$ and $R > 0$. By the definition of $\| \cdot \|_{\mathcal{M}_{\phi}^{p}}$, we have

$$\|\chi_{B_0}\|_{\mathcal{M}_{\phi}^{p}} = \sup_{a \in \mathbb{R}^d,\ r>0} \frac{1}{\phi(r)} \left(\frac{\int_{B(a,r)} |\chi_{B_0}(x)|^p \ dx}{|B(a,r)|} \right)^{\frac{1}{p}} \geq \frac{1}{\phi(R)} \left(\frac{|B_0 \cap B_0|}{|B_0|} \right)^{\frac{1}{p}} = \frac{1}{\phi(R)}.$$

To prove the other inequality, we consider two cases.

Case 1: If $r \leq R$, then $\phi(r) \geq \phi(R)$, so that

$$\frac{1}{\phi(r)} \left(\frac{\int\limits_{B(a,r)} |\chi_{B_0}(x)|^p \ dx}{|B(a,r)|} \right)^{\frac{1}{p}} \leq \frac{1}{\phi(R)} \left(\frac{|B(a,r) \cap B_0|}{|B(a,r)|} \right)^{\frac{1}{p}} \leq \frac{1}{\phi(R)},$$

for every $a \in \mathbb{R}^d$.

Case 2: If $r \geq R$, then $R^{\frac{d}{p}} \phi(R) \leq r^{\frac{d}{p}} \phi(r)$, so that

$$\begin{aligned} \frac{1}{\phi(r)} \left(\frac{\int\limits_{B(a,r)} |\chi_{B_0}(x)|^p \ dx}{|B(a,r)|} \right)^{\frac{1}{p}} &\leq \frac{r^{\frac{d}{p}} R^{-\frac{d}{p}}}{\phi(R)} \left(\frac{|B(a,r) \cap B_0|}{|B(a,r)|} \right)^{\frac{1}{q}} \\ &\leq \frac{r^{\frac{d}{p}} R^{-\frac{d}{p}}}{\phi(R)} \left(\frac{|B_0|}{|B(a,r)|} \right)^{\frac{1}{p}} = \frac{1}{\phi(R)}, \end{aligned}$$

for every $a \in \mathbb{R}^d$.

From both cases, we conclude that $\|\chi_{B_0}\|_{\mathcal{M}_{\phi}^{p}} \leq \frac{1}{\phi(R)}$. This proves the first part of the lemma.

To prove the second part, we first note that by Lemma 1.3 and Lemma 3.1, we have

$$\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \le \|\chi_{B_0}\|_{\mathcal{M}^p_\phi} \le \frac{1}{\phi(R)}.$$

Next, by using the definition of $\|\cdot\|_{w\mathcal{M}^p_\phi(\mathbb{R}^d)}$, we have

$$\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \ge \frac{\gamma}{\phi(R)}\left(\frac{|\{x\in B(a,R):\ |\chi_{B_0}(x)|>\gamma\}|}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(R)}\left(\frac{|B_0|}{|B_0|}\right)^{\frac{1}{p}} = \frac{\gamma}{\phi(R)},$$

for every $a\in\mathbb{R}^d$ and $\gamma\in(0,1)$. Therefore $\|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \ge \frac{1}{\phi(R)}$, and the lemma is proved. □

3.1 The proof of Theorem 2.1

Proof. ($1 \Rightarrow 2$) Let $\sum_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ and $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$ hold. We take arbitrary $B = B(a,R)$ and $f_i \in \mathcal{M}^{p_i}_{q_i}\left(\mathbb{R}^d\right)$, where $i = 1,\dots,m$.

Case 1: For $\sum_{i=1}^m \frac{1}{p_i} = \frac{1}{p}$, by the generalized Hölder's inequality on Lebesgue spaces [2], we have

$$|B|^{\frac{1}{q}-\frac{1}{p}}\left(\int_B \prod_{i=1}^m |f_i(x)|^p dx\right)^{\frac{1}{p}} \le \prod_{i=1}^m |B|^{\frac{1}{q_i}-\frac{1}{p_i}}\left(\int_B |f(x)|^{p_i} dx\right)^{\frac{1}{p_i}}.$$

By taking the supremum over B, we obtain $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$.

Case 2: For $\sum_{i=1}^m \frac{1}{p_i} < \frac{1}{p}$, we define $\frac{1}{p^*} = \sum_{i=1}^m \frac{1}{p_i}$. Using Hölder's inequality and the generalized Hölder's inequality on Lebesgue spaces, we have

$$\begin{aligned}|B|^{\frac{1}{q}-\frac{1}{p}}\left(\int_B \prod_{i=1}^m |f_i(x)|^p dx\right)^{\frac{1}{p}} &\le |B|^{\frac{1}{q}-\frac{1}{p^*}}\left(\int_B \prod_{i=1}^m |f_i(x)|^{p^*} dx\right)^{\frac{1}{p^*}}\\ &\le \prod_{i=1}^m |B|^{\frac{1}{q_i}-\frac{1}{p_i}}\left(\int_B |f(x)|^{p_i} dx\right)^{\frac{1}{p_i}}.\end{aligned}$$

Now take the supremum over B to obtain $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$.

($2 \Rightarrow 1$) Suppose that $\left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|f_i\|_{\mathcal{M}^{p_i}_{q_i}}$, for every $f_i \in \mathcal{M}^{p_i}_{q_i}(\mathbb{R}^d)$. Take an arbitrary $R > 0$ and choose $f_i := \chi_{B(0,R)}$ for $i = 1,\dots,m$. Observe that

$$\|\chi_{B(0,R)}\|_{\mathcal{M}^p_q} = \left\|\prod_{i=1}^m f_i\right\|_{\mathcal{M}^p_q} \le \prod_{i=1}^m \|\chi_{B(0,R)}\|_{\mathcal{M}^{p_i}_{q_i}}.$$

Hence we have $R^{\frac{d}{q}-\sum_{i=1}^m \frac{d}{q_i}} \le 1$. Since $R > 0$ is arbitrary, we conclude that $\sum_{i=1}^m \frac{1}{q_i} = \frac{1}{q}$.

Now, take an arbitrary $K \in \mathbb{N}$ with $K \gg 1$. Define $h_j(x) := \chi_{\{j \le |x| \le j + j^{-1/2}\}}(x)$ with $j \in \mathbb{N}$, and consider $f_i(x) := \chi_{\{0 \le |x| < 1\}}(x) + \sum_{j=1}^{K} h_j(x)$ for $i = 1, \ldots, m$. We observe that

$$\begin{aligned} \left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_q^p} &\geq |B(0, K + K^{-1/2})|^{\frac{1}{q} - \frac{1}{p}} \left(\int_{B(0,K+K^{-1/2})} |\prod_{i=1}^{m} f_i(x)|^p \, dx \right)^{\frac{1}{p}} \\ &\geq C(K + K^{-1/2})^{\frac{d}{q} - \frac{d}{p}} (K + K^{-1/2})^{\frac{d}{p} - \frac{1}{2p}} = C(K + K^{-1/2})^{\frac{d}{q} - \frac{1}{2p}}. \end{aligned}$$

Meanwhile, for $i = 1, 2, \ldots, m$, we obtain

$$\begin{aligned} \|f_i\|_{\mathcal{M}_{q_i}^{p_i}} &\leq C |B(0, K + K^{-1/2})|^{\frac{1}{q_i} - \frac{1}{p_i}} \left(\int_{B(0,K+K^{-1/2})} |f_i(x)|^{p_i} \, dx \right)^{\frac{1}{p_i}} \\ &\leq C(K + K^{-1/2})^{\frac{d}{q_i} - \frac{d}{p_i}} (K + K^{-1/2})^{\frac{d}{p_i} - \frac{1}{2p_i}} = C(K + K^{-1/2})^{\frac{d}{q_i} - \frac{1}{2p_i}}. \end{aligned}$$

Because $\sum_{i=1}^{m} \frac{d}{q_i} = \frac{d}{q}$ and $\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_q^p} \leq \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}_{q_i}^{p_i}}$, we have $(K + K^{-1/2})^{-\frac{1}{2p} + \sum_{i=1}^{m} \frac{1}{2p_i}} \leq C$, for arbitrary $K > 1$. Thus $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$, as desired. □

`Remark`. For $m = 2$, we obtain the proof of Theorem 1.1.

3.2 The proof of Theorem 2.2

Proof. If (1) holds, then by similar arguments as in [10] we can prove that (1) implies (2). It thus remains to prove that (2) implies (1). To do so, take an aribtrary $R > 0$ and let $f_i := \chi_{B(0,R)}$ for $i = 1, \ldots, m$. We then have

$$\|\chi_{B(0,R)}\|_{w\mathcal{M}_q^p} = \left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_q^p} \leq m \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{w\mathcal{M}_{q_i}^{p_i}}.$$

Hence $R^{\frac{d}{q} - \sum_{i=1}^{m} \frac{d}{q_i}} \leq m$. Since this holds for every $R > 0$, it follows that $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.

Now, take an aribtrary $K \in \mathbb{N}$ with $K \gg 1$. Define $h_j(x) := \chi_{\{j \le |x| \le j + j^{-1/2}\}}(x)$ with $j \in \mathbb{N}$. As before, let $f_i(x) := \chi_{\{0 \le |x| < 1\}}(x) + \sum_{j=1}^{K} h_j(x)$, for $i = 1, \ldots, m$. Then we have

$$\begin{aligned} \left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_q^p} &\geq |B(0, K + K^{-1/2})|^{\frac{1}{q} - \frac{1}{p}} \frac{1}{2} \left(\left| \{x \in B(a, r) \colon |f_i(x)| > \frac{1}{2}\} \right|^{\frac{1}{p}} \right) \\ &\geq C(K + K^{-1/2})^{\frac{d}{q} - \frac{d}{p}} (K + K^{-1/2})^{\frac{d}{p} - \frac{d}{2p}} = C(K + K^{-1/2})^{\frac{d}{q} - \frac{1}{2p}}. \end{aligned}$$

Next, we use Lemma 1.2 to obtain $\|f_i\|_{w\mathcal{M}^{p_i, q_i}} \leq C(K + K^{-1/2})^{\frac{n}{q_i} - \frac{1}{2p_i}}$ for $i = 1, 2, \ldots, m$. Since $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ and $\left\| \prod_{i=1}^{m} f_i \right\|_{w\mathcal{M}_q^p} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}_{q_i}^{p_i}}$, we have

$$(K + K^{-1/2})^{-\frac{1}{2p} + \sum_{i=1}^{m} \frac{1}{2p_i}} \leq Cm,$$

for arbitrary $K > 1$. Hence we must have $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$. □

3.3 The proof of Theorem 2.3

Proof. $(1 \Rightarrow 2)$ Suppose that $\sum_{i=1}^{m} \frac{1}{p_i} \le \frac{1}{p}$ and $\prod\limits_{i=1}^{m} \phi_i(r) \le \phi(r)$, for every $r > 0$. Take arbitrary $B = B(a,R) \subseteq \mathbb{R}^d$ and write $f_i \in \mathcal{M}_{\phi_i}^{p_i}\left(\mathbb{R}^d\right)$ for $i = 1, \dots, m$. We consider two cases.

Case 1: For $\sum_{i=1}^{m} \frac{1}{p_i} = \frac{1}{p}$, we use the generalized Hölder's inequality on Lebesgue spaces [2] to get

$$\frac{1}{\phi(R)} \left(\frac{\int\limits_B \prod\limits_{i=1}^{m} |f_i(x)|^p dx}{|B|} \right)^{\frac{1}{p}} \le \prod_{i=1}^{m} \frac{1}{\phi_i(R)} \left(\frac{\int\limits_B |f_i(x)|^{p_i} dx}{|B|} \right)^{\frac{1}{p_i}}.$$

We then take the supremum over B to obtain $\left\| \prod\limits_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \le \prod\limits_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$.

Case 2: For $\sum_{i=1}^{m} \frac{1}{p_i} < \frac{1}{p}$, we define $\frac{1}{p^*} = \sum_{i=1}^{m} \frac{1}{p_i}$. Using the generalized Hölder's inequality on Lebesgue spaces, we have

$$\left(\frac{\int\limits_B \prod\limits_{i=1}^{m} |f_i(x)|^p dx}{\phi^p(R)|B|} \right)^{\frac{1}{p}} \le \left(\frac{\int\limits_B \prod\limits_{i=1}^{m} |f_i(x)|^{p^*} dx}{\phi^{p^*}(R)|B|} \right)^{\frac{1}{p^*}} \le \prod_{i=1}^{m} \left(\frac{\int\limits_B |f_i(x)|^{p_i} dx}{\phi^{p_i}(R)|B|} \right)^{\frac{1}{p_i}}.$$

We can now take the supremum over B to obtain $\left\| \prod\limits_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \le \prod\limits_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$.

$(2 \Rightarrow 1)$ Let $\left\| \prod\limits_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \le \prod\limits_{i=1}^{m} \|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}}$ for every $f_i \in \mathcal{M}_{\phi_i}^{p_i}(\mathbb{R}^d)$ where $i = 1, \dots, m$. Take an arbitrary $R > 0$ and define $f_i := \chi_{B(0,R)}$ for $i = 1, \dots, m$. We have

$$\frac{1}{\phi(R)} = \|\chi_{B(0,R)}\|_{\mathcal{M}_\phi^p} \le \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{\mathcal{M}_{\phi_i}^{p_i}} = \prod_{i=1}^{m} \frac{1}{\phi_i(R)}.$$

Thus $\prod\limits_{i=1}^{m} \phi_i(R) \le \phi(R)$.

Now, take an arbitrary $K \in \mathbb{N}$ with $K \gg 1$. Choose $C_1 > 0$ such that $C_1 \phi^p(j) < 1$ for every $j \in \mathbb{N}$. Now, we define $h_j(x) := \chi_{\{j \le |x| \le j + C_1 j^{-1/2} \phi^p(j)\}}(x)$ where $j \in \mathbb{N}$. Next, let $f_i(x) := \chi_{\{0 \le |x| < 2\}}(x) + \sum_{j=2}^{K} h_j(x)$, for $i = 1, \dots, m$. It is easy to check that

$$\left\| \prod_{i=1}^{m} f_i \right\|_{\mathcal{M}_\phi^p} \ge \frac{1}{\phi\left(K + C_1 K^{-1/2} \phi^p(K)\right)} \left(\frac{\int\limits_{B(0, K + C_1 K^{-1/2} \phi^p(K))} |f_i(x)|^p \, dx}{|B(0, K + C_1 K^{-1/2} \phi^p(K))|} \right)^{\frac{1}{p}} \ge C\, K^{-\frac{1}{2p}}.$$

Meanwhile, for $i = 1, 2, \dots, m$, we also observe that

$$\|f_i\|_{\mathcal{M}_{\phi_i}^{p_i}} \le \frac{1}{\phi_i\left(K + C_1 K^{-1/2} \phi^p(K)\right)} \left(\frac{\int\limits_{B(0, K + C_1 K^{-1/2} \phi^p(K))} |f_i(x)|^{p_i} \, dx}{|B(0, K + C_1 K^{-1/2} \phi^p(K))|} \right)^{\frac{1}{p_i}} \le C \, \frac{\phi^{\frac{p}{p_i}}(K)}{K^{\frac{1}{2p_i}} \phi_i(K)}.$$

Because $\left\|\prod_{i=1}^{m} f_i\right\|_{\mathcal{M}^p_\phi} \le \prod_{i=1}^{m} \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}}$ and $\prod_{i=1}^{m} \phi_i(K) \le \phi(K) \le \phi^b(K)$ for $K \gg 1$ and $0 < b \le 1$, we get $C\, K^{-\frac{1}{2p}+\sum_{i=1}^m \frac{1}{2p_i}} \le \frac{[\phi(K)]^{\sum_{i=1}^m p/p_i}}{\prod_{i=1}^{m} \phi_i(K)}$. Consequently, $\sum_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$, as desired. □

3.4 The proof of Theorem 2.4

Proof. ($1 \Rightarrow 2$) Suppose that $\sum_{i=1}^m \frac{1}{p_i} \le \frac{1}{p}$ and $\prod_{i=1}^{m} \phi_i(r) \le \phi(r)$, for every $r > 0$. Let $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$, where $i = 1, 2, \ldots, m$. Take an arbitrary $B = B(a, R) \subseteq \mathbb{R}^d$ and $\gamma > 0$. Define $A(\gamma, B) = \left[\frac{\gamma^p \left|\left\{x \in B : \prod_{i=1}^{m} \left|\frac{f_i(x)}{\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma\right\}\right|}{\phi^p(R)|B|}\right]^{\frac{1}{p}}$ and observe that

$$A(\gamma, B) \le \left[\frac{\gamma^p \left|\left\{x \in B : \prod_{i=1}^{m} \left|\frac{f_i(x)}{\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma\right\}\right|}{|B|(\prod_{i=1}^{m} \phi_i(R))^p}\right]^{\frac{1}{p}} = \left[\frac{\gamma_0^p \left|\left\{x \in B : \prod_{i=1}^{m} \left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma_0\right\}\right|}{|B|}\right]^{\frac{1}{p}},$$

where $\gamma_0 := \frac{\gamma}{\prod_{i=1}^{m} \phi_i(R)}$. Furthermore, by using Young's inequality, we have

$$\begin{aligned}
A(\gamma, B) &\le \left[\frac{\gamma_0^p |\{x \in B : \prod_{i=1}^{m} \left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right| > \gamma_0\}|}{|B|}\right]^{\frac{1}{p}} \\
&\le \left[\frac{\gamma_0^p \left|\left\{x \in B : \sum_{i=1}^{m} \frac{p}{p_i}\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right|^{\frac{p_i}{p}} > \gamma_0\right\}\right|}{|B|}\right]^{\frac{1}{p}} \\
&\le \left[\sum_{i=1}^{m} \frac{\gamma_0^p \left|\left\{x \in B : \frac{p}{p_i}\left|\frac{f_i(x)}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}}\right|^{\frac{p_i}{p}} > \frac{\gamma_0}{m}\right\}\right|}{|B|}\right]^{\frac{1}{p}} \\
&= \left[\sum_{i=1}^{m} \frac{\left(\frac{\gamma_i (mp)^{\frac{p}{p_i}}}{\phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}} (p_i)^{\frac{p}{p_i}}}\right)^{p_i} |\{x \in B : |f_i(x)| > \gamma_i\}|}{|B|}\right]^{\frac{1}{p}} \\
&\le \left[\sum_{i=1}^{m} \left(\frac{(mp)^{\frac{p}{p_i}}}{\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}} (p_i)^{\frac{p}{p_i}}}\right)^{p_i} \|f_i\|^{p_i}_{w\mathcal{M}^{p_i}_{\phi_i}}\right]^{\frac{1}{p}} \le m,
\end{aligned}$$

for $\gamma_i = \left(\frac{p_i\gamma_0}{mp}\right)^{\frac{p}{p_i}} \phi_i(R)\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$, where $i = 1, \ldots, m$. We then take the supremum over B and $\gamma > 0$ to obtain $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^{pm}_{\phi_m}} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$.

$(2 \Rightarrow 1)$ Let $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^{p}_{\phi}} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$, for every $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$ where $i = 1, \ldots, m$. Take an arbitrary $R > 0$ and define $f_i := \chi_{B(0,R)}$ for $i = 1, \ldots, m$. We observe that

$$\|\chi_{B(0,R)}\|_{w\mathcal{M}^{p}_{\phi}} \leq \prod_{i=1}^{m} \|\chi_{B(0,R)}\|_{w\mathcal{M}^{p_i}_{\phi_i}}.$$

By using Lemma 3.1, we get $\prod_{i=1}^{m} \phi_i(R) \leq \phi(R)$.

As before, take an arbitrary $K \in \mathbb{N}$ with $K \gg 1$, and define $h_j(x) := \chi_{\{j \leq |x| \leq j + j^{-1/2}\phi^p(j)\}}(x)$ with $j \in \mathbb{N}$. Now choose $f_i(x) := \chi_{\{0 \leq |x| < 2\}}(x) + \sum_{j=2}^{K} h_j(x)$, for $i = 1, \ldots, m$. Here we have

$$\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^{p}_{\phi}} \geq \left[\frac{(1/2)^p \left|\left\{x \in B(0, K + K^{-1/2}\phi^p(K)) : |f_i(x)| > 1/2\right\}\right|}{\phi^p(K + K^{-1/2}\phi^p(K))|B(0, K + K^{-1/2}\phi^p(K))|}\right]^{\frac{1}{p}} \geq C\, K^{-\frac{1}{2p}}.$$

Next, for $i = 1, \ldots, m$, we have $\|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}} \leq \|f_i\|_{\mathcal{M}^{p_i}_{\phi_i}} \leq C\, K^{-\frac{1}{2p_i}} \frac{\phi^{\frac{p}{p_i}}(K)}{\phi_i(K)}$. Since $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^{p}_{\phi}} \leq m \prod_{i=1}^{m} \|f_i\|_{w\mathcal{M}^{p_i}_{\phi_i}}$ and $\prod_{i=1}^{m} \phi_i(K) \leq \phi(K) \leq \phi^b(K)$ for $K \gg 1$ and $0 < b \leq 1$, it follows that

$$C\, K^{-\frac{1}{2p} + \sum_{i=1}^{m} \frac{1}{2p_i}} \leq \frac{m[\phi(K)]^{\sum_{i=1}^{m} \frac{p}{p_i}}}{\prod_{i=1}^{m} \phi_i(K)}.$$

We therefore conclude that $\sum_{i=1}^{m} \frac{1}{p_i} \leq \frac{1}{p}$. □

`Remark`. For $\phi_i(r) = r^{-\frac{n}{q_i}}$, where $i = 1, \ldots, m$, we get the proof of Theorem 2.2.

4 Concluding Remarks

We have shown the sufficient and necessary conditions for generalized Hölder's inequality on several spaces, namely Morrey spaces, generalized Morrey spaces, and their weak type versions. By Theorem 2.2 and Theorem 2.4, we obtain the following corollary.

Corollary 4.1. *For $m \geq 2$, the following statements are equivalent:*

(1) $\left\|\prod_{i=1}^{m} f_i\right\|_{\mathcal{M}^{p}_{\phi}} \leq \prod_{i=1}^{m} \| f_i \|_{\mathcal{M}^{p_i}_{\phi_i}}$, *for every $f_i \in \mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$ where $i = 1, \ldots, m$.*

(2) $\left\|\prod_{i=1}^{m} f_i\right\|_{w\mathcal{M}^{p}_{\phi}} \leq m \prod_{i=1}^{m} \| f_i \|_{w\mathcal{M}^{p_i}_{\phi_i}}$, *for every $f_i \in w\mathcal{M}^{p_i}_{\phi_i}(\mathbb{R}^d)$ where $i = 1, \ldots, m$.*

We prove this corollary via the parameters of these spaces as the sufficient and necessary conditions. Consequently, the generalized Hölder's inequality on Morrey spaces is equivalent to the generalized Hölder's inequality on weak Morrey spaces. Furthermore, the generalized Hölder's inequality on Lebegue spaces is also equivalent to the generalized Hölder's inequality on weak Lebesgue spaces.

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