Angles between Subspaces of an *n*-Inner Product Space

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Abstract

We discuss angles between two subspaces of an inner product space. This paper is an extension of the work by Gunawan et al [7]. We present an explicit formula for angles between two subspaces of an n-inner product space. Moreover, we study its connection with angles in an inner product space.

Key Words: *angles, inner product space, n-inner product space.*

1. Introduction

In an inner product space, we can calculate angles between two subspaces. Since the $1950'$s, the concept of angles between two subspaces of the Euclidean space $\mathbb{R}^d$ has been studied by many researchers [2]. Application of angles between two subspaces in an inner product space can be found in the fields of computing and statistics. For example, measuring the similarity of images of three-dimensional objects is invariant under the displacement of the object and the physic of the camera [8]. In statistics, the angle between two subspaces is related to canonical (or principal) angles which are measures of dependency of one set of random variables on another [1]. In 2001, Risteksi and Trencevski [11] introduced a definition of angles between two subspaces of $\mathbb{R}^d$ using determinant Gram and explained their connection with canonical angles. Gunawan *et al.* [6,7] refined their definition and gave the formulas for angles between two subspaces in an inner product space of arbitrary dimension. They also explained the connection with canonical angles by using elementary calculus and linear algebra and some application examples.

Let $(X,\langle\cdot,\cdot\rangle)$ be a real inner product space. If $U = span\{u\}$ is a 1-dimensional subspace and $V = span\{v_1,\cdots,v_q\}$ is a q-dimensional subspace of X, then the angle between subspaces U and V isdefined by θ with $0 \le \theta \le \frac{\pi}{2}$ and $\cos^2\theta = \frac{\langle u,u_V\rangle^2}{\|u\|^2\|u_V\|^2}$. In formula, u_V denotes the (orthogonal) projection of u on V and $\|\cdot\| = \langle\cdot,\cdot\rangle^2$. Gunawan *et al* [8] showed that the value of $\cos\,\theta$ is equal to the ratio between the length of the projection of u on V and the length of u $\left(\cos^2\theta = \frac{\|u_V\|^2}{\|u\|^2}\right)$. Likewise, if $U = span\{u,w_2,\cdots,w_p\}$ and $V = span\{v,w_2,\cdots,w_p\}$ are p-dimensional subspaces of X that intersects on $(p-1)$-dimensional subspace $W = span\{w_2,\cdots,w_p\}$ with $p \ge 2$ then the angle between U and V is defined by θ with $0 \le \theta \le \frac{\pi}{2}$ and $\cos^2\theta = \frac{\langle u_W^\perp,v_W^\perp\rangle^2}{\|u_W^\perp\|^2\|v_W^\perp\|^2}$ with $u_W^\perp, v_W^\perp$ are the orthogonal complement of u and v, respectively, on W. Gunawan *et al.* showed that the value of $\cos\,\theta$ is equal to the ratio between the volume of the p-dimensional parallelepiped spanned by the projection of $u,w_2,\cdots,w_p$ on V and the volume of the p-dimensional parallelepiped spanned by $u,w_2,\cdots,w_p$.

In this paper, we will give some explicit formulas for angles between two subspaces in various cases of an inner product space of arbitrary dimension. This research is a further development of the work of Gunawan *et al*. In the next section, we will formulate angles in an n-inner product space and will show the connection between angles in an inner product space and in an n-inner product space.

2. Main Results

2.1 Angles between subspaces in an inner product space

In this subsection, we will discuss angles between subspaces in an inner product space. Let $(X,\langle\cdot,\cdot\rangle)$ be an inner product space and consider the standar n-inner product

$$\langle x_0,x_1|x_2,\dots,x_n\rangle = \begin{vmatrix} \langle x_{0,}x_1\rangle & \langle x_{0,}x_2\rangle & \cdots & \langle x_{0,}x_n\rangle \\ \langle x_{2,}x_1\rangle & \langle x_2,x_2\rangle & \cdots & \langle x_{2,}x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_{n,}x_1\rangle & \langle x_{n,}x_2\rangle & & \langle x_{n,}x_n\rangle \end{vmatrix}$$

as in [4,10]. Then, the following function $\|x_1,\dots,x_n\|:=\langle x_1,x_1|x_2,\dots,x_n\rangle^{\frac{1}{2}}$ defines the standard n-norm. Geometrically, if $n=1$ then $\|x_1\|=\langle x,x\rangle^{\frac{1}{2}}$ is the length of x. If $n=2$ then $\|x_1,x_2\|=(\|x_1\|^2\|x_2\|^2-\langle x_1,x_2\rangle^2)^{\frac{1}{2}}$ is the area of the parallelogram spanned by the vectors x_1 and x_2 in X. Thus, in general $\|x_1,\dots,x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1,\dots,x_n$ (see[4,5]).

As in [7], we define the angle between subspaces of an inner product space by using the standard n-inner product, as follows.

Definition 1. [7] Let $(X,\langle\cdot,\cdot\rangle)$ be a real inner product space. If $U=span\{u_1,\cdots,u_p\}$ is a p-dimensional subspace and $V=span\{v_1,\cdots,v_q\}$ is a q-dimensional subspace of X with $p\le q$, then the angle between subspaces U and V is defined by θ with $0\le\theta\le\frac{\pi}{2}$ and $\cos^2\theta=\frac{\|u_1^V,\cdots,u_p^V\|_p^2}{\|u_1,\cdots,u_p\|_p^2}$, where u_i^V denote the projection of $u_i$on V for each $i=1,\cdots,p$ and $\|\cdot,\cdots,\cdot\|_p$ denotes the standard p-norm on X.

According to Definition 1 for case $p=q$, we have an explisit formula the cosine of the angle θ that can be obtained as follows.

Proposition 2. Let $(X,\langle\cdot,\cdot\rangle)$ be a real inner product space. If $U=span\{u_1,\cdots,u_p\}$ and $V=span\{v_1,\cdots,v_p\}$ are p-dimensional subspaces of X then the angle between subspaces U and V is θ with $\cos^2\theta=\frac{(det\,[\langle u_{i,}v_j\rangle])^2}{\|u_1,\cdots,u_p\|_p^2\|v_1,\cdots,v_p\|_p^2}$.

Proof. The projection of u_i on V for $i=1,\cdots,p$ may be expressed as $u_i^V=\sum_{k=1}^{p}\alpha_{ik}\,v_k$. Observe that

$$\langle u_i^V,u_j^V\rangle=\langle u_i,u_j^V\rangle=\sum_{k=1}^{p}\alpha_{ik}\,\langle u,v_k\rangle$$

for $i,j=1,\cdots,p$. Hence we have

$$\left\|u_1^V,\cdots,u_p^V\right\|_p^2=\begin{vmatrix}\sum_{k=1}^{p}\alpha_{1k}\langle u_1,v_k\rangle & \cdots & \sum_{k=1}^{p}\alpha_{pk}\langle u_1,v_k\rangle\\ \vdots & \ddots & \vdots\\ \sum_{k=1}^{p}\alpha_{1k}\langle u_p,v_k\rangle & \cdots & \sum_{k=1}^{p}\alpha_{pk}\langle u_p,v_k\rangle\end{vmatrix}$$

$$=\begin{vmatrix}\langle u_1,v_1\rangle & \cdots & \langle u_1,v_p\rangle\\ \vdots & \ddots & \vdots\\ \langle u_p,v_1\rangle^* & \cdots & \langle u_p,v_p\rangle\end{vmatrix}\begin{vmatrix}\alpha_{11} & \cdots & \alpha_{p1}\\ \vdots & \ddots & \vdots\\ \alpha_{1p} & \cdots & \alpha_{pp}\end{vmatrix}.$$

Next observe that

$$\begin{vmatrix}\langle u_{1,}v_1\rangle & \cdots & \langle u_{1,}v_p\rangle\\ \vdots & \ddots & \vdots\\ \langle u_{p,}v_1\rangle & \cdots & \langle u_{p,}v_p\rangle\end{vmatrix}=\begin{vmatrix}\langle u_1^V,v_1\rangle & \cdots & \langle u_1^V,v_p\rangle\\ \vdots & \ddots & \vdots\\ \langle u_p^V,v_1\rangle & \cdots & \langle u_p^V,v_p\rangle\end{vmatrix}$$

$$=\begin{vmatrix}\sum_{i=1}^{p}\alpha_{1i}\langle v_i,v_1\rangle & \cdots & \sum_{i=1}^{p}\alpha_{1i}\langle v_i,v_p\rangle\\ \vdots & \ddots & \vdots\\ \sum_{i=1}^{p}\alpha_{pi}\langle v_i,v_1\rangle & \cdots & \sum_{i=1}^{p}\alpha_{pi}\langle v_i,v_p\rangle\end{vmatrix}$$

$$=\begin{vmatrix}\alpha_{11} & \cdots & \alpha_{1p}\\ \vdots & \ddots & \vdots\\ \alpha_{p1} & \cdots & \alpha_{pp}\end{vmatrix}\begin{vmatrix}\langle v_1,v_1\rangle & \cdots & \langle v_1,v_p\rangle^*\\ \vdots & \ddots & \vdots\\ \langle v_p,v_1\rangle^* & \cdots & \langle v_p,v_p\rangle\end{vmatrix}.$$

Consequently, we have

$$\frac{\left\|u_1^V,\cdots,u_p^V\right\|_p^2}{\left\|u_1,\cdots,u_p\right\|_p^2}=\frac{\begin{vmatrix}\langle u_{1,}v_1\rangle & \cdots & \langle u_{1,}v_p\rangle\\ \vdots & \ddots & \vdots\\ \langle u_{p,}v_1\rangle & \cdots & \langle u_{p,}v_p\rangle\end{vmatrix}^2}{\left\|u_1,\cdots,u_p\right\|^2\left\|v_1,\cdots,v_p\right\|^2}.$$

As a consequence of this formula, we have Kurepa's generalization of the Cauchy-Schwarz inequality (see [9]). Next, we will determine an formula for the cosine of the angle between subspaces U and V that intersects on subspaces W of $(X,\langle\cdot,\cdot\rangle)$. The formula for the angle $U=span\{u,w_1,\cdots,w_r\}$ and $V=span\{v_1,\cdots,v_q,w_1,\cdots,w_r\}$ can be obtained as follows.

Proposition 3. Let $(X,\langle\cdot,\cdot\rangle)$ be a real inner product space. If $U=span\{u,w_1,\cdots,w_r\}$ is a $1+r$-dimensional subspace and $V=span\{v_1,\cdots,v_q,w_1,\cdots,w_r\}$ is a $(q+r)$-dimensional subspace of X that intersects on r-dimensional subspace $W=span\{w_1,\cdots,w_r\}$ with $r\geq 1$ then the angle between subspaces U and V is θ with $\cos^2\theta=\frac{\|(u^V)_W^\perp\|^2}{\|u_W^\perp\|^2}$ where $(u^V)_W^\perp$ and $u_W^\perp$ are the orthogonal complement of u^V and u, respectively, on W.

Proof. The projection of u on V is u^V. Next, we may write $u^V=(u^V)_W+(u^V)_W^\perp$ where $(u^V)_W$ is the projection of u^V on W and $(u^V)_W^\perp$ is the orthogonal complement of u^V on W. In line with this, we may write $u=u_W+u_W^\perp$ where u_W is the projection of u on W and $u_W^\perp$ is the orthogonal complement of u on W. Using the standard $(1+r)$-norm, we obtain

$$\cos^2\theta=\frac{\|u^V,w_1,\cdots,w_r\|_{1+r}^2}{\|u,w_1,\cdots,w_r\|_{1+r}^2}=\frac{\left\|(u^V)_W+(u^V)_W^\perp,w_1,\cdots,w_r\right\|_{1+r}^2}{\|u_W+u_W^\perp,w_1,\cdots,w_r\|_{1+r}^2}$$

$$= \frac{\left\|(u^V)_W^{\perp}, w_1, \cdots, w_r\right\|_{1+r}^2}{\|u_W^{\perp}, w_1, \cdots, w_r\|_{1+r}^2} = \frac{\left\|(u^V)_W^{\perp}\right\|^2}{\|u_W^{\perp}\|^2}. \quad \blacksquare$$

This formula tells us that the value of $\cos^2\theta$ is equal to the ratio between the length of the orthogonal complement of u^V on W and the length of the projection of u on W. More generally, the angle that intersects on subspaces $W = span\{w_1, \cdots, w_r\}$ of X is poured in following theorem:

Theorem 4. Let $(X, \langle\cdot,\cdot\rangle)$ be a real inner product space. If $U = span\{u_1, \cdots, u_p, w_1, \cdots, w_r\}$ is a $(p + r)$-dimensional subspace and $V = span\{v_1, \cdots, v_q, w_1, \cdots, w_r\}$ is a $(q + r)$-dimensional subspace on X with $p \le q$ that intersects on r-dimensional subspace $W = span$ $\{w_1, \cdots, w_r\}$ with $r \ge 1$ then the angle between subspaces U and V is θ with $\cos^2\theta = \frac{\|(u_1^V)_W^{\perp}, \cdots, (u_p^V)_W^{\perp}\|_p^2}{\|(u_1)_W^{\perp}, \cdots, (u_p)_W^{\perp}\|_p^2}$, where $(u_i^V)_W^{\perp}$ and $(u_i)_W^{\perp}$ are the orthogonal complement of u_i^V and u_i, respectively, on W for $i = 1, \cdots, p$.

Proof. The projection of u_i on V is u_i^V. Next, we may write $u_i^V = (u_i^V)_W + (u_i^V)_W^{\perp}$ where $(u_i^V)_W$ is the projection of u_i^V on W and $(u_i^V)_W^{\perp}$ is the orthogonal complement of u_i^V on W. In line with this, we may write $u_i = (u_i)_W + (u_i)_W^{\perp}$ where $(u_i)_W$ is the projection of u_i on W and $(u_i)_W^{\perp}$ is the orthogonal complement of u_i on W for $i = 1, \cdots, p$. Using the standard $(p + r)$-norm, we obtain

$$\cos^2\theta = \frac{\left\|u_1^V, \cdots, u_p^V, w_1, \cdots, w_r\right\|_{p+r}^2}{\left\|u_1, \cdots, u_p, w_1, \cdots, w_r\right\|_{p+r}^2}$$

$$= \frac{\left\|(u_1^V)_W + (u_1^V)_W^{\perp}, \cdots, (u_p^V)_W + (u_p^V)_W^{\perp}, w_1, \cdots, w_r\right\|_{p+r}^2}{\left\|(u_1)_W + (u_1)_W^{\perp}, \cdots, (u_p)_W + (u_p)_W^{\perp}, w_1, \cdots, w_r\right\|_{p+r}^2}$$

$$= \frac{\left\|(u_1^V)_W^{\perp}, \cdots, (u_p^V)_W^{\perp}\right\|_p^2 \|w_1, \cdots, w_r\|_r^2}{\left\|(u_1)_W^{\perp}, \cdots, (u_p)_W^{\perp}\right\|_p^2 \|w_1, \cdots, w_r\|_r^2} = \frac{\left\|(u_1^V)_W^{\perp}, \cdots, (u_p^V)_W^{\perp}\right\|_p^2}{\left\|(u_1)_W^{\perp}, \cdots, (u_p)_W^{\perp}\right\|_p^2}. \quad \blacksquare$$

2.2. Angles between subspaces in an n-inner product space

In this subsection, we will discuss angles between subspaces in an n-inner product space and its connection with angles in an inner product space. Let $(X, \langle\cdot,\cdot \mid \cdot, \ldots, \cdot\rangle)$ be a real n-inner product space with $n \ge 2$. Fix a linearly independent set $\{a_1, \ldots, a_n\}$ in X with respect to $\{a_1, \ldots, a_n\}$, define the function $\langle\cdot,\cdot\rangle^*$ by

$$\langle x, y\rangle^* = \sum_{\{i_2,\ldots,i_n\}\subseteq\{1,\ldots,n\}} \langle x, y | a_{i_2}, \ldots, a_{i_n}\rangle$$

for each $x, y \in X$. Then we have the following proposition.

Proposition 5. [3] The function $\langle\cdot,\cdot\rangle^*$ defines an inner product on X.

Corollary 6. Let $\|\cdot, \ldots, \cdot\|$ be a n-norm that induced from an n-inner product. The following function

$$\|x\|^* = \left[\sum_{\{i_2,\ldots,i_n\}\subseteq\{1,\ldots,n\}} \|x, a_{i_2}, \ldots, a_{i_n}\|^2\right]^{\frac{1}{2}}$$

defines a norm that corresponds to an inner product $\langle\cdot,\cdot\rangle^*$ pada X.

Using an inner product $\langle\cdot,\cdot\rangle^*$, we have a new standard n-inner product $\langle\cdot,\cdot\mid\cdot,\ldots,\cdot\rangle^*$ on X, namely

$$\langle x_0,x_1|x_2,\ldots,x_n\rangle^* = \begin{vmatrix} \langle x_0,x_1\rangle^* & \langle x_0,x_2\rangle^* & \cdots & \langle x_0,x_n\rangle^* \\ \langle x_2,x_1\rangle^* & \langle x_2,x_2\rangle^* & \cdots & \langle x_2,x_n\rangle^* \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle^* & \langle x_n,x_2\rangle^* & \cdots & \langle x_n,x_n\rangle^* \end{vmatrix}$$

and a new standard n-norm $\|x_1,\ldots,x_n\|^* := (\langle x_1,x_1|x_2,\ldots,x_n\rangle^*)^{\frac{1}{2}}$. Furthermore, one may also use an inner product $\langle\cdot,\cdot\rangle^*$ and its induced norm to study the angle between subspaces in an n-inner product space. Inspired by Definition 1 with the new standard n-norm, we define the angle between subspaces in an n-inner product space.

Definition 7. Let $(X,\langle\cdot,\cdot\mid\cdot,\ldots,\cdot\rangle)$ be a real n-inner product space. If $U = span\{u_1,\cdots,u_p\}$ is a p-dimensional subspace and $V = span\{v_1,\cdots,v_q\}$ is a q-dimensional subspace of X with $p \le q$, then the angle between subspaces U and V is defined by θ with

$$\cos^2\theta = \frac{\left(\left\|u_1^V,\cdots,u_p^V\right\|_p^*\right)^2}{\left(\left\|u_1,\cdots,u_p\right\|_p^*\right)^2},$$

where u_i^V denote the projection of $u_i$on V for each $i = 1,\cdots,p$ and $\|\cdot,\cdots,\cdot\|_p^*$ denotes the standard p-norm on $(X,\langle\cdot,\cdot\rangle^*)$.

According to Definition 7, we will determine an formula for the cosine of the angle between subspaces U and V that intersects on subspaces W of $(X,\langle\cdot,\cdot\mid\cdot,\ldots,\cdot\rangle)$. The formula for the angle $U = span\{u,w_1,\cdots,w_r\}$ and $V = span\{v_1,\cdots,v_q,w_1,\cdots,w_r\}$ can be obtained as follows.

Proposition 8. Let $(X,\langle\cdot,\cdot\mid\cdot,\ldots,\cdot\rangle)$ be a real n-inner product space. If $U = span\{u,w_1,\cdots,w_r\}$ is a $(1+r)$-dimensional subspace and $V = span\{v_1,\cdots,v_q,w_1,\cdots,w_r\}$ is a $(q+r)$-dimensional subspace on X that intersects on r-dimensional subspace $W = span\{w_1,\cdots,w_r\}$ with $r \ge 1$ then the angle between subspaces U and V is $(0 \le \theta \le \frac{\pi}{2})$, with $\cos^2\theta = \frac{\left(\|(u^V)_W^\perp\|^*\right)^2}{\left(\|u_W^\perp\|^*\right)^2}$ where $(u^V)_W^\perp$ and $u_W^\perp$ are the orthogonal complement of u^V and u, respectively, on W.

Proof. Writing $u^V = (u^V)_W + (u^V)_W^\perp$ and $u = u_W + u_W^\perp$ and using the new standard $(1+r)$-norm, we obtain

$$\cos^2\theta = \frac{(\|u^V,w_1,\cdots,w_r\|_{1+r}^*)^2}{(\|u,w_1,\cdots,w_r\|_{1+r}^*)^2} = \frac{\left\|(u^V)_W + (u^V)_W^\perp,w_1,\cdots,w_r\right\|_{1+r}^2}{\|u_W + u_W^\perp,w_1,\cdots,w_r\|_{1+r}^2}$$
$$= \frac{\left(\left\|(u^V)_W^\perp,w_1,\cdots,w_r\right\|_{1+r}^*\right)^2}{(\|u_W^\perp,w_1,\cdots,w_r\|_{1+r}^*)^2} = \frac{\left(\|(u^V)_W^\perp\|^*\right)^2}{(\|u_W^\perp\|^*)^2}. \quad \blacksquare$$

Using Definition 7 and following the proof of Theorem 4, the angle $U = span\{u_1,\cdots.u_p,w_1,\cdots,w_r\}$ and $V = span\{v_1,\cdots,v_q,w_1,\cdots,w_r\}$ with $p \le q$ can be obtained as follows.

Theorem 9. Let $(X,\langle\cdot,\cdot\mid\cdot,\ldots,\cdot\rangle)$ be a real n-inner product space. If $U = span\{u_1,\cdots,u_p,w_1,\cdots,w_r\}$ is a $(p+r)$-dimensional subspace and

$V = span\{v_1, \cdots, v_q, w_1, \cdots, w_r\}$ is a $(q+r)$-dimensional subspace on X with $p \leq q$ that intersects on r-dimensional subspace $W = span\{w_1, \cdots, w_r\}$ with $r \geq 1$ then the angle between subspaces U and V is $(0 \leq \theta \leq \frac{\pi}{2})$, with

$$\cos^2\theta = \frac{\left(\left\|(u_1^V)_W^\perp, \cdots, (u_p^V)_W^\perp\right\|_p^*\right)^2}{\left(\left\|(u_1)_W^\perp, \cdots, (u_p)_W^\perp\right\|_p^*\right)^2},$$

where $(u_i^V)_W^\perp$ and $(u_i)_W^\perp$ are the orthogonal complement of u_i^V and u_i, respectively, on W for $i = 1, \cdots, p$.

Before we discuss the connection between angles in an inner product space and in an n-inner product space, we have equivalent norm on $(X, \langle\cdot,\cdot\rangle)$ with norm $\|\cdot\|^*$ where $\{a_1, \ldots, a_n\}$ are an orthonormal set on $(X, \langle\cdot,\cdot \mid \cdot, \ldots, \cdot\rangle)$ as follows.

Proposition 10. [6] Norm $\|\cdot\|^*$ equivalent with norm that corresponds to the inner product $\|\cdot\|$ on $(X, \langle\cdot,\cdot\rangle)$. Namely,

$$\|x\| \leq \|x\|^* \leq \sqrt{n}\|x\|$$

for every $x \in X$.

From this proposition, we have the connection between angles in an inner product space and in an n-inner product space, namely

Theorem 11. If θ_1 is the angle between subspaces U and V of $(X, \langle\cdot,\cdot \mid \cdot, \ldots, \cdot\rangle)$ and θ_2 is the angle between subspaces U and V of $(X, \langle\cdot,\cdot\rangle)$ with dim $U = 1 + r$, dim $V = q + r$ for $q \geq 1$ and dim $(U \cap V) = r$ for $r \geq 1$ then

$$\frac{1}{n}\cos^2\theta_2 \leq \cos^2\theta_1 \leq n\cos^2\theta_2.$$

Proof. Writing $U = span\{u, w_1, \cdots, w_r\}$, $V = span\{v_1, \cdots, v_q, w_1, \cdots, w_r\}$ and $U \cap V$ $= span\{w_1, \cdots, w_r\}$. Next, we observe that

$$\frac{\left\|(u^V)_W^\perp\right\|^2}{n\|u_W^\perp\|^2} \leq \frac{\left(\left\|(u^V)_W^\perp\right\|^*\right)^2}{(\|u_W^\perp\|^*)^2} \leq \frac{n\left\|(u^V)_W^\perp\right\|^2}{\|u_W^\perp\|^2}.$$

According to Proposition 3 and 8, we have $\cos^2\theta_1 = \frac{\left(\|(u^V)_W^\perp\|^*\right)^2}{\left(\|u_W^\perp\|^*\right)^2}$ and $\cos^2\theta_2 = \frac{\|(u^V)_W^\perp\|^2}{\|u_W^\perp\|^2}$.

Hence, we obtain $\frac{1}{n}\cos^2\theta_2 \leq \cos^2\theta_1 \leq n\cos^2\theta_2$. ∎

By Theorem 11 for $= 1$, the value of $\cos^2\theta_1$ is equal to the value of $\cos^2\theta_2$. Nevertheless, the upper bound of $\cos\ \theta_1$ for $n \geq 2$ is inappropriate because its value is greater than 1. If $\cos\ \theta_2 = 1$ then the lower bound of $\cos\ \theta_1$ is $\frac{1}{n}$.

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