The Boundedness of Generalized Bessel-Riesz Operators on Generalized Morrey Spaces

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***Abstract*—The purpose of this paper is to prove the boundedness of generalized Bessel-Riesz operators on generalized Morrey spaces. The kernel of the operators contain some parameters, one of which is related to Bessel decay. As usual, we use the usual dyadic decomposition, Hölder's inequality, a Hedberg-type inequality for the operators, and the boundedness of Hardy-Littlewood maximal operator in the proofs. In addition, we also exploit the relationship between the parameters of the kernel and of the space. We obtain that the norm of the operators is dominated by the norm of the kernels.**

***Keywords*: generalized Bessel-Riesz operators, Hardy-Littlewood maximal operator, generalized Morrey spaces.**

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I. INTRODUCTION

In this paper, we shall discuss about an integral operator. The following definition formulates the operator.

Definition I.1. Let $\gamma \in (0,\infty)$ and $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function satisfying the doubling condition ($\frac{1}{2} \le \frac{r_1}{r_2} \le 2$ implies $C_1 \le \frac{H(r_1)}{H(r_2)} \le C_2$, where $C_1, C_2 > 0$). *The generalized Bessel-Riesz operator* $I_{\rho,\gamma}$ is defined by

$$I_{\rho,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\rho,\gamma}(x-y) f(y)\, dy$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, $1 \le p < \infty$, where

$$K_{\rho,\gamma}(x) := \frac{\rho(|x|)}{|x|^n (1+|x|)^\gamma},$$

$x \in \mathbb{R}^n$.

Here, $K_{\rho,\gamma}$ is called *generalized Bessel-Riesz kernel*. If $\rho(r) := r^\alpha$, for every $r > 0$, where $0 < \alpha < n$, then we have $I_{\rho,\gamma} = I_{\alpha,\gamma}$ (Bessel-Riesz operator [6]).

Definition I.2. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function satisfying the doubling condition. For $1 \le p < \infty$, *The generalized Morrey spaces* $L^{p,\phi}(\mathbb{R}^n)$ is defined to be the set of all functions $f \in L^p_{loc}(\mathbb{R}^n)$ such that

$$\|f\|_{L^{p,\phi}} := \sup_{B=B(a,R)} \frac{\left(\int_B |f(x)|^p\, dx\right)^{1/p}}{\phi(R)\, |B|^{1/p}} < \infty,$$

where $|B|$ denotes *Lebesgue measure* of ball $B(a,R)$, where $a \in \mathbb{R}^n, R > 0$.

If $\phi(r) = r^{-n/q}$, $p \le q < \infty$, for every $r > 0$, then we have the Morrey space $L^{p,q}(\mathbb{R}^n)$. In particular, we know that $L^{p,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ is the Lebesgue space. Moreover, one may check that $r \mapsto r^{-n/q}$ satisfy the doubling condition.

On Lebesgue spaces, the boundedness of $I_{\alpha,\gamma}$ could be shown via Young's inequality [4]. For $0 < \gamma$ and $0 < \alpha < n$, we have

$$\|I_{\alpha,\gamma} f\|_{L^q} \le \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^n)$, $1 \le p < t'$, where $\frac{1}{q} = \frac{1}{p} - \frac{1}{t'}$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. The membership of $K_{\alpha,\gamma}$ on Lebesgue spaces was also proved [6].

On Morrey spaces, the boundedness of $I_{\alpha,\gamma}$ was proved by Hardy-Littlewood maximal operator. The proof is explained in [6]. Furthermore, $I_{\alpha,\gamma}$ is also bounded on generalized Morrey spaces [7]. To show the boundedness of $I_{\alpha,\gamma}$ on (generalized) Morrey spaces, $K_{\alpha,\gamma}$ can be viewed as a member of Lebesgue spaces or Morrey spaces.

Meanwhile for $\gamma = 0$, we have $I_{\rho,0} = I_\rho$, *the generalized fractional integral operator [5]*. In 2002, Eridani and Gunawan [2] proved the boundedness of I_ρ on generalized Morrey spaces. Using different assumptions, Eridani, Gunawan, and Nakai [3] also proved it.

The Hardy-Littlewood maximal operator M, which is used to prove the boundedness of $I_{\alpha,\gamma}$ and I_ρ on Morrey spaces and generalized Morrey spaces, is defined by

$$Mf(x) := \sup_{x\in B=B(a,R)} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$. We know that M is bounded on Morrey spaces [1]. In [8], Nakai proved the boundedness of M on generalized Morrey spaces. We shall use this fact to prove our results.

II. MAIN RESULTS

In this section, the boundedness of $I_{\rho,\gamma}$ on generalized Morrey spaces will be proved. We must show that $K_{\rho,\gamma}$ is contained in Lebesgue spaces.

Lemma II.1. *Let* $0<\gamma$ *and* ρ *satisfy*

$$\int_{0<r\le R}\frac{\rho^t(r)}{r^{(nt+1)-n}}dr+\int_{R<r<\infty}\frac{\rho^t(r)}{r^{(nt+\gamma t+1)-n}}dr<\infty,$$

for some $t\in[1,\infty)$, *for every* $R>0$. *Then* $K_{\rho,\gamma}$ *is a member of* $L^t(\mathbb{R}^n)$.

Proof: Suppose that $0<\gamma$ and ρ satisfy

$$\int_{0<r\le R}\frac{\rho^t(r)}{r^{(nt+1)-n}}dr+\int_{R<r<\infty}\frac{\rho^t(r)}{r^{(nt+\gamma t+1)-n}}dr<\infty,$$

for some $t\in[1,\infty)$, for every $R>0$. Observe that

$$\begin{aligned}\int_{\mathbb{R}^n}K^t_{\rho,\gamma}(x)dx &= \int_{|x|\le R}\frac{\rho^t(|x|)}{|x|^{nt}(1+|x|)^{\gamma t}}dx\\ &\quad+\int_{R<|x|}\frac{\rho^t(|x|)}{|x|^{nt}(1+|x|)^{\gamma t}}dx\end{aligned}$$

and we estimate

$$\begin{aligned}\int_{\mathbb{R}^n}K^t_{\rho,\gamma}(x)dx &\le C\int_{0<r\le R}\frac{\rho^t(r)}{r^{(nt+1)-n}}dr\\ &\quad+C\int_{R<r<\infty}\frac{\rho^t(r)}{r^{(nt+\gamma t+1)-n}}dr\\ &< \infty.\end{aligned}$$

So we have $\int_{\mathbb{R}^n}K^t_{\rho,\gamma}(x)dx<\infty$. Hence $K_{\rho,\gamma}\in L^t(\mathbb{R}^n)$. ■

By Lemma II.1 and the inclusion property of Morrey spaces, we obtain

$$K_{\rho,\gamma}\in L^t(\mathbb{R}^n)=L^{t,t}(\mathbb{R}^n)\subseteq L^{s,t}(\mathbb{R}^n),$$

where $1\le s\le t$. Consequently, we also have

$$\frac{\left(\int_{2^kR<|x|\le 2^{k+1}R}K^s_{\rho,\gamma}(x)dx\right)^{1/s}}{(2^kR)^{n/s-n/t}}\le C\|K_{\rho,\gamma}\|_{L^{s,t}}$$

for every integer k and $R>0$. Because ρ satisfies the doubling condition, we obtain

$$\frac{\left(\sum_{k=-1}^{-\infty}K^s_{\rho,\gamma}\left(2^kR\right)\left(2^kR\right)^n\right)^{1/s}}{R^{n/s-n/t}}\le C\|K_{\rho,\gamma}\|_{L^{s,t}}$$

for every $R>0$. Here, one may check that

$$1-\frac{\ln\rho(R_1)}{n\ln R_1}<\frac{1}{t}<1+\frac{\gamma}{n}-\frac{\ln\rho(R_1)}{n\ln R_1},$$

for every $R_1>1$. If $\rho(R)=R^\alpha$ for every $R>0$ where $0<\alpha<n$, then $1-\frac{\alpha}{n}<\frac{1}{t}<1+\frac{\gamma-\alpha}{n}$.

Theorem II.2. *Let* $1<p_1<p_2<\infty$. *Let* $0<\gamma$ *and* ρ *satisfy*

$$\frac{\int_{0<r\le R}\frac{\rho^t(r)r^n}{r^{(nt+1)}}dr}{R^{\gamma t}}+\int_{R<r<\infty}\frac{\rho^t(r)r^{n-1}}{r^{(n+\gamma)t}}dr\le\frac{\rho^t(R)R^n}{R^{(n+\gamma)t}},$$

for some $t\in[1,\infty)$, *for every* $R>0$. *Suppose that* ϕ *is surjective and* $\int_{0<r\le R}r^{\frac{n}{t'}-1}dr+\frac{\int_{R<r<\infty}\phi(r)r^{\frac{n}{t'}-1}dr}{\phi(R)}\le\phi^{\frac{p_1}{p_2}-1}(R)$. *Then for every* $f\in L^{p_1,\phi}(\mathbb{R}^n)$, *we have*

$$\|I_{\rho,\gamma}f\|_{L^{p_2,\phi^{p_1/p_2}}}\le C_{p_1,\phi}\|K_{\rho,\gamma}\|_{L^{s,t}}\|f\|_{L^{p_1,\phi}},$$

where $1\le s\le t$.

Proof: Consider $1<p_1<p_2<\infty$. Suppose that $0<\gamma$ and ρ satisfy $\frac{\int_{0<r\le R}\frac{\rho^t(r)r^n}{r^{(nt+1)}}dr}{R^{\gamma t}}+\int_{R<r<\infty}\frac{\rho^t(r)r^{n-1}}{r^{(n+\gamma)t}}dr\le\frac{\rho^t(R)R^n}{R^{(n+\gamma)t}}$, for some $t\in[1,\infty)$, for every $R>0$, then $K_{\rho,\gamma}$ is a member of $L^{s,t}(\mathbb{R}^n)$, where $1\le s\le t$.

Next, take $f\in L^{p_1,\phi}(\mathbb{R}^n)$ where ϕ is surjective and $\int_{0<r\le R}r^{\frac{n}{t'}-1}dr+\frac{\int_{R<r<\infty}\phi(r)r^{\frac{n}{t'}-1}dr}{\phi(R)}\le\phi^{\frac{p_1}{p_2}-1}(R)$. Split the following formula into two parts, namely

$$I_{\rho,\gamma}f(x):=I_1(x)+I_2(x),$$

where

$$I_1(x):=\int_{|x-y|<R}K_{\rho,\gamma}(x-y)f(y)dy$$

and

$$I_2(x):=\int_{|x-y|\ge R}K_{\rho,\gamma}(x-y)f(y)dy.$$

Each part will be estimated by dyadic decomposition. The positive constants are denoted by the letter C which may vary from line to line. To estimate I_1, we have

$$|I_1(x)|\le C\sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)\int_{2^kR\le|x-y|<2^{k+1}R}|f(y)|dy}{\left(2^kR\right)^n\left(1+2^kR\right)^\gamma}.$$

Use maximal operator to obtain

$$|I_1(x)|\le Mf(x)\sum_{k=-\infty}^{-1}\frac{\rho\left(2^kR\right)\left(2^kR\right)^{n/s}}{\left(2^kR\right)^n\left(1+2^kR\right)^\gamma}\left(2^kR\right)^{n/s'}.$$

Use Hölder's inequality to obtain

$$\begin{aligned}|I_1(x)| &\le CMf(x)\left(\sum_{k=-\infty}^{-1}\left(2^kR\right)^n\right)^{1/s'}\\ &\quad\times\left(\sum_{k=-\infty}^{-1}K^s_{\rho,\gamma}\left(2^kR\right)\left(2^kR\right)^n\right)^{1/s}.\end{aligned}$$

Now, we state

$$\begin{aligned}|I_1(x)| &\le CMf(x)\frac{\left(\sum_{k=-\infty}^{-1}\left(2^kR\right)^n\right)^{1/s'}}{R^{n/t-n/s}}\\ &\quad\times\frac{\left(\sum_{k=-\infty}^{-1}K^s_{\rho,\gamma}\left(2^kR\right)\left(2^kR\right)^n\right)^{1/s}}{R^{n/s-n/t}}.\end{aligned}$$

Because $K_{\rho,\gamma}\in L^{s,t}(\mathbb{R}^n)$ and $R^{n/t'}\le\phi^{\frac{p_1-p_2}{p_2}}(R)$, we get $|I_1(x)|\le C\|K_{\rho,\gamma}\|_{L^{s,t}}Mf(x)\,\phi^{\frac{p_1-p_2}{p_2}}(R)$.

Meanwhile for I_2, we have

$$|I_2(x)|\le C\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)\int_{2^kR\le|x-y|<2^{k+1}R}|f(y)|dy}{\left(2^kR\right)^n\left(1+2^kR\right)^\gamma}.$$

Use again Hölder inequality, so we get

$$\begin{aligned}|I_2(x)| &\leq C\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)\left(2^kR\right)^{n/p_1'}}{\left(2^kR\right)^n\left(1+2^kR\right)^{\gamma}}\\ &\quad\times\left(\int_{2^kR\leq|x-y|<2^{k+1}R}|f(y)|^{p_1}\,dy\right)^{1/p_1}\end{aligned}$$

and the norm of f dominates I_2

$$|I_2(x)|\leq C\,\|f\|_{L^{p_1,\phi}}\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)\phi\left(2^kR\right)\left(2^kR\right)^n}{\left(2^kR\right)^n\left(1+2^kR\right)^{\gamma}}.$$

Observe that

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)}{\left(2^kR\right)^n\left(1+2^kR\right)^{\gamma}}\phi\left(2^kR\right)\left(2^kR\right)^n\\ \leq&\sum_{k=0}^{\infty}K_{\rho,\gamma}\left(2^kR\right)\phi\left(2^kR\right)\left(2^kR\right)^n\\ &\times\frac{\left(\int_{2^kR\leq|x-y|<2^{k+1}R}\right)^{1/s}}{\left(2^kR\right)^{n/s}}.\end{aligned}$$

We know that ρ satisfies the doubling condition, so

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{\rho\left(2^kR\right)}{\left(2^kR\right)^n\left(1+2^kR\right)^{\gamma}}\phi\left(2^kR\right)\left(2^kR\right)^n\\ \leq&\ C\sum_{k=0}^{\infty}\frac{\phi\left(2^kR\right)\left(2^kR\right)^{n(1-1/t)}}{\left(2^kR\right)^{n/s-n/t}}\\ &\times\left(\int_{2^kR\leq|x-y|<2^{k+1}R}K_{\rho,\gamma}^s(x-y)dy\right)^{1/s}\\ \leq&\ C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\sum_{k=0}^{\infty}\phi\left(2^kR\right)\left(2^kR\right)^{n/t'}.\end{aligned}$$

Because $\sum_{k=0}^{\infty}\phi\left(2^kR\right)\left(2^kR\right)^{n/t'}\leq\phi^{\frac{p_1}{p_2}}(R)$, we obtain

$$|I_2(x)|\leq C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}\,\phi^{\frac{p_1}{p_2}}(R).$$

By summing of the estimates of I_1 and I_2, we have

$$\begin{aligned}|I_{\rho,\gamma}f(x)| &\leq C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\phi^{\frac{p_1}{p_2}}(R)\\ &\quad\times\left(\frac{Mf(x)}{\phi(R)}+\|f\|_{L^{p_1,\phi}}\right),\end{aligned}$$

for each $x\in\mathbb{R}^n$. We know that ϕ is surjective. So, choose $R>0$, such that $\phi(R)=\frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$. Now, we get

$$|I_{\rho,\gamma}f(x)|\leq C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}^{1-p_1/p_2}\,Mf(x)^{p_1/p_2}.$$

Take arbitrary $r>0$ and $a\in\mathbb{R}^n$ to obtain

$$\begin{aligned}&\left(\int_{|x-a|<r}|I_{\rho,\gamma}f(x)|^{p_2}\,dx\right)^{1/p_2}\\ \leq&\ C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}^{1-p_1/p_2}\\ &\times\left(\int_{|x-a|<r}|Mf(x)|^{p_1}\,dx\right)^{(1/p_2)}.\end{aligned}$$

The two sides are divided by $\phi(r)^{p_1/p_2}\,r^{n/p_2}$ and take supremum over $r>0$ and $a\in\mathbb{R}^n$ to get

$$\begin{aligned}\|I_{\rho,\gamma}f\|_{L^{p_2,\phi^{p_1/p_2}}} &\leq C\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}^{1-p_1/p_2}\\ &\quad\times\|Mf\|_{L^{p_1,\phi}}^{p_1/p_2}.\end{aligned}$$

By the boundedness of maximal operator on generalized Morrey spaces, $\|I_{\rho,\gamma}f\|_{L^{p_2,\phi^{p_1/p_2}}}\leq C_{p_1,\phi}\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}$ holds. ■

By Theorem II.2 and the inclusion property of Morrey spaces,

$$\begin{aligned}\|I_{\rho,\gamma}f\|_{L^{p_2,\phi^{p_1/p_2}}} &\leq C_{p_1,\phi}\,\|K_{\rho,\gamma}\|_{L^{s,t}}\,\|f\|_{L^{p_1,\phi}}\\ &\leq C_{p_1,\phi}\,\|K_{\rho,\gamma}\|_{L^{t}}\,\|f\|_{L^{p_1,\phi}}.\end{aligned}$$

The above inequality shows that norm of $I_{\rho,\gamma}$ is dominated by $\|K_{\rho,\gamma}\|_{L^{s,t}}$ and $\|K_{\rho,\gamma}\|_{L^t}$. Furthermore, we want to obtain the best upper bound. We shall reinvestigate $K_{\rho,\gamma}$. The following Lemma explains that $K_{\rho,\gamma}$ is a member of generalized Morrey spaces.

Lemma II.3. *If $0<\gamma$ and ρ,σ satisfy*

$$\int_{0<r\leq R}\rho^s(r)r^{-(ns+1)+n}dr\leq C\sigma^s(R)R^n,$$

for some $s\in[1,\infty)$, for every $R>0$, then $K_{\rho,\gamma}$ is a member of $L^{s,\sigma}(\mathbb{R}^n)$.

Proof: Suppose that $0<\gamma$. Suppose also that ρ,σ satisfies $\int_{0<r\leq R}\rho^s(r)r^{-(ns+1)+n}dr\leq C\sigma^s(R)R^n$, for some $s\in[1,\infty)$, for every $R>0$. Now, observe that

$$\begin{aligned}\int_{|x-0|\leq R}K_{\rho,\gamma}^s(x)dx &\leq C\int_{0<r\leq R}\frac{\rho^s(r)}{r^{(ns+1)-n}}dr\\ &\leq C\sigma^s(R)R^n.\end{aligned}$$

Take supremum over $R>0$ to get

$$\sup_{R>0}\frac{\left(\int_{|x-0|\leq R}K_{\rho,\gamma}^s(x)dx\right)^{1/s}}{\sigma(R)R^{n/s}}<\infty.$$

Hence $K_{\rho,\gamma}\in L^{s,\sigma}(\mathbb{R}^n)$. ■

By Lemma II.3 we also have

$$\frac{\left(\int_{2^kR<|x|\leq 2^{k+1}R}K_{\rho,\gamma}^s(x)dx\right)^{1/s}}{\sigma(2^kR)\left(2^kR\right)^{n/s}}\leq C\,\|K_{\rho,\gamma}\|_{L^{s,\sigma}},$$

for every integer k and $R>0$. Moreover, we obtain

$$\frac{\left(\sum_{k=-1}^{-\infty}K_{\rho,\gamma}^s\left(2^kR\right)\left(2^kR\right)^n\right)^{1/s}}{\sigma(R)\left(R\right)^{n/s}}\leq C\,\|K_{\rho,\gamma}\|_{L^{s,\sigma}}.$$

Observe that $1\leq s\leq\frac{n\ln R_1}{-\ln\sigma(R_1)}$ for every $R_1>1$. In particular, for $\sigma(R)=R^{-n/t}$, we have $K_{\rho,\gamma}\in L^{s,t}(\mathbb{R}^n)$, where $1\leq s\leq t$.

Theorem II.4. *Let $1 < p_1 < p_2$. Let $0 < \gamma$ and ρ, σ satisfies $\int_{0<r\leq R} \frac{\rho^s(r)}{r^{(ns+1)-n}} dr \leq C\sigma^s(R)R^n$, for some $s \in [1,\infty)$, for every $R > 0$. If ϕ is surjective and*

$$\int_{0<r\leq R} \frac{\sigma(r)}{r^{-n+1}} dr + \frac{\int_{R<r<\infty} \frac{\phi(r)\sigma(r)}{r^{-n+1}} dr}{\phi(R)} \leq \phi^{\frac{p_1}{p_2}-1}(R).$$

Then for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$, we have

$$\|I_{\rho,\gamma} f\|_{L^{p_2,\phi^{p_1/p_2}}} \leq C_{p_1,\phi} \|K_{\rho,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}.$$

Proof: To process the proof, we do similar steps as in the proof of Theorem II.2. ■

The norm of $I_{\rho,\gamma}$ have some upper bounds. Choosing the membership of the kernels on the function spaces influences the upper bound of the norm. The upper bound in Theorem II.4 is smaller than in Theorem II.2, while $R^{-n/t} < \sigma(R)$ for every $R > 0$. So we can choose the best of σ to obtain the smallest upper bound.

III. Concluding Remarks

In this paper, we have shown that the boundedness of $I_{\rho,\gamma}$ from $L^{p_1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\phi^{p_1/p_2}}(\mathbb{R}^n)$ where $1 < p_1 < p_2 < \infty$. In the next study, the boundedness of $I_{\rho,\gamma}$ will be investigated from $L^{1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\phi^{p_1/p_2}}(\mathbb{R}^n)$ using different methods.

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