An Inclusion Property of Orlicz-Morrey Spaces

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***Abstract*—The Orlicz-Morrey spaces $L_{\phi,\Phi}$ (where Φ is a Young function and ϕ is a parameter for the Morrey spaces) are generalizations of Orlicz spaces and Morrey spaces. Inclusion properties between Orlicz spaces L_Φ and between Morrey spaces $\mathcal{M}^p_\psi$ are well known. In this study we will investigate the inclusion relation between Orlicz-Morrey spaces L_{ϕ_1,Φ_1} and L_{ϕ_2,Φ_2} with respect to Young functions Φ_1, Φ_2 and parameters ϕ_1, ϕ_2. Also, we give sufficient and necessary conditions for the inclusion property of these spaces, which are obtained through norm estimates for the characteristic functions of balls in $\mathbb{R}^n$. In addition, we shall give a sufficient and necessary condition for generalized Hölder's inequality.**

I. Introduction

Orlicz-Morrey spaces are generalizations of Orlicz spaces and Morrey spaces. There are two versions of Orlicz-Morrey spaces. One is defined by Nakai [2], [9] and another by Sawano, Sugano, and Tanaka [2], [12]. Here we are interested in studying the inclusion property of Orlicz-Morrey spaces which were introduced by Nakai.

First, we recall the definiton of Young functions (see [11], [9]). A function $\Phi : [0,\infty) \to [0,\infty)$ is called a Young function if Φ is convex, left-continuous, $\lim_{t\to 0} = \Phi(t) = \Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Let G_1 be the set of all functions $\phi : (0,\infty) \to (0,\infty)$ such that $\phi(r)$ is nondecreasing but $\frac{\phi(r)}{r}$ is nonincreasing. For $\phi_1, \phi_2 \in G_1$, we write $\phi_1 \sim \phi_2$ if there exists a constant $C > 1$ such that

$$C^{-1}\phi_1(t) \le \phi_2(t) \le C\phi_1(t)$$

for all $t > 0$.

Let Φ be a Young function and $\phi \in G_1$. The Orlicz-Morrey space $L_{\phi,\Phi}(\mathbb{R}^n)$ is the set of measurable functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that for every open ball B in $\mathbb{R}^n$, the following

$$\| f \|_{(\phi,\Phi,B)} := \inf\{b > 0 : \frac{\phi(|B|)}{|B|} \int_B \Phi(\frac{|f(x)|}{b})dx \le 1\}$$

is finite. We use the notation $\mathfrak{B}$ to denote the family of all open balls B in $\mathbb{R}^n$, and $|B|$ for its Lebesgue measure. $L_{\phi,\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\| f \|_{L_{\phi,\Phi}(\mathbb{R}^n)} := \sup_{B\in\mathfrak{B}} \| f \|_{(\phi,\Phi,B)} .$$

For $\phi(t) = t$, $L_{\phi,\Phi}(\mathbb{R}^n) = L_\Phi(\mathbb{R}^n)$ is the Orlicz space. Meanwhile, for $\phi(t) = \frac{1}{\psi(t^{1/n})^p}$ (where $\psi : (0,\infty) \to (0,\infty)$ is almost decreasing and $t^{\frac{n}{p}}\psi(t)$ is almost increasing) and $\Phi(t) = t^p$ we have $L_{\phi,\Phi}(\mathbb{R}^n) = \mathcal{M}^p_\psi(\mathbb{R}^n)$, the generalized Morrey space introduced by Nakai in 1994.

Gunawan *et al.* [3] have proved an inclusion property of generalized Morrey spaces: If $1 \le p_1 \le p_2 < \infty$, then $\mathcal{M}^{p_2}_{\psi_2} \subset \mathcal{M}^{p_1}_{\psi_1}$ if and only if $\psi_2(t) \le C\psi_1(t)$ for all $t > 0$ and some $C > 0$. On the other hand, inclusion properties between Orlicz spaces $L_\Phi(\mathbb{R}^n)$ and between weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ are well known (see [5], [6]).

Motivated by these results, the purpose of this study is to get the inclusion property of Orlicz-Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$.

The main results are presented in Section 2. In particular, Theorem II.2 contains a necessary and sufficient condition for the inclusion relation between Orlicz-Morrey spaces. In Section 3, we have also given sufficent and necessary conditions for generalized Hölder's inequality.

To prove the results, we will use the same method as in [3] and [6] which pay attention to the characteristic functions of balls in $\mathbb{R}^n$ and use the inverse function of Φ, namely $\Phi^{-1}(s) := \inf\{r \ge 0 : \Phi(r) > s\}$.

In the following, we recall several lemmas which will be used in the next section.

Lemma I.1. *[11], [6] Suppose that Φ is a Young function and $\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\}$. We have*
(1) $\Phi^{-1}(0) = 0$.
(2) $\Phi^{-1}(s_1) \le \Phi^{-1}(s_2)$ *for* $s_1 \le s_2$.
(3) $\Phi(\Phi^{-1}(s)) \le s \le \Phi^{-1}(\Phi(s))$ *for* $0 \le s < \infty$.
(4) *Let $C > 0$ be a positive real number. Then $\Phi_1(t) \le \Phi_2(Ct)$ if and only if $C\Phi_1^{-1}(t) \ge \Phi_2^{-1}(t)$, for every $t \ge 0$.*
(5) *Let $C > 0$ be a positive real number. Then $\Phi_1(t) \le C\Phi_2(t)$ if and only if $\Phi_1^{-1}(Ct) \ge \Phi_2^{-1}(t)$, for every $t \ge 0$.*

Lemma I.2. *[2] Let Φ be a Young function, $a \in \mathbb{R}^n$ and $r > 0$. Then $\| \chi_{B(a,r)} \|_{L_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r)|)})}$, where $|B(a,r)|$ denotes the volume of $B(a,r)$.*

Lemma I.3. *If $f \in L_{\phi,\Phi}(\mathbb{R}^n)$, then*

$$\frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\| f \|_{(\phi,\Phi,B)}}\right)dx \le 1$$

for any open ball $B \in \mathfrak{B}$. Furthermore, $\| f \|_{(\phi,\Phi,B)} \le 1$ if and only if $\frac{\phi(|B|)}{|B|}\int_B \Phi(|f(x)|)dx \le 1$ for any open ball $B \in \mathfrak{B}$.

Proof.

Let f be an element of $L_{\phi,\Phi}(\mathbb{R}^n)$ and take an arbitrary $\epsilon > 0$, then there exists $b_\epsilon > 0$ such that $b_\epsilon \le \| f \|_{(\phi,\Phi,B)} + \epsilon$ and $\frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{b_\epsilon}\right)dx \le 1$ for any open ball $B \in \mathfrak{B}$. Because, Φ is increasing, we have

$$\frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\|f\|_{(\phi,\Phi,B)}+\epsilon}\right)dx \le \int_B \Phi\left(\frac{|f(x)|}{b_\epsilon}\right)dx \le 1.$$

Since $\epsilon > 0$ is arbitrary, we can conclude

$$\frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\| f \|_{(\phi,\Phi,B)}}\right)dx \le 1$$

for any open ball $B \in \mathfrak{B}$.

Next, if $\| f \|_{(\phi,\Phi,B)} \le 1$ for any open ball $B \in \mathfrak{B}$, then $\frac{\phi(|B|)}{|B|}\int_B \Phi(|f(x)|)dx \le \frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\|f\|_{(\phi,\Phi,B)}}\right)dx \le 1$. Now, assume that $\frac{\phi(|B|)}{|B|}\int_B \Phi(|f(x)|)dx \le 1$ holds for any open ball $B \in \mathfrak{B}$. Clearly, we have $\| f \|_{(\phi,\Phi,B)} \le 1$. □

Corollary I.4. *If $f \in L_{\phi,\Phi}(\mathbb{R}^n)$, then*

$$\frac{\phi(|B|)}{|B|}\int_B \Phi\left(\frac{|f(x)|}{\| f \|_{L_{\phi,\Phi}(\mathbb{R}^n)}}\right)dx \le 1$$

for any open ball $B \in \mathfrak{B}$. Furthermore, $\| f \|_{L_{\phi,\Phi}(\mathbb{R}^n)} \le 1$ if and only if $\frac{\phi(|B|)}{|B|}\int_B \Phi(|f(x)|)dx \le 1$ for any open ball $B \in \mathfrak{B}$.

II. Main Results

Now, we come to the inclusion relation between Orlicz-Morrey space L_{ϕ_1,Φ_1} and L_{ϕ_2,Φ_2} with respect to Young functions Φ_1, Φ_2 and parameters ϕ_1, ϕ_2.

Theorem II.1. *[7] Let Φ_1, Φ_2 be Young functions and $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \sim \phi_2$. If $\Phi_1(x) \le \Phi_2(Kx)$ for some $K > 0$ then $L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n)$.*

We can prove the Lemma II.1 by using similar arguments in the proof of [9, Proposition 3.2].

Theorem II.2. *Let Φ_1, Φ_2 be Young functions and $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \sim \phi_2$. Then the following statements are equivalent:*
(1) *$\Phi_1(t) \le \Phi_2(Ct)$, for every $t > 0$.*
(2) *$L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n)$.*
(3) *There exists a constant $C > 0$ such that*

$$\| f \|_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \le C \| f \|_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)},$$

for every $f \in L_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Proof.

Assume that (1) holds. Let f be an element of $L_{\phi_2,\Phi_2}(\mathbb{R}^n)$. Since $\phi_1 \sim \phi_2$ and $\Phi_1(t) \le \Phi_2(Ct)$, for every $t > 0$, by Theorem II.1, we have $L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n)$. Next, since $(L_{\phi_2,\Phi_2}(\mathbb{R}^n), L_{\phi_1,\Phi_1}(\mathbb{R}^n))$ is a Banach pair, it follows from [4, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1). Assume that (3) holds. By Lemma I.2, for every $r_0 > 0$ we have

$$\begin{aligned}\frac{1}{\Phi_1^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)})} &= \| \chi_{B(a,r_0)} \|_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \\ &\le C \| \chi_{B(a,r_0)} \|_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)} \\ &= \frac{C}{\Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})}\end{aligned}$$

or $C\Phi_1^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)}) \ge \Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})$. By Lemma I.1 (4) and $\phi_1 \sim \phi_2$, we have

$$\Phi_1(\tfrac{1}{\phi_1(|B(a,r_0)|)}) \le \Phi_2(\tfrac{C}{\phi_2(|B(a,r_0)|)}) \le \Phi_2(\tfrac{CK}{\phi_1(|B(a,r_0)|)})$$

for $K > 1$. Since r_0 is an arbitrary positive real number, we get $\Phi_1(t) \le \Phi_2(Ct)$, for every $t > 0$. □

Corollary II.3. *[6] Let Φ_1, Φ_2 be Young functions and $\phi_1(t) = \phi_2(t) = t$ for $t > 0$. Then the following statements are equivalent:*
(1) *$\Phi_1(t) \le \Phi_2(Ct)$, for every $t > 0$.*
(2) *$L_{\Phi_2}(\mathbb{R}^n) \subseteq L_{\Phi_1}(\mathbb{R}^n)$.*
(3) *There exists a constant $C > 0$ such that*

$$\| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \le C \| f \|_{L_{\Phi_2}(\mathbb{R}^n)}$$

for every $f \in L_{\Phi_2}(\mathbb{R}^n)$.

`Remark`. Corollary II.3 refines Corollary 2.11 in [10] which only states that (1) implies (2) for $w_1(x) = w_2(x) = 1$ and $X = \mathbb{R}^n$.

III. Generalized Hölder Inequality

In the following, we will give sufficient and necessary conditions for generalized Hölder inequality on Orlicz-Morrey spaces. To get the result we will give attention to estimate the norm of the characteristic function of ball in $\mathbb{R}^n$.

Theorem III.1. *For $m \ge 2$. Let Φ_i be Young functions and $\phi_i \in G_1$, for $i = 1, 2, 3, ..., m$. If there exists a constant $C > 0$ such that $\prod_{i=1}^{m-1} \Phi_i^{-1}(\frac{s}{\phi_i(t)}) \le C\Phi_m^{-1}(\frac{s}{\phi_m(t)})$ for $s, t > 0$, then for $f_i \in L_{\phi_i,\Phi_i}(\mathbb{R}^n)$, $i = 1, 2, 3, ..., m-1$ we have*

$$\prod_{i=1}^{m-1} f_i \in L_{\phi_m,\Phi_m}(\mathbb{R}^n)$$

with

$$\| \prod_{i=1}^{m-1} f_i \|_{L_{\phi_m,\Phi_m}(\mathbb{R}^n)} \le (m-1)C \prod_{i=1}^{m-1} \| f_i \|_{L_{\phi_i,\Phi_i}(\mathbb{R}^n)} .$$

Proof.

Let f_i be an element of $L_{\phi_i,\Phi_i}(\mathbb{R}^n)$ for $i=1,2,3,...,m-1$. Then

$$\frac{\phi_i(|B|)}{|B|}\int_B \Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}})dx \le 1$$

for any ball $B\in\mathfrak{B}$ and $i=1,2,3,...,m-1$. Now fix B. For each $x\in B$, let

$$M(x)=\max\left\{\phi_i(|B|)\Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}}) : 1\le i\le m-1\right\}.$$

From $\Phi_i(\frac{|f_i(x)|}{\|f_i\|_{(\phi_i,\Phi_i,B)}})\le\frac{M(x)}{\phi_i(|B|)}$ and Lemma I.1(3), we have

$$\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}}\le\Phi_i^{-1}(\Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}}))\le\Phi_i^{-1}(\frac{M(x)}{\phi_i(|B|)})$$

for $i=1,2,3,...,m-1$. Hence

$$\prod_{i=1}^{m-1}\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}}\le\prod_{i=1}^{m-1}\Phi_i^{-1}(\frac{M(x)}{\phi_i(|B|)})\le C\Phi_m^{-1}(\frac{M(x)}{\phi_m(|B|)})$$

and

$$\Phi_m(\frac{1}{(m-1)C}\prod_{i=1}^{m-1}\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}})\le\frac{\Phi_m(\Phi_m^{-1}(\frac{M(x)}{\phi_m(|B|)}))}{(m-1)}$$
$$\le\frac{M(x)}{(m-1)\phi_m(|B|)}.$$

On the other hand, we have

$$\frac{M(x)}{(m-1)\phi_m(|B|)}\le\frac{\sum_{i=1}^{m-1}\phi_i(|B|)\Phi_i(\frac{|f_i(x)|}{\|f_i\|_{(\phi_i,\Phi_i,B)}})}{(m-1)\phi_m(|B|)}$$

and

$$\sum_{i=1}^{m-1}\phi_i(|B|)\int_B\Phi_i(\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}})dx\le(m-1)|B|.$$

Therefore

$$\int_B\Phi_m(\frac{1}{(m-1)C}\prod_{i=1}^{m-1}\frac{|f_i(x)|}{\| f_i \|_{(\phi_i,\Phi_i,B)}})dx\le\frac{(m-1)|B|}{(m-1)\phi_m(|B|)}$$
$$=\frac{|B|}{\phi_m(|B|)}.$$

This shows that

$$\| \prod_{i=1}^{m-1} f_i \|_{(\phi_m,\Phi_m,B)}\le(m-1)C\prod_{i=1}^{m-1}\| f_i \|_{(\phi_i,\Phi_i,B)}$$

for every open ball $B\in\mathfrak{B}$. Hence, we conclude that

$$\| \prod_{i=1}^{m-1} f_i \|_{L_{\phi_m,\Phi_m}(\mathbb{R}^n)}\le(m-1)C\prod_{i=1}^{m-1}\| f_i \|_{L_{\phi_i,\Phi_i}(\mathbb{R}^n)}.$$

□

Remark III.2. For $m=3$, Theorem III.1 reduces to Theorem 4.1 (p.200) in [7].

Corollary III.3. *(Hölder's inequality on Orlicz spaces) Let Φ_i be Young functions, $i=1,2,3$. If there exists a constant $C>0$ such that $\Phi_1^{-1}(t)\Phi_2^{-1}(t)\le C\Phi_3^{-1}(t)$ for $t>0$, then for $f\in L_{\Phi_1}(\mathbb{R}^n)$ and $g\in L_{\Phi_2}(\mathbb{R}^n)$ we have $f\cdot g\in L_{\Phi_3}(\mathbb{R}^n)$ with*

$$\| f\cdot g \|_{L_{\Phi_3}(\mathbb{R}^n)}\le 2C\| f \|_{L_{\Phi_1}(\mathbb{R}^n)}\| g \|_{L_{\Phi_2}(\mathbb{R}^n)}.$$

Theorem III.4. *For $m\ge 2$. Let Φ_i be Young functions and $\phi_i\in G_1$, for $i=1,2,3,...,m$. If there exists a constant $K>0$ such that $\| \prod_{i=1}^{m-1} f_i \|_{L_{\frac{\phi_m}{s},\Phi_m}(\mathbb{R}^n)}\le K\prod_{i=1}^{m-1}\| f_i \|_{L_{\frac{\phi_i}{s},\Phi_i}(\mathbb{R}^n)}$ for $f_i\in L_{\frac{\phi_i}{s},\Phi_i}(\mathbb{R}^n)$, $i=1,2,3,...,m-1$, then we have $\prod_{i=1}^{m-1}\Phi_i^{-1}(\frac{s}{\phi_i(t)})\le K\Phi_m^{-1}(\frac{s}{\phi_m(t)})$ for every $s,t>0$.*

Proof.

Observe that, for every open ball $B(a,r)$ we have

$$\| \chi_{B(a,r)} \|_{L_{\frac{\phi_m}{s},\Phi_m}(\mathbb{R}^n)}=\| \prod_{i=1}^{m-1}\chi_{B(a,r)} \|_{L_{\frac{\phi_m}{s},\Phi_m}(\mathbb{R}^n)}$$
$$\le K\prod_{i=1}^{m-1}\| \chi_{B(a,r)} \|_{L_{\frac{\phi_i}{s},\Phi_i}(\mathbb{R}^n)}.$$

By Lemma I.2 we have

$$\frac{1}{\Phi_m^{-1}(\frac{s}{\phi_m(|B(a,r)|)})}\le\frac{K}{\prod_{i=1}^{m-1}\Phi_i^{-1}(\frac{s}{\phi_i(|B(a,r)|)})}$$

or

$$\prod_{i=1}^{m-1}\Phi_i^{-1}(\frac{s}{\phi_i(|B(a,r)|)})\le K\Phi_m^{-1}(\frac{s}{\phi_m(|B(a,r)|)}).$$

Since $B(a,r)$ is arbitrary, we get

$$\prod_{i=1}^{m-1}\Phi_i^{-1}(\frac{s}{\phi_i(t)})\le K\Phi_m^{-1}(\frac{s}{\phi_m(t)})$$

for $s,t>0$. □

Corollary III.5. *Let $p_i\ge 1$ be real numbers, $i=1,2,3,...,m$. If $f_i\in L_{p_i}(\mathbb{R}^n)$, $i=1,2,3,...,m$ such that*

$$\| \prod_{i=1}^{m} f_i \|_{L_1(\mathbb{R}^n)}\le\prod_{i=1}^{m}\| f_i \|_{L_{p_i}(\mathbb{R}^n)},$$

then $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$.

Proof.

By Theorem III.4, we have $|x|^{\frac{1}{p_1}}|x|^{\frac{1}{p_2}}|x|^{\frac{1}{p_3}}|x|^{\frac{1}{p_m}}\le|x|$ for every $x>0$. Since x is an arbitrary positive real number, we conclude that $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$. □

`Remark`. Corollary III.5 completes Theorem 2.11 in [1]. Theorem 2.1 states that $\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_m}=1$ is a sufficient condition for generalized Hölder's inequality on Lebesgue spaces, while Corollary III.5 states that it is also a necessary condition.

Corollary III.6. *Let Φ_1, Φ_2, Φ_3 be Young functions. If there exists a constant $C > 0$ such that*

$$\| f \cdot g \|_{L_{\Phi_3}(\mathbb{R}^n)} \leq C \| f \|_{L_{\Phi_1}(\mathbb{R}^n)} \| g \|_{L_{\Phi_2}(\mathbb{R}^n)}$$

for $f \in L_{\Phi_1}(\mathbb{R}^n)$ and $g \in L_{\Phi_2}(\mathbb{R}^n)$, then

$$\Phi_1^{-1}(t)\Phi_2^{-1}(t) \leq C\Phi_3^{-1}(t)$$

for every $t > 0$.

For each Young function Φ, we can define another convex function $\tilde{\Phi} : \mathbb{R} \to (0, \infty)$ having similiar properties, by $\tilde{\Phi}(y) = \sup_{x>0} \{x|y| - \Phi(x)\}$, $y \in \mathbb{R}$. Then $\tilde{\Phi}$ is called the complementary function to Φ.

Corollary III.7. *If Φ is Young function and $\tilde{\Phi}$ is complementary function of Φ, then*

$$\Phi^{-1}(t)\tilde{\Phi}^{-1}(t) \leq 2t$$

for every $t > 0$.

Proof.

It follows from [7, Theorem 2.3] that, for any $f \in L_{\Phi}(\mathbb{R}^n)$ and $g \in L_{\tilde{\Phi}}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |f \cdot g| dx \leq 2 \| f \|_{L_{\Phi}(\mathbb{R}^n)} \| g \|_{L_{\tilde{\Phi}}(\mathbb{R}^n)} .$$

By Corollary III.5, we have $\Phi^{-1}(t)\tilde{\Phi}^{-1}(t) \leq 2t$. □

`Remark`. We can prove the Corollary III.7 directly by definition of Φ^{-1} and $\tilde{\Phi}^{-1}$. What we showed here is that we can obtain the result through the lens of Orlicz-Morrey spaces.

IV. Conclusion

In this paper, we have discussed the inclusion relation between Orlicz-Morrey spaces for Nakai version. By estimating the norm of characteristic function of balls in $\mathbb{R}^n$, we obtain sufficient and necessary conditions for inclusion relation between Orlicz-Morrey spaces (Theorem II.2), which were generalized for inclusion property of Orlicz spaces in [6]. Furthermore, by estimating the characteristic function of balls in $\mathbb{R}^n$ we obtain sufficient and necessary conditions for generalized Hölder's inequality on Orlicz-Morrey spaces (Theorem III.1 and Theorem III.4). Theorem III.4 generalized Hölder's inequality in [7].

Acknowledgment

The first and second authors are supported by ITB Research and Innovation Program 2016.

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