The Boundedness of Bessel-Riesz Operators On Morrey Spaces

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Abstract. In this paper, we shall discuss about Bessel-Riesz operators. Kurata *et. al* [6] have investigated their boundedness on generalized Morrey spaces with weight. The boundedness of these operators on Lebesgue spaces and Morrey spaces will be reproved using a different approach. Moreover, we also find the norm of the operators are bounded by the norm of the kernels.

Keywords: Bessel-Riesz operators, Hardy-Littlewood maximal operator, Morrey spaces.

INTRODUCTION

Let $0 < \gamma$, $0 < \alpha < n$ and define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y) f(y)\, dy$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, where $p \geq 1$, $K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^{\gamma}}$, $x \in \mathbb{R}^n$. Here, $K_{\alpha,\gamma}$ can be viewed as multiple of two kernels, $K_{\alpha,\gamma}(x) = J_\gamma(x) K_\alpha(x)$ for every $x \in \mathbb{R}^n$. In [9], J_γ and K_α are known as *Bessel kernel* and *Riesz kernel*. So, $K_{\alpha,\gamma}$ is called *Bessel-Riesz kernel* and $I_{\alpha,\gamma}$ is called *Bessel-Riesz operator*.

For $\gamma = 0$, we have $I_{\alpha,0} = I_\alpha$ (is called *fractional integral operator* or *Riesz potential* [9]). Studies about I_α were started since 1920's. Hardy-Littlewood [4, 5] and Sobolev [8] proved the boundedness of I_α on *Lebesgue spaces* through the inequality $\|I_\alpha f\|_{L^q} \leq C_p \|f\|_{L^p}$, for every $f \in L^p(\mathbb{R}^n)$, $1 < p < \frac{\alpha}{n}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

For $1 \leq p \leq q$, *the (classical) Morrey space* $L^{p,q}(\mathbb{R}^n)$ is defined by

$$L^{p,q}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty \right\},$$

where $\|f\|_{L^{p,q}} := \sup_{r>0, a\in\mathbb{R}^n} r^{n(1/q-1/p)} \left(\int_{|x-a|<r} |f(x)|^p\, dx \right)^{1/p}$. We have an inclusion property for Morrey spaces $L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n)$.

On Morrey spaces, Spanne [7] has shown that I_α is bounded form $L^{p_1,q_1}(\mathbb{R}^n)$ to $L^{p_2,q_2}(\mathbb{R}^n)$ for $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$. Furthermore, Adams [1] and Chiarenza-Frasca [2] obtained a stronger result.

Theorem 1 *[Adams, Chiarenza-Frasca] If* $0 < \alpha < n$ *then we have*

$$\|I_\alpha f\|_{L^{p_2,q_2}} \leq C_{p_1,q_1} \|f\|_{L^{p_1,q_1}},$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ *where* $1 < p_1 < q_1 < \frac{n}{\alpha}, \frac{1}{p_2} = \frac{1}{p_1}\left(1 - \frac{\alpha q_1}{n}\right)$, *and* $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$.

Meanwhile, we have $\left|I_{\alpha,\gamma} f(x)\right| \leq |I_\alpha f(x)|$, for every $f \in L^p_{loc}(\mathbb{R}^n)$. Using this inequality, $I_{\alpha,\gamma}$ is bounded on these spaces. In 1999, Kurata *et. al* [6] have proved that $W \cdot I_{\alpha,\gamma}$ is bounded on generalized Morrey spaces where W is a multiplication operator. Here, we shall discuss about the boundedness of $I_{\alpha,\gamma}$ on Lebesgue spaces and Morrey spaces. We shall see the influence of $K_{\alpha,\gamma}$ for the boundedness of $I_{\alpha,\gamma}$.

PRELIMINARY RESULTS

We can see that the kernel of Bessel-Riesz vanishes faster at infinity than that of the fractional integral operator. From this fact, we can show that the kernel of Bessel-Riesz is a member of some Lebesgue spaces. We begin with the following lemma.

Lemma 2 *If $b > a > 0$ then $\sum_{k\in\mathbb{Z}} \frac{(u^k R)^a}{(1+u^k R)^b} < \infty$, for every $u > 1$ and $R > 0$.*

Proof. Let $b > a > 0$, so that $b - a > 0$. We write $\sum_{k\in\mathbb{Z}} \frac{(u^k R)^a}{(1+u^k R)^b} = \sum_{k=-1}^{-\infty} \frac{(u^k R)^a}{(1+u^k R)^b} + \sum_{k=0}^{\infty} \frac{(u^k R)^a}{(1+u^k R)^b}$. Next, we estimate $\sum_{k=-1}^{-\infty} \frac{(u^k R)^a}{(1+u^k R)^b} \le \sum_{k=-1}^{-\infty} \left(u^k R\right)^a < \infty$ and $\sum_{k=0}^{\infty} \frac{(u^k R)^a}{(1+u^k R)^b} \le \sum_{k=0}^{\infty} \left(u^k R\right)^{a-b} < \infty$. Hence, we obtain $\sum_{k\in\mathbb{Z}} \frac{(u^k R)^a}{(1+u^k R)^b} < \infty$. ■

Lemma 2 is useful to prove the membership of $K_{\alpha,\gamma}$ in some Lebesgue spaces.

Theorem 3 *If $0 < \alpha < n$ and $0 < \gamma$ then $K_{\alpha,\gamma} \in L^t(\mathbb{R}^n)$ and $\left\|K_{\alpha,\gamma}\right\|_{L^t} \sim \left(\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}}\right)^{\frac{1}{t}}$, for $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$.*

Proof. Suppose $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$ where $0 < \gamma,\ 0 < \alpha < n$, so that $(\alpha - n)t + n > 0$. For arbitrary $R > 0$, write

$$\int_{\mathbb{R}^n} \left|K_{\alpha,\gamma}(y)\right|^t dy = \int_{\mathbb{R}^n} \frac{|y|^{(\alpha-n)t}}{(1+|y|)^{\gamma t}} dy = C_1 \int_{\mathbb{R}^+} \frac{r^{(\alpha-n)t+n-1}}{(1+r)^{\gamma t}} dr = C_1 \sum_{k\in\mathbb{Z}} \int_{2^k R \le r < 2^{k+1} R} \frac{r^{(\alpha-n)t+n-1}}{(1+r)^{\gamma t}} dr, \quad C_1 > 0.$$

We obtain $\int_{\mathbb{R}^n} \left|K_{\alpha,\gamma}(y)\right|^t dy \le C_1 \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma t}} \int_{2^k R \le r < 2^{k+1} R} r^{(\alpha-n)t+n-1} dr = C_2 \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}}, C_2 = \frac{C_1(2^{(\alpha-n)t+n}-1)}{(\alpha-n)t+n}$ and $\int_{\mathbb{R}^n} \left|K_{\alpha,\gamma}(y)\right|^t dy \ge \frac{C_1}{2^{\gamma t}} \sum_{k\in\mathbb{Z}} \frac{1}{(1+2^k R)^{\gamma t}} \int_{2^k R \le r < 2^{k+1} R} r^{(\alpha-n)t+n-1} dr = C_3 \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}}, \quad C_3 = \frac{C_1(2^{(\alpha-n)t+n}-1)}{2^{\gamma s}[(\alpha-n)t+n]}$. Therefore $\int_{\mathbb{R}^n} \left|K_{\alpha,\gamma}(y)\right|^t dy \backsim \sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}}$ for every $R > 0$. Using Lemma 2, take $t \in \left(\frac{n}{n+\gamma-\alpha}, \frac{n}{n-\alpha}\right)$, choose $u = 2$, and define $a := (\alpha - n)t + n$, $b := \gamma t$. We get $\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} < \infty$. Hence $K_{\alpha,\gamma} \in L^t(\mathbb{R}^n)$. ■

In this study, the membership of $K_{\alpha,\gamma}$ in Lebesgue spaces is an important result. With the result, we can use *Young inequality* [3] to prove the boundedness of $I_{\alpha,\gamma}$ on Lebesgue spaces.

Theorem 4 *(Young's inequality) Let $1 \le p, q, t \le \infty$ satisfy $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$, then we have*

$$\|g * f\|_{L^q} \le \|g\|_{L^t} \|f\|_{L^p}$$

for every $g \in L^t(\mathbb{R}^n)$, $f \in L^p(\mathbb{R}^n)$.

Corollary 5 *For $0 < \alpha < n$, $\gamma > 0$, we have*

$$\left\|I_{\alpha,\gamma} f\right\|_{L^q} \le \left\|K_{\alpha,\gamma}\right\|_{L^t} \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^n)$ where $1 \le p < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$.

By the above corollary, we can say that $I_{\alpha,\gamma}$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. Moreover, norm of kernel Bessel-Riesz dominates norm of $I_{\alpha,\gamma} f$. Consequently in Lebesgue spaces, we obtain $\left\|I_{\alpha,\gamma}\right\| \le \left\|K_{\alpha,\gamma}\right\|_{L^t}$.

We shall extend the boundedness of $I_{\alpha,\gamma}$ on Morrey spaces, but Young's inequality is not available on Morrey spaces. Using the *Hardy-Littlewood maximal operator* M, the boundedness of $I_{\alpha,\gamma}$ can be reproved on Lebesgue spaces and Morrey spaces. The operator M is defined by

$$Mf(x) := \sup_{x\in B} \frac{1}{|B|} \int_B |f(y)|\, dy, x \in \mathbb{R}^n,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$ where $|B|$ denotes *Lebesgue measure* of ball $B = B(a, r)$ (centered at $a \in \mathbb{R}^n$ with radius $r > 0$). The supremum is taken over all open balls in $\mathbb{R}^n$. It is well known that the operator M is bounded on Lebesgue spaces $(L^p(\mathbb{R}^n),\ p > 1)$ [9, 10] and Morrey spaces [2].

MAIN RESULTS

In this section, we are going to discuss about the boundedness of the Bessel-Riesz operators on Morrey spaces. In the previous section, we have $K_{\alpha,\gamma} \in L^t(\mathbb{R}^n)$ where $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$ and the inclusion property of Morrey spaces, so $K_{\alpha,\gamma} \in L^{s,t}(\mathbb{R}^n)$ where $1 \le s \le t$. Accordingly, we have the following theorem.

Theorem 6 *Let $0 < \alpha < n$, $0 < \gamma$, then we have*

$$\left\|I_{\alpha,\gamma} f\right\|_{L^{p_2,q_2}} \le C_{p_1,q_1} \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ where $1 < p_1 < q_1 < t'$, $1 \le s \le t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{q_1}{p_1 t'}$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{t'}$.

Proof. Suppose $0 < \alpha < n$, $0 < \gamma$ and take $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $1 \le s \le t$. Let $f \in L^{p_1,q_1}(\mathbb{R}^n)$, $1 < p_1 < q_1 < t'$. For every $x \in \mathbb{R}^n$, write $I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x)$ where $I_1(x) := \int_{|x-y|<R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^{\gamma}} dy$ and $I_2(x) := \int_{|x-y|\ge R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^{\gamma}} dy$, $R > 0$. To estimate I_1 and I_2, we use dyadic decomposition. Now, estimate I_1

$$|I_1(x)| \le C_1 \sum_{k=-1}^{-\infty} \frac{\left(2^k R\right)^{\alpha-n}}{(1+2^k R)^{\gamma}} \int_{2^k R \le |x-y| < 2^{k+1} R} |f(y)|\, dy \le C_2 M f(x) \sum_{k=-1}^{-\infty} \frac{\left(2^k R\right)^{\alpha-n+n/s} \left(2^k R\right)^{n/s'}}{(1+2^k R)^{\gamma}}.$$

By using Hölder's inequality, we get

$$\begin{aligned} |I_1(x)| &\le C_3 M f(x) \left(\sum_{k=-1}^{-\infty} \frac{\left(2^k R\right)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-1}^{-\infty} \left(2^k R\right)^{n} \right)^{1/s'} \\ &\le C_4 M f(x) \frac{\left(\int_{|x-y|<R} K_{\alpha,\gamma}^s (x-y)\, dy \right)^{1/s}}{R^{n(1/s-1/t)}} R^{n(1/s-1/t)} R^{n/s'} \le C_4 \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} M f(x) R^{n/t'}. \end{aligned}$$

Hölder's inequality is used again to estimate I_2:

$$|I_2(x)| \le C_5 \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha-n}}{(1+2^k R)^{\gamma}} \int_{2^k R \le |x| < 2^{k+1} R} |f(y)|\, dy \le C_5 \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha-n}}{(1+2^k R)^{\gamma}} \left(\int_{2^k R \le |x| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1} \left(2^k R\right)^{n/p_1'}.$$

Next, we write

$$|I_2(x)| \le C_6 \|f\|_{L^{p_1,q_1}} \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha-n+n-n/q_1}}{(1+2^k R)^{\gamma}} \frac{\left(\int_{2^k R \le |x| < 2^{k+1} R} dy \right)^{1/s}}{\left(2^k R\right)^{n/s}} \le C_6 \|f\|_{L^{p_1,q_1}} \sum_{k=0}^{\infty} \frac{\left(\int_{2^k R \le |x| < 2^{k+1} R} \frac{|x-y|^{(\alpha-n)s}}{(1+|x-y|)^{\gamma s}} dy \right)^{1/s}}{\left(2^k R\right)^{n/q_1-n} \left(2^k R\right)^{n/s}},$$

and we obtain $|I_2(x)| \le C_6 \|f\|_{L^{p_1,q_1}} \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \sum_{k=0}^{\infty} \left(2^k R\right)^{n/t'-n/q_1} \le C_7 \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}} R^{n(1/t'-1/q_1)}$. Summing the two estimates, we get $\left|I_{\alpha,\gamma} f(x)\right| \le C \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \left(M f(x) R^{n/t'} + \|f\|_{L^{p_1,q_1}} R^{n/t'-n/q_1} \right)$, for each $x \in \mathbb{R}^n$.

Assume that f is not identically 0 and Mf is finite everywhere. Choose $R > 0$ such that $R^{n/q_1} = \frac{\|f\|_{L^{p_1,q_1}}}{Mf(x)}$. We get $\left|I_{\alpha,\gamma} f(x)\right| \le C \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}^{q_1/t'} M f(x)^{1-q_1/t'}$. Define $\frac{1}{p_2} := \frac{(t'-q_1)}{p_1 t'}$ and $\frac{1}{q_2} := \frac{1}{q_1} - \frac{1}{t'}$. For arbitrary $r > 0$, we have

$$\left(\int_{|x|<r} \left|I_{\alpha,\gamma} f(x)\right|^{p_2} dx \right)^{1/p_2} \le C \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}^{1-p_1/p_2} \left(\int_{|x|<r} |M f(x)|^{p_1} dx \right)^{(1/p_2)}.$$

Divide by $r^{n/p_2-n/q_2}$ and take supremum to get

$$\begin{aligned} \left\|I_{\alpha,\gamma} f\right\|_{L^{p_2,q_2}} &= \sup_{r>0} \frac{\left(\int_{|x|<r} \left|I_{\alpha,\gamma} f(x)\right|^{p_2} dx \right)^{1/p_2}}{r^{n/p_2-n/q_2}} \\ &\le C \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}^{1-p_1/p_2} \sup_{r>0} \frac{\left(\int_{|x|<r} |M f(x)|^{p_1} dx \right)^{(1/p_2)}}{r^{n/p_2-n/q_2}} = C \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,q_1}}^{p_1/p_2}. \end{aligned}$$

Using the boundedness of M on Morrey spaces (Chiarenza-Frasca's Theorem [2]), we obtain an inequality $\left\|I_{\alpha,\gamma}f\right\|_{L^{p_2,q_2}} \leq C_{p_1,q_1} \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}}$. ∎

By Theorem 6 and the inclusion property of Morrey spaces, for $1 \leq s \leq t$, we have

$$\left\|I_{\alpha,\gamma}f\right\|_{L^{p_2,q_2}} \leq C_{p_1,q_1} \left\|K_{\alpha,\gamma}\right\|_{L^{s,t}} \|f\|_{L^{p_1,q_1}} \leq C_{p_1,q_1} \left\|K_{\alpha,\gamma}\right\|_{L^{t}} \|f\|_{L^{p_1,q_1}}$$

where $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. We also obtain $\frac{p_2}{q_2} = \frac{p_1}{q_1}$. It is similar with Chiarenza-Frasca's result for the boundedness of fractional integral operators on Morrey spaces.

CONCLUDING REMARK

From the results of this study, we have seen that the norm of the Bessel-Riesz kernel domines the norm of $I_{\alpha,\gamma}f$ for every f in Morrey space $L^{p,q}(\mathbb{R}^n)$ (p and q are suitable number). Moreover, using $K_{\alpha,\gamma} \in L^{s,t}(\mathbb{R}^n)$, $1 \leq s < t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, the norm of the Bessel-Riesz kernel closer to the norm of $I_{\alpha,\gamma}f$ than using $K_{\alpha,\gamma} \in L^{t}(\mathbb{R}^n)$. In the future, we shall continue this study to prove the boundedness of generalized Bessel-Riesz operators on Morrey spaces and generalized Morrey spaces.

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