Comparison of Inclusion Properties of Two Versions of Orlicz-Morrey Spaces

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Abstract

There are two versions of Orlicz-Morrey spaces as defined by Nakai in 2004 and by Sawano, Sugano, and Tanaka in 2012. In this paper we discuss the comparison of inclusion properties of these two spaces. In the proof of our results, we compute the norm of the characteristic functions of balls in $\mathbb{R}^n$.

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1 Introduction

Orlicz-Morrey spaces are generalizations of Orlicz spaces and Morrey spaces. There are two versions of Orlicz-Morrey spaces. One is defined by Nakai [2, 9] and another by Sawano, Sugano, and Tanaka [2, 13].

Let $\Phi : [0, \infty) \to [0, \infty)$ be a Young function, that is, Φ is convex, left-continuous, $\Phi(0) = 0$, and $\lim_{t\to\infty} \Phi(t) = \infty$. Let Φ, Ψ be Young functions, we denote $\Phi \approx \Psi$ if there exists a constant $C > 0$ such that $\Phi(C^{-1}t) \leq \Psi(t) \leq \Phi(Ct)$ for $t > 0$. For Φ is Young function, we define the inverse function of Φ, namely $\Phi^{-1}(s) := \inf\{r \geq 0 : \Phi(r) > s\}$, for every $s \geq 0$.

Let G_1 be the set of all functions $\phi : (0, \infty) \to (0, \infty)$ such that $\phi(r)$ is nondecreasing but $\frac{\phi(r)}{r}$ is nonincreasing. For Ψ is Young function, let G_2 be the set of all functions $\psi : (0, \infty) \to (0, \infty)$ such that $\psi(r)$ is nondecreasing but $\frac{\psi((r+s)^n)}{\Psi^{-1}((\frac{s+t}{s})^n)}$ is nonincreasing for every $s > 0$. If $\phi_1, \phi_2 \in G_1$, we denote $\phi_1 \sim \phi_2$ if there exists a constant $C > 0$ such that $C^{-1}\phi_1(t) \leq \phi_2(t) \leq C\phi_1(t)$ for $t > 0$.

Let Φ be a Young function and $\phi \in G_1$. The Orlicz-Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$ (Nakai version) is the set of measurable functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that for every $a \in \mathbb{R}^n$ and $r > 0$, the following

$$\parallel f \parallel_{(\phi,\Phi,B(a,r))} := \inf\Big\{b > 0 : \frac{\phi(|B(a,r)|)}{|B(a,r)|} \int_{B(a,r)} \Phi\Big(\frac{|f(x)|}{b}\Big)dx \leq 1\Big\}$$

is finite. We use the notation $B(a,r)$ to denote the open ball in $\mathbb{R}^n$ centered at a and radius r, and $|B(a,r)|$ for its Lebesgue measure. $L_{\phi,\Phi}(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\parallel f \parallel_{L_{\phi,\Phi}(\mathbb{R}^n)} := \sup_{r>0, a\in\mathbb{R}^n} \parallel f \parallel_{(\phi,\Phi,B(a,r))} .$$

The space $L_{\phi,\Phi}(\mathbb{R}^n)$ generalizes Orlicz space $L_\Phi(\mathbb{R}^n)$ where $\phi(r) = r$. On the other hand, the space $L_{\phi,\Phi}(\mathbb{R}^n)$ generalizes Morrey space $L_{p,\lambda}(\mathbb{R}^n)$ where $\Phi(r) = r^p$ and $\phi(r) = r^{1-\frac{\lambda}{n}}$ for $0 \leq \lambda \leq n$.

Now, let Ψ be a Young function and $\psi \in G_2$. Sawano, Sugano, and Tanaka defined the Orlicz-Morrey space $M_{\psi,\Psi}(\mathbb{R}^n)$ be the set of measurable functions $f \in L^1_{loc}(\mathbb{R}^n)$ such that

$$\parallel f \parallel_{M_{\psi,\Psi}(\mathbb{R}^n)} := \sup_{r>0, a\in\mathbb{R}^n} \psi(|B(a,r)|) \parallel f \parallel_{(\Psi,B(a,r))} < \infty,$$

where $\parallel f \parallel_{(\Psi,B(a,r))} := \inf\Big\{b > 0 : \frac{1}{|B(a,r)|} \int_{B(a,r)} \Psi\Big(\frac{|f(x)|}{b}\Big)dx \leq 1\Big\}$.

In 2014, Gunawan, et al. [3] have gave sufficient and necessary condition for inclusion properties of generalized Morrey and generalized weak Morrey spaces in the following theorem.

Teorema 1.1. *Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:*

(a) $\phi_2 \leq C\phi_1$.

(b) $\mathcal{M}^{p_2}_{\phi_2} \subset \mathcal{M}^{p_1}_{\phi_1}$.

(c)*For every $f \in \mathcal{M}^{p_2}_{\phi_2}$, we have $\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}$.*

(d) $w\mathcal{M}^{p_2}_{\phi_2} \subset w\mathcal{M}^{p_1}_{\phi_1}$.

(e) *For every $f \in w\mathcal{M}^{p_2}_{\phi_2}$, we have $\|f\|_{w\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$.*

On the other hand, inclusion properties between Orlicz spaces $L_\Phi(\mathbb{R}^n)$ and between weak Orlicz spaces $wL_\Phi(\mathbb{R}^n)$ are well known (see [5, 6]).

In 2016, Masta, et al. [7] have proved the inclusion properties for Orlicz-Morrey space $L_{\phi,\Phi}(\mathbb{R}^n)$, in the following theorem.

Teorema 1.2. *Let Φ_1, Φ_2 be Young functions and $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \sim \phi_2$. Then the following statements are equivalent:*

(1) *$\Phi_1(t) \leq \Phi_2(Ct)$, for every $t > 0$.*

(2) *$L_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq L_{\phi_1,\Phi_1}(\mathbb{R}^n)$.*

(3) *There exists a constant $C > 0$ such that*

$$\parallel f \parallel_{L_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C \parallel f \parallel_{L_{\phi_2,\Phi_2}(\mathbb{R}^n)},$$

for every $f \in L_{\phi_2,\Phi_2}(\mathbb{R}^n)$.

Motivated by these results, the purpose of this study is to get the comparison of inclusion property of Orlicz-Morrey spaces $L_{\phi,\Phi}(\mathbb{R}^n)$ and $M_{\phi,\Phi}(\mathbb{R}^n)$. For another result we will extend the results to weak Orlicz-Morrey spaces.

To prove the results, we will use the similar method in [3, 6, 7] which pay attention to the characteristic functions of balls in $\mathbb{R}^n$ and use the inverse function of Φ. The reader will find the following lemma useful.

Lemma 1.3. *[7, 9, 12] Suppose that Φ is a Young function and $\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}$. We have*

(1) $\Phi^{-1}(0) = 0$.

(2) $\Phi^{-1}(s_1) \leq \Phi^{-1}(s_2)$ *for* $s_1 \leq s_2$.

(3) $\Phi(\Phi^{-1}(s)) \leq s \leq \Phi^{-1}(\Phi(s))$ *for* $0 \leq s < \infty$.

(4) *Let* $C > 0$. *Then* $\Phi_1(t) \leq \Phi_2(Ct)$ *if only if* $C\Phi_1^{-1}(t) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.

(5) *Let* $C > 0$. *Then* $\Phi_1(t) \leq C\Phi_2(t)$ *if only if* $\Phi_1^{-1}(Ct) \geq \Phi_2^{-1}(t)$, *for every* $t \geq 0$.

More lemmas (and their proofs) will be presented in the next sections.

2 An Inclusion Property Of Orlicz-Morrey Spaces

In [7], we have proved the inclusion relation between Orlicz-Morrey space $L_{\Phi_1,\phi_1}(\mathbb{R}^n)$ and $L_{\Phi_2,\phi_2}(\mathbb{R}^n)$ with respect to Young function Φ_1, Φ_2 and parameters ϕ_1, ϕ_2 (Theorem 1.2).

Remark 2.1. For $\phi_1(t) = \phi_2(t) = t$, Theorem 1.2 reduces to Theorem 2.15 in [6]. Furthermore, for $\phi_1(t) = \phi_2(t) = t$ and $w_1(x) = w_2(x) = 1$, Theorem 1.2 completes Corollary 2.11 in [11]. Corollary 2.1 states that $\Phi_1(t) \leq \Phi_2(Ct)$ is a sufficient condition for inclusion property on Orlicz spaces, while Theorem 1.2 states that it is also a necessary condition.

Now, we will come to inclusion properties of Orlicz-Morrey spaces which were introduced by Sawano, et al.

Teorema 2.2. *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \approx \Psi_2$ and $\psi_1, \psi_2 \in G_2$. If $\psi_1(x) \leq K\psi_2(x)$ for some $K > 0$ then $M_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq M_{\psi_1,\Psi_1}(\mathbb{R}^n)$.*

Proof.

Given $f \in M_{\psi_2,\Psi_2}(\mathbb{R}^n)$, let $A_{(\Psi_1,B(a,r))} = \left\{b > 0 : \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_1\left(\frac{\frac{1}{C}|f(x)|}{b}\right)dx \leq 1\right\}$

and $A_{(\Psi_2,B(a,r))} = \left\{b > 0 : \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{|f(x)|}{b}\right)dx \leq 1\right\}$.

Observe that, $\Psi_1 \approx \Psi_2$ then there exists a constant $C > 0$ such that $\Psi_2(C^{-1}t) \leq \Psi_1(t) \leq \Psi_2(Ct)$

for arbitrary $t>0$. For arbitrary $b \in A_{(\Psi_2,B(a,r))}$ we have

$$\begin{aligned}
\frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_1\left(\frac{|f(x)|}{Cb}\right)dx &\leq \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_1\left(\frac{|f(x)|}{Cb}\right)dx \\
&\leq \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{C|f(x)|}{Cb}\right)dx \\
&= \frac{1}{|B(a,r)|}\int_{B(a,r)} \Psi_2\left(\frac{|f(x)|}{b}\right)dx \\
&\leq 1.
\end{aligned}$$

Hence, it follows that $b \in A_{(\Psi_1,B(a,r))}$ and so we conclude that $A_{(\Psi_2,B(a,r))} \subseteq A_{(\Psi_1,B(a,r))}$. Accordingly, we have

$$\| \frac{f}{C} \|_{(\Psi_1,B(a,r))} = \inf A_{(\Psi_1,B(a,r))} \leq \inf A_{(\Psi_2,B(a,r))} = \| f \|_{(\Psi_2,B(a,r))} .$$

Since $r>0$ is arbitrary and $\psi_1(x) \leq K\psi_2(x)$ for some $K>0$, we obtain

$$\begin{aligned}
\| f \|_{M_{\psi_1,\Psi_1}(\mathbb{R}^n)} &= \sup_{r>0,a\in\mathbb{R}^n} \psi_1(|B(a,r)|) \| f \|_{(\Psi_1,B(a,r))} \\
&\leq \sup_{r>0,a\in\mathbb{R}^n} CK\psi_2(|B(a,r)|) \| f \|_{(\Psi_2,B(a,r))} \\
&= CK \| f \|_{M_{\psi_2,\Psi_2}(\mathbb{R}^n)}
\end{aligned}$$

which also proves that $M_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq M_{\phi_1,\Phi_1}(\mathbb{R}^n)$. □

Lemma 2.3. *[2]* $\| \chi_{B(a,r_0)} \|_{M_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$, *for every* $r_0>0$.

Teorema 2.4. *Let* Ψ_1,Ψ_2 *be Young functions such that* $\Psi_1 \approx \Psi_2$ *and* $\psi_1,\psi_2 \in G_2$. *Then the following statements are equivalent:*

(1) $\psi_1(t) \leq C\psi_2(t)$, *for every* $t>0$.

(2) $M_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq M_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

(3) *There exists a constant* $C>0$ *such that* $\| f \|_{M_{\psi_1,\Psi_2}(\mathbb{R}^n)} \leq C \| f \|_{M_{\psi_2,\Psi_2}(\mathbb{R}^n)}$ *for every* $f \in M_{\psi_2,\Psi_2}(\mathbb{R}^n)$.

Proof.

Assume that (1) holds. Let f be an element of $M_{\psi_2,\Psi_2}(\mathbb{R}^n)$. Since $\Psi_1 \approx \Psi_2$ and $\psi_1(t) \leq C\psi_2(t)$, for every $t>0$, by Theorem 2.2, we have $M_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq M_{\psi_1,\Psi_1}(\mathbb{R}^n)$. Next, since $(M_{\psi_2,\Psi_2}(\mathbb{R}^n), M_{\psi_1,\Psi_1}(\mathbb{R}^n))$ is a Banach pair, it follows from [4, Lemma 3.3] that (2) and (3) are equivalent. It thus remains to show that (3) implies (1).

Assume that (3) holds. For every $r>0$, and by Lemma 2.3, we have

$$\frac{\psi_1(|B(a,r)|)}{\Psi_1^{-1}(1)} = \| \chi_{B(a,r)} \|_{\psi_1,\Psi_1} \leq C \| \chi_{|B(a,r)|} \|_{\psi_2,\Psi_2} = \frac{C\psi_2(|B(a,r)|)}{\Psi_2^{-1}(1)} \leq \frac{C\psi_2(|B(a,r)|)}{\Psi_2^{-1}(1)}.$$

or $\psi_1(|B(a,r)|) \leq C\psi_2(|B(a,r)|))$. Since $r>0$ arbitrary, we get $\psi_1(t) \leq C\psi_2(t)$, for every $t>0$. □

Remark 2.5. For $\Phi(t)=|t|^p$, $p \geq 1$, Theorem 2.4 reduce to Theorem 3.3 in [3].

3 An Inclusion Property Of Weak Orlicz-Morrey Spaces

First, we recall the definition of weak Orlicz-Morrey spaces $wL_{\phi,\Phi}(\mathbb{R}^n)$ [10]. Let Φ be a Young function and $\phi \in G_1$. Weak Orlicz spaces $wL_{\phi,\Phi}(\mathbb{R}^n)$ to be the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \sup\limits_{a\in\mathbb{R}^n, r>0} \|f\|_{wL_{\phi,\Phi,B(a,r)}} < \infty$, where

$$\|f\|_{wL_{\phi,\Phi,B(a,r)}} := \inf\left\{ b > 0 : \sup_{t>0} \frac{\Phi(t)\phi(|B(a,r)|)\left|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \leq 1 \right\}.$$

The relation between $wL_{\phi,\Phi}(\mathbb{R}^n)$ and $L_{\phi,\Phi}(\mathbb{R}^n)$ is clear, as presented in the following lemma.

Lemma 3.1. *Let Φ be a Young function and $\phi \in G_1$. Then $L_{\phi,\Phi}(\mathbb{R}^n) \subset wL_{\phi,\Phi}(\mathbb{R}^n)$ with $\|f\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} \leq \|f\|_{L_{\phi,\Phi}(\mathbb{R}^n)}$ for every $f \in L_{\phi,\Phi}(\mathbb{R}^n)$.*

Lemma 3.2. *Let Φ be a Young function, $\phi \in G_1$, $a \in \mathbb{R}^n$, and $r_0 > 0$ be arbitrary. Then we have $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$.*

Proof. Since Φ is a Young function and $\phi \in G_1$, then $\|\chi_{B(a,r_0)}\|_{L_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$ (see [2]). By Lemma 3.1, we have $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} \leq \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$. On the other hand,

$$\begin{aligned}
\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} &= \sup_{a\in\mathbb{R}^n, r>0} \|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi,B(a,r)}} \\
&= \sup_{a\in\mathbb{R}^n, r>0} \frac{1}{\Phi^{-1}\Big(\frac{|B|}{|B(a,r)\cap B(a,r_0)|\phi(|B(a,r)|)}\Big)} \\
&\geq \frac{1}{\Phi^{-1}\big(\frac{1}{\phi(|B(a,r_0)|)}\big)}.
\end{aligned}$$

As a result, we have $\|\chi_{B(a,r_0)}\|_{wL_{\phi,\Phi}(\mathbb{R}^n)} = \frac{1}{\Phi^{-1}(\frac{1}{\phi(|B(a,r_0)|)})}$. □

Now we come to the inclusion property of weak Orlicz-Morrey spaces $wL_{\phi,\Phi}(\mathbb{R}^n)$.

Teorema 3.3. *Let Φ_1, Φ_2 be Young functions, $\phi_1, \phi_2 \in G_1$ such that $\phi_1 \sim \phi_2$. Then the following statements are equivalent:*

(1) *$\Phi_1(t) \leq \Phi_2(Ct)$ for every $t > 0$.*

(2) *$wL_{\phi_2,\Phi_2}(\mathbb{R}^n) \subseteq wL_{\phi_1,\Phi_1}(\mathbb{R}^n)$.*

(3) *There exists a constant $C > 0$ such that $\|f\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq \|Cf\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)}$ for every $f \in wL_{\phi_2,\Phi_2}(\mathbb{R}^n)$.*

Proof. Assume that (1) holds, and let $f \in wL_{\phi_2,\Phi_2}(\mathbb{R}^n)$. Since $\phi_1 \sim \phi_2$ then there exists a constant $K \geq 1$ such that $K^{-1}\phi_1(t) \leq \phi_2(t) \leq K\phi_1(t)$ for arbitrary $t > 0$. Let

$$A_{\phi_1,\Phi_1,B(a,r)} = \Big\{b>0 : \sup_{t>0} \frac{\Phi_1(\frac{t}{K})\phi_1(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|} \le 1\Big\}$$
$$= \Big\{b>0 : \sup_{y>0} \frac{\Phi_1(y)\phi_1(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > yK\}\Big|}{|B(a,r)|} \le 1\Big\}$$
$$= \Big\{b>0 : \sup_{y>0} \frac{\Phi_1(y)\phi_1(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|\frac{f(x)}{K}|}{b} > y\}\Big|}{|B(a,r)|} \le 1\Big\}$$

and

$$A_{\phi_2,\Phi_2,B(a,r)} = \Big\{b>0 : \sup_{t>0} \frac{\Phi_2(Ct)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|} \le 1\Big\}.$$
$$= \Big\{b>0 : \sup_{y>0} \frac{\Phi_2(y)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > \frac{y}{C}\}\Big|}{|B(a,r)|} \le 1\Big\}.$$
$$= \Big\{b>0 : \sup_{y>0} \frac{\Phi_2(y)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|Cf(x)|}{b} > y\}\Big|}{|B(a,r)|} \le 1\Big\}.$$

Then $\|\frac{f}{K}\|_{wL_{\phi_1,\Phi_1,B(a,r)}} = \inf A_{\phi_1,\Phi_1,B(a,r)}$ and $\|Cf\|_{wL_{\phi_2,\Phi_2,B(a,r)}} = \inf A_{\phi_2,\Phi_2,B(a,r)}$. Observe that, for arbitrary $b \in A_{(\phi_2,\Phi_2,B(a,r))}$ and $t>0$, we have

$$\begin{aligned}\frac{\Phi_1(\frac{t}{K})\phi_1(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|} &\le \frac{\Phi_1(t)\frac{\phi_1(|B(a,r)|)}{K}\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|}\\ &\le \frac{\Phi_2(Ct)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|}\\ &\le \frac{\Phi_2(Ct)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|}\\ &\le \frac{\Phi_2(y)\phi_2(|B(a,r)|)\Big|\{x\in B(a,r) : \frac{|Cf(x)|}{b} > y\}\Big|}{|B(a,r)|}\\ &\le 1\end{aligned}$$

Thus, $\sup_{t>0} \frac{\Phi_1(\frac{t}{K})\phi_1(|B(a,r)|)\Big|\{x\in\mathbb{R}^n : \frac{|f(x)|}{b} > t\}\Big|}{|B(a,r)|} \le 1$. Hence it follows that $b \in A_{\phi_1,\Phi_1,B(a,r)}$, and so we conclude that $A_{\phi_2,\Phi_2,B(a,r)} \subseteq A_{\phi_1,\Phi_1,B(a,r)}$. Accordingly, we obtain

$$\|\frac{f}{K}\|_{wL_{\phi_1,\Phi_1,B(a,r)}} = \inf A_{\phi_1,\Phi_1,B(a,r)} \le \inf A_{\phi_2,\Phi_2,B(a,r)} = \|Cf\|_{wL_{\phi_2,\Phi_2,B(a,r)}}.$$

Since, $a \in \mathbb{R}^n$ and $r>0$ are arbitratry, we conclude that $\|f\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)} \le CK\|f\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)}$ which also prove that $wL_{\phi_2,\Phi_2}(\mathbb{R}^n) \subset wL_{\phi_1,\Phi_1}(\mathbb{R}^n)$.

As mention in [10, Appendix G], we are aware that [4, Lemma 3.3] still holds for quasi-Banach spaces, and so (2) and (3) are equivalent. Now, we will show (3) implies (1). Assume now that (3) holds. By Lemma 3.2, we have

$$\frac{1}{\Phi_1^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)})} = \|\chi_{B(a,r_0)}\|_{wL_{\phi_1,\Phi_1}(\mathbb{R}^n)} \leq C\|\chi_{B(a,r_0)}\|_{wL_{\phi_2,\Phi_2}(\mathbb{R}^n)} = \frac{C}{\Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})},$$

or $C\Phi_1^{-1}(\frac{1}{\phi_1(|B(a,r_0)|)}) \geq \Phi_2^{-1}(\frac{1}{\phi_2(|B(a,r_0)|)})$, for arbitrary $a \in \mathbb{R}^n$ and $r_0 > 0$. By using Lemma 1.3, we have

$$\Phi_1\Big(\frac{1}{\phi_1(|B(a,r_0)|)}\Big) \leq \Phi_2\Big(\frac{C}{\phi_2(|B(a,r_0)|)}\Big) \leq \Phi_2\Big(\frac{CK}{\phi_1(|B(a,r_0)|)}\Big).$$

Since $a \in \mathbb{R}^n$ and $r_0 > 0$ are arbitrary, we conclude that $\Phi_1(t) \leq \Phi_2(Ct)$. $\square$

Remark 3.4. For $\phi(r) = r$, Theorem 3.3 reduce to Theorem 3.3 in [6].

Now, we will study inclusion properties of weak Orlicz-Morrey spaces $wM_{\psi,\Psi}(\mathbb{R}^n)$. Let Ψ be a Young function and $\psi \in G_2$, weak Orlicz-Morrey spaces $wM_{\psi,\Psi}(\mathbb{R}^n)$ is the set of measurable functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $\|f\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} = \sup\limits_{a\in\mathbb{R}^n, r>0} \psi(|B(a,r)|)\|f\|_{wM_{\Psi,B(a,r)}} < \infty$

where $\|f\|_{wM_{\Psi,B(a,r)}} := \inf\left\{b > 0 : \sup\limits_{t>0} \frac{\Psi(t)\left|\{x\in B(a,r): \frac{|f(x)|}{b} > t\}\right|}{|B(a,r)|} \leq 1\right\}$.

The relation between $wM_{\psi,\Psi}(\mathbb{R}^n)$ and $M_{\psi,\Psi}(\mathbb{R}^n)$ is clear, as presented in the following lemma.

Lemma 3.5. *Let Ψ be a Young function and $\psi \in G_2$. Then $M_{\psi,\Psi}(\mathbb{R}^n) \subset wM_{\psi,\Psi}(\mathbb{R}^n)$ with $\|f\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} \leq \|f\|_{M_{\psi,\Psi}(\mathbb{R}^n)}$ for every $f \in M_{\psi,\Psi}(\mathbb{R}^n)$.*

Lemma 3.6. *Let Ψ be a Young function, $a \in \mathbb{R}^n$, and $r_0 > 0$ be arbitrary. Then we have $\|\chi_{B(a,r_0)}\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$.*

Proof. Since Ψ is a Young function and $\psi \in G_2$, by Lemma 2.3 and Lemma 3.5, we have $\|\chi_{B(a,r_0)}\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} \leq \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$. On the other hand, we have

$$\begin{aligned}
\|\chi_{B(a,r_0)}\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} &= \sup_{r>0} \|\chi_{B(a,r_0)}\|_{wM_{\psi,\Psi,B(a,r)}} \\
&= \sup_{r>0} \frac{\psi(|B(a,r)|)}{\Psi^{-1}\Big(\frac{|B(a,r)|}{|B(a,r)\cap B(a,r_0)|}\Big)} \\
&\geq \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}
\end{aligned}$$

As a result, we have $\|\chi_{B(a,r_0)}\|_{wM_{\psi,\Psi}(\mathbb{R}^n)} = \frac{\psi(|B(a,r_0)|)}{\Psi^{-1}(1)}$. $\square$

Now we come to the inclusion property of weak Orlicz-Morrey spaces $wM_{\psi,\Psi}(\mathbb{R}^n)$.

Teorema 3.7. *Let Ψ_1, Ψ_2 be Young functions such that $\Psi_1 \approx \Psi_2$ and $\psi_1, \psi_2 \in G_2$. Then the following statements are equivalent:*

(1) $\psi_1(t) \leq C\psi_2(t)$ *for every* $t > 0$.

(2) $wM_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq wM_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

(3) *There exists a constant* $C > 0$ *such that* $\|f\|_{wM_{\psi_1,\Psi_1}(\mathbb{R}^n)} \leq \|f\|_{wM_{\psi_2,\Psi_2}(\mathbb{R}^n)}$ *for every* $f \in wM_{\psi_2,\Psi_2}(\mathbb{R}^n)$.

Proof.

Assume that (1) hold. Let $f \in wM_{\psi_2,\Psi_2}(\mathbb{R}^n)$. Observe that, for every $r > 0$ we have

$$\begin{aligned}\| f \|_{wM_{\psi_1,\Psi_1}(\mathbb{R}^n)} &= \sup_{r>0, a\in\mathbb{R}^n} \psi_1(|B(a,r)|) \| f \|_{wM_{\psi_1,\Psi_1,B(a,r)}} \\ &\leq \sup_{r>0, a\in\mathbb{R}^n} C\psi_2(|B(a,r)|) \| f \|_{wM_{\psi_1,\Psi_1,B(a,r)}} \\ &\leq \sup_{r>0, a\in\mathbb{R}^n} C\psi_2(|B(a,r)|) \| f \|_{wM_{\psi_2,\Psi_2,B(a,r)}} \\ &= C \| f \|_{wM_{\psi_2,\Psi_2}(\mathbb{R}^n)}\end{aligned}$$

So, we conclude that $wM_{\psi_2,\Psi_2}(\mathbb{R}^n) \subseteq wM_{\psi_1,\Psi_1}(\mathbb{R}^n)$.

As mention in [10, Appendix G], we are aware that [4, Lemma 3.3] still holds for quasi-Banach spaces, and so (2) and (3) are equivalent. Now, we will show (3) implies (1). Assume now that (3) holds. By Lemma 3.6, we have

$$\frac{\psi_1(|B(a,r_0)|)}{\Psi_1^{-1}(1)} = \|\chi_{B(a,r_0)}\|_{wM_{\psi_1,\Psi_1}(\mathbb{R}^n)} \leq C\|\chi_{B(a,r_0)}\|_{wM_{\psi_2,\Psi_2}(\mathbb{R}^n)} = \frac{C\psi_2(|B(a,r_0)|)}{\Psi_2^{-1}(1)} \leq \frac{C\psi_2(|B(a,r_0)|)}{\Psi_1^{-1}(1)},$$

or $\psi_1(|B(a,r_0)|) \leq C\psi_2(|B(a,r_0)|)$, for arbitrary $a \in \mathbb{R}^n$ and $r_0 > 0$. We conclude that

$$\psi_1(t) \leq C\psi_2(t)$$

for every $t > 0$. □

Remark 3.8. For $\Phi(t) = |t|^p$, $p \geq 1$, Theorem 3.7 reduce to Theorem 4.4 in [3].

4 Concluding Remarks

We have shown the inclusion property of (strong) Orlicz-Morrey spaces and of weak Orlicz-Morrey spaces for Nakai version and Sawano, et al. version. In the proof of the inclusion property we use the norm of the characteristic functions of the balls in $\mathbb{R}^n$. As our final conclusion, the inclusion property of (strong) Orlicz-Morrey spaces are equivalent to that of weak Orlicz-Morrey spaces. We obtain necessary and sufficient condition for inclusion relation between Orlicz-Morrey space $L_{\phi,\Phi}(\mathbb{R}^n)$ and weak Orlicz-Morrey space $wL_{\phi,\Phi}(\mathbb{R}^n)$ (Theorem 1.2 and Theorem 3.3), which were generalized for inclusion property of Orlicz spaces and weak Orlicz spaces in [5, 6].

On the other hand, we obtain necessary and sufficient condition for inclusion relation between Orlicz-Morrey space $M_{\psi,\Psi}(\mathbb{R}^n)$ and weak Orlicz-Morrey space $wM_{\psi,\Psi}(\mathbb{R}^n)$ (Theorem 2.4 and Theorem 3.7), which were generalized for inclusion property of generalized Morrey spaces and generalized weak Morrey spaces in [3].

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