BOUNDED LINEAR FUNCTIONALS ON THE n-NORMED SPACE OF p-SUMMABLE SEQUENCES

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ABSTRACT. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a real n-normed space, as introduced by S. Gähler in 1969. We shall be interested in bounded linear functionals on X, using the n-norm as our main tool. We study the duality properties and show that the space X' of bounded linear functionals on X also forms an n-normed space. We shall present more results on bounded multilinear n-functionals on the space of p-summable sequences being equipped with an n-norm. Open problems are also posed.

1. Introduction

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$. A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties:

N.1. $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,
N.2. $\|x_1, \dots, x_n\|$ is invariant under permutation,
N.3. $\|\alpha x_1, \dots, x_n\| = |\alpha| \; \|x_1, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$,
N.4. $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

In an n-normed space $(X, \|\cdot, \dots, \cdot\|)$, one may observe that $\|x_1, \dots, x_n\| \geq 0$ and

$$\|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2, \dots, x_n\| = \|x_1, x_2, \dots, x_n\| \tag{1.1}$$

for every $x_1, \dots, x_n \in X$ dan $\alpha_2, \dots, \alpha_n \in \mathbb{R}$.

If $(X, \|\cdot\|)$ is a normed space and X' is its dual (consisting of bounded linear functionals on X), the following function defines an n-norm on X:

$$\|x_1, \dots, x_n\|^G := \sup_{f_i \in X', \; \|f_i\| \leq 1} \begin{vmatrix} f_1(x_1) & \cdots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{vmatrix}. \tag{1.2}$$

Meanwhile, if $(X, \langle \cdot, \cdot \rangle)$ is an inner product space, we can define the standard n-norm on X by

$$\|x_1, \dots, x_n\|^S := \sqrt{\det(\langle x_i, x_j \rangle)}. \tag{1.3}$$

Generally, the value of $\|x_1, \dots, x_n\|$ may be interpreted as the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ [5].

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The concept of n-normed spaces was initially introduced by Gähler [1, 2, 3, 4] in the 1960's. Recent results and related topics may be found in [7, 8, 9, 10, 11].

In this paper, we shall be interested in studying bounded linear functionals on X, using the n-norm as our main tool. We prove an analog of the Riesz-Fréchet Theorem and show that the dual space X', consisting of all bounded linear functionals on X, also forms an n-normed space. We shall present more results when X is the space of p-summable sequences being equipped with an n-norm. In addition, some open problems will be posed.

2. Bounded Linear Functionals

Let $(X, \|\cdot,\dots,\cdot\|)$ be a real n-normed space and $f : X \to \mathbb{R}$ be a linear functional on X. We may define bounded linear functionals on X by using the n-norm in several ways as follows.

2.1. **Bounded linear functionals (of 1st index).** Fix a linearly independent set $Y := \{y_1,\dots,y_n\}$ in X. We say that f is *bounded with respect to* Y if and only if there exists $K > 0$ such that

$$|f(x)| \le K \sum \|x, y_{i_2},\dots,y_{i_n}\| \tag{2.1}$$

for all $x \in X$, where the sum is taken over $\{i_2,\dots,i_n\} \subset \{1,\dots,n\}$ with $i_2 < \cdots < i_n$. [One might ask why we do not just take a linearly independent set $\{y_2,\dots,y_n\}$ in X and put $|f(x)| \le K\,\|x, y_2,\dots,y_n\|$ for all $x \in X$. The drawback with this is that for a nonzero vector x in the linear span of $\{y_2,\dots,y_n\}$, we have $\|x, y_2,\dots,y_n\| = 0$ while $f(x) \ne 0$. This problem is overcome by taking a set of n linearly independent vectors and form the sum as in (2.1). Indeed, one might observe that the sum is equal to 0 if and only if $x = 0$.]

For simplicity, we shall say 'bounded' instead of 'bounded with respect to Y'. Clearly the set X'_1 of all linear functionals which are bounded on X forms a vector space. Now, for $f \in X'_1$, we define

$$\|f\|_1 := \inf\{K > 0 : (2.1) \text{ holds}\}. \tag{2.2}$$

It is easy to see that

$$\|f\|_1 = \sup\{|f(x)| : \sum \|x, y_{i_2},\dots,y_{i_n}\| \le 1\}$$

Moreover, the formula (2.2) defines a norm on X'_1.

To give an example, we invoke the notion of n-inner product spaces [11]. Assume that X is of dimension $d \ge n+1$. A real-valued function $\langle\cdot,\cdot|\cdot,\dots,\cdot\rangle$ on X^{n+1} satisfying the following properties:

- I.1. $\langle x_1, x_1|x_2,\dots,x_n\rangle \ge 0$ and it is equal to 0 if and only if $x_1,\dots,x_n$ are linearly dependent,
- I.2. $\langle x_{i_1}, x_{i_1}|x_{i_2},\dots,x_{i_n}\rangle = \langle x_1, x_1|x_2,\dots,x_n\rangle$ for any permutation $\{i_1,\dots,i_n\}$ of $\{1,\dots,n\}$,
- I.3. $\langle x, y|x_2,\dots,x_n\rangle = \langle y, x|x_2,\dots,x_n\rangle$,
- I.4. $\langle \alpha x, y|x_2,\dots,x_n\rangle = \alpha\langle x, y|x_2,\dots,x_n\rangle$ for any $\alpha \in \mathbb{R}$,
- I.5. $\langle x + x', y|x_2,\dots,x_n\rangle = \langle x, y|x_2,\dots,x_n\rangle + \langle x', y|x_2,\dots,x_n\rangle$,

is called an *n-inner product* on X, and the pair $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ is called an *n-inner product space.*

Note that if $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ is an n-inner product space, then we can define an n-norm $\|\cdot,\ldots,\cdot\|$ on X by

$$\|x_1, x_2, \ldots, x_n\| := \langle x_1, x_1|x_2, \ldots, x_n\rangle^{1/2}.$$

Here we have the Cauchy-Schwarz inequality:

$$|\langle x, y|x_2, \ldots, x_n\rangle| \le \|x, x_2, \ldots, x_n\| \, \|y, x_2, \ldots, x_n\|.$$

Now we give an example of bounded linear functionals on X. Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle)$ be an n-inner product space, and $\|\cdot,\ldots,\cdot\| := \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle^{1/2}$ be the induced n-norm on X. With respect to the set $Y = \{y_1, \ldots, y_n\}$, define $f : X \to \mathbb{R}$ by

$$f(x) := \sum \langle x, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle, \tag{2.3}$$

where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$ and $i_1 \in \{1, \ldots, n\} \setminus \{i_2, \ldots, i_n\}$. Clearly f is linear. Furthermore, we have:

Fact 2.1 *The linear functional f defined by (2.3) is bounded with $\|f\|_1 = \|y_1, \ldots, y_n\|$.*

Proof. We observe that for every $x \in X$, we have

$$\begin{aligned} |f(x)| &\le \sum |\langle x, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle| \\ &\le \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \, \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| \\ &= \|y_1, \ldots, y_n\| \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \end{aligned}$$

where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. Thus f is bounded with $\|f\|_1 \le \|y_1, \ldots, y_n\|$.

To show that $\|f\|_1 = \|y_1, \ldots, y_n\|$, just take $x := \|y_1, \ldots, y_n\|^{-1} y_1$. Then we see that $\sum \|x, y_{i_2}, \ldots, y_{i_n}\| = 1$ and

$$\begin{aligned} |f(x)| &= \|y_1, \ldots, y_n\|^{-1} f(y_1) \\ &= \|y_1, \ldots, y_n\|^{-1} \sum \langle y_1, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle \\ &= \|y_1, \ldots, y_n\|^{-1} \langle y_1, y_1|y_2, \ldots, y_n\rangle \\ &= \|y_1, \ldots, y_n\|^{-1} \|y_1, \ldots, y_n\|^2 \\ &= \|y_1, \ldots, y_n\|. \end{aligned}$$

[Note that when $i_1 \ne 1$ and $\{i_2, \ldots, i_n\} = \{1, \ldots, n\} \setminus \{i_1\}$, we have

$$|\langle y_1, y_{i_1}|y_{i_2}, \ldots, y_{i_n}\rangle| \le \|y_1, y_{i_2}, \ldots, y_{i_n}\| \, \|y_{i_1}, y_{i_2}, \ldots, y_{i_n}\| = 0$$

because one of $y_{i_2}, \ldots, y_{i_n}$ must be equal to y_1.] □

2.2. **Bounded linear functionals of p-th index.** Fix a linearly independent set $Y := \{y_1, \ldots, y_n\}$ in X and $1 \le p \le \infty$. We say that f is *bounded of p-th index* (with respect to Y) if and only if there exists $K > 0$ such that

$$|f(x)| \le K \left(\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p\right)^{1/p} \tag{2.4}$$

where the sum is taken over $\{i_2, \ldots, i_n\} \subset \{1, \ldots, n\}$ with $i_2 < \cdots < i_n$. [If $p = \infty$, then the sum is the maximum of all possible values of $\|x, y_{i_2}, \ldots, y_{i_n}\|$.]

As in the case where $p = 1$, the set X'_p of all linear functionals which are bounded of p-index on X forms a vector space. Now, for $f \in X'_p$, we define

$$\|f\|_p := \inf\{K > 0 \,:\, (2.4) \text{ holds}\}. \tag{2.5}$$

One then has

$$\|f\|_p = \sup\{|f(x)| \,:\, \sum \|x, y_{i_2}, \dots, y_{i_n}\|^p \le 1\}.$$

Moreover, the formula (2.5) defines a norm on X'_p.

Fact 2.2 *The linear functional f defined by (2.3) is bounded of p-th index with $\|f\|_p = n^{1/p'}\,\|y_1, \dots, y_n\|$.*

Proof. For every $x \in X$, we have

$$|f(x)| \le \sum \|x, y_{i_2}, \dots, y_{i_n}\|\, \|y_1, \dots, y_n\| \le n^{1/p'}\, \|y_1, \dots, y_n\| \left(\sum \|x, y_{i_2}, \dots, y_{i_n}\|^p\right)^{1/p},$$

whence $\|f\|_p \le n^{1/p'}\, \|y_1, \dots, y_n\|$.

To obtain the equality, take $x := n^{-1/p}\|y_1, \dots, y_n\|^{-1}(y_1 + \dots + y_n)$. Then, using (1.1), one may verify that $\sum \|x, y_{i_2}, \dots, y_{i_n}\|^p = 1$. Moreover, we have

$$\begin{aligned} f(x) &= n^{-1/p}\|y_1, \dots, y_n\|^{-1} \sum \langle y_1 + \dots + y_n, y_{i_1} | y_{i_2}, \dots, y_{i_n}\rangle \\ &= n^{-1/p}\|y_1, \dots, y_n\|^{-1} \sum \langle y_{i_1}, y_{i_1} | y_{i_2}, \dots, y_{i_n}\rangle \\ &= n^{-1/p}\|y_1, \dots, y_n\|^{-1} \cdot n\, \|y_1, \dots, y_n\|^2 \\ &= n^{1/p'}\, \|y_1, \dots, y_n\|. \end{aligned}$$

This convinces us that $\|f\|_p = n^{1/p'}\, \|y_1, \dots, y_n\|$. □

The following theorem tells us that X'_1 and X'_p are identical as a set.

Theorem 2.3 *Let f be a linear functional on X. If f is bounded of 1st index, then f is bounded of p-th index; and vice versa. In other words, $X'_1 = X'_p$.*

Proof. Suppose that f is bounded of p-index (with respect to $Y = \{y_1, \dots, y_n\}$). If x satisfies $\sum \|x, y_{i_2}, \dots, y_{i_n}\| \le 1$, then each term of the sum is less than 1, i.e., $\|x, y_{i_2}, \dots, y_{i_n}\| \le 1$. Hence $\|x, y_{i_2}, \dots, y_{i_n}\|^p \le \|x, y_{i_2}, \dots, y_{i_n}\|$, and so

$$\sum \|x, y_{i_2}, \dots, y_{i_n}\|^p \le \sum \|x, y_{i_2}, \dots, y_{i_n}\| \le 1.$$

Consequently, $|f(x)| \le \|f\|_p$, and thus f is bounded of 1st index with $\|f\|_1 \le \|f\|_p$.

Conversely, suppose that f is bounded of 1st index. If x satisfies $\sum \|x, y_{i_2}, \dots, y_{i_n}\|^p \le 1$, then $\sum \|x, y_{i_2}, \dots, y_{i_n}\| \le n^{1/p'}$, where p' is the dual exponent of p. Hence

$$\sum \left\|\frac{x}{n^{1/p'}}, y_{i_2}, \dots, y_{i_n}\right\| \le 1,$$

and so $\left|f\left(\frac{x}{n^{1/p'}}\right)\right| \le \|f\|_1$ or $|f(x)| \le n^{1/p'}\|f\|_1$. We therefore conclude that f is bounded of p-th index with $\|f\|_p \le n^{1/p'}\|f\|_1$. □

`Remark`. Unless we need to specify the index explicitly, we may simply use the word 'bounded' instead of 'bounded of p-th index'. We also denote by X' the set of all bounded linear functionals on X and call it the *dual space* of X (with respect to Y).

Theorem 2.3 states further that, on X', the norms $\|\cdot\|_p$ are all equivalent to $\|\cdot\|_1$, with

$$\|f\|_1 \le \|f\|_p \le n^{1/p'}\,\|f\|_1,$$

for every $f \in X'$.

2.3. **Duality properties for $p = 2$.** Let us now discuss another example of bounded linear functionals on the n-inner product space X, using the linearly independent set $Y = \{y_1, \dots, y_n\}$. Let $y \ne y_i$ for $i = 1, \dots, n$. Define $f_y : X \to \mathbb{R}$ by

$$f_y(x) := \sum \langle x, y | y_{i_2}, \dots, y_{i_n}\rangle, \tag{2.6}$$

where the sum is taken over $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ with $i_2 < \cdots < i_n$. Then f_y is linear. Moreover, we have:

Fact 2.4 *The linear functional f_y defined by (2.6) is bounded of 2nd index with $\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}$.*

Proof. For every $x \in X$, it follows from Cauchy-Schwarz inequalities that

$$\begin{aligned}|f_y(x)| &\le \sum |\langle x, y | y_{i_2}, \dots, y_{i_n}\rangle| \\ &\le \sum \|x, y_{i_2}, \dots, y_{i_n}\|\,\|y, y_{i_2}, \dots, y_{i_n}\| \\ &\le \left(\sum \|x, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2} \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2},\end{aligned}$$

whence $\|f_y\|_2 \le \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}$.

Now, if we take $x := \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{-1/2} y$, we get

$$\begin{aligned}f_y(x) &= \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{-1/2} f_y(y) \\ &= \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{-1/2} \sum \langle y, y | y_{i_2}, \dots, y_{i_n}\rangle \\ &= \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{-1/2} \sum \|y, y_{i_2}, \dots, y_{i_n}\|^2 \\ &= \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}.\end{aligned}$$

We must therefore have $\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}$. □

It is desirable to have an analog of the Riesz-Fréchet Theorem for linear functionals which are bounded of 2nd index on an n-inner product space. For that, we import the following theorem from [8].

Theorem 2.5 [8] *Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot\rangle)$ be an n-inner product space and $\|\cdot, \dots, \cdot\| = \langle \cdot, \cdot | \cdot, \dots, \cdot\rangle^{1/2}$ be the induced n-norm on X. With respect to the linearly independent set $Y = \{y_1, \dots, y_n\}$, the mapping $\langle \cdot, \cdot\rangle : X \times X \to \mathbb{R}$ given by*

$$\langle x, y\rangle := \sum \langle x, y | y_{i_2}, \dots, y_{i_n}\rangle \tag{2.7}$$

defines an inner product on X, and its induced norm $\|\cdot\|_2 : X \to \mathbb{R}$ is given by

$$\|x\|_2 := \left(\sum \|x, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}. \tag{2.8}$$

Corollary 2.6 *If $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ is complete with respect to the norm $\|\cdot\|_2$ in (2.8), then for every linear functional f which is bounded of 2nd index on X there exists a unique $y\in X$ such that*

$$f(x) = \langle x, y\rangle, \quad x\in X,$$

where $\langle\cdot,\cdot\rangle$ is the inner product in (2.7). Moreover, we have $\|y\|_2 = \|f\|_2$.

Theorem 2.7 *Let $(X, \|\cdot,\dots,\cdot\|)$ be an n-normed space, X' be the dual space of X (with respect to Y), and $\|\cdot\|_2$ be the derived norm on X given by*

$$\|x\|_2 := \left(\sum \|x, y_{i_2},\dots,y_{i_n}\|^2\right)^{1/2}.$$

Then, the function $\|\cdot,\dots,\cdot\|' : (X')^n \to \mathbf{R}$ given by

$$\|f_1,\dots,f_n\|' := \sup_{x_i\in X,\ \|x_i\|_2\le 1} \begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X'.

Proof. Similar to the proof of Fact 2 in [6]. □

3. Bounded Multilinear n-Functionals on ℓ^p

In this section, we shall focus on the space of p-summable sequences, denoted by ℓ^p, where $1 \le p < \infty$. Recall that a sequence $u := \{u_k\}$ (of real numbers) belongs ℓ^p space if $\|u\|_p := \left(\sum_{k=1}^{\infty} |u_k|^p\right)^{1/p} < \infty$. The dual space of ℓ^p is $\ell^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

3.1. **Several n-norms on ℓ^p.** Using the formula (1.2), ℓ^p may be equipped with the following n-norm:

$$\|x_1,\dots,x_n\|_p^G := \sup_{y_i\in\ell^{p'},\ \|y_i\|_{p'}\le 1} \left| \begin{matrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{matrix} \right|, \tag{3.1}$$

where p' denotes the dual exponent of p. But there is another formula of n-norm that we can define on ℓ^p, namely

$$\|x_1,\dots,x_n\|_p^H := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n} \left| \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \right|^p \right]^{\frac{1}{p}}, \tag{3.2}$$

where $x_i = \{x_{ij}\}$, $i = 1,\dots,n$. As shown in [12], the two n-norms are equivalent:

$$(n!)^{1/p-1}\|x_1,\dots,x_n\|_p^H \le \|x_1,\dots,x_n\|_p^G \le (n!)^{1/p}\|x_1,\dots,x_n\|_p^H.$$

On ℓ^2, both n-norms coincide with the standard n-norm given by (1.3) [6].

Next observe that the determinant on the right hand side of (3.1) can be rewritten as

$$\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix}.$$

By Hölder's inequality, we find that it is dominated by

$$\|x_1,\dots,x_n\|_p^H \|y_1,\dots,y_n\|_{p'}^H .$$

This inspires us to define another n-norm on ℓ^p, namely

$$\|x_1,\dots,x_n\|_p^I := \sup_{y_i\in\ell^{p'},\, \|y_1,\dots,y_n\|_{p'}^H\le 1} \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{vmatrix} . \tag{3.3}$$

Theorem 3.1 *The three n-norms on ℓ^p, namely $\|\cdot,\dots,\cdot\|_p^I$, $\|\cdot,\dots,\cdot\|_p^H$, and $\|\cdot,\dots,\cdot\|_p^G$, are equivalent.*

Proof. By the observation above, we have $\|x_1,\dots,x_n\|_p^I \le \|x_1,\dots,x_n\|_p^H$. By Theorem 2.3 of [12], we have $\|x_1,\dots,x_n\|_p^H \le (n!)^{1/p'}\|x_1,\dots,x_n\|_p^G$. Now, using the inequality

$$\|y_1,\dots,y_n\|_{p'}^H \le (n!)^{1/p}\|y_1\|_{p'}\cdots\|y_n\|_{p'}$$

(see Fact 3.1 of [10]), we see that if $\|y_i\|_{p'}\le 1$ for $i=1,\dots,n$, then $\|y_1,\dots,y_n\|_{p'}^H \le (n!)^{1/p}$. Hence we obtain

$$\|x_1,\dots,x_n\|_p^G \le (n!)^{1/p}\|x_1,\dots,x_n\|_p^I.$$

The chain of these inequalities shows that the three n-norms are equivalent. □

3.2. **Multilinear n-functionals on ℓ^p.** By a *multilinear n-functional* on a real vector space X we mean a mapping $F : X^n \to \mathbb{R}$ which is linear in each variable. A multilinear n-functional F is *bounded* on an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ if and only if there exists $K>0$ such that

$$|F(x_1,\dots,x_n)| \le K\,\|x_1,\dots,x_n\| \tag{3.4}$$

for every $x_1,\dots,x_n\in X$. No that for a bounded multilinear n-functional F on an n-normed space $(X,\|\cdot,\dots,\cdot\|)$, we have $F(x_1,\dots,x_n)=0$ when $x_1,\dots,x_n$ are linearly dependent. Moreover, we have the following proposition.

Proposition 3.2 *If F is a bounded multilinear n-functional on an n-normed space $(X,\|\cdot,\dots,\cdot\|)$, then F is antisymmetric, that is*

$$F(x_1,\dots,x_n) = \operatorname{sgn}(\sigma)\,F(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

for any $x_1,\dots,x_n\in X$ and any permutation σ of $(1,\dots,n)$. [Here $\operatorname{sgn}(\sigma)=1$ if σ is an even permutation and $\operatorname{sgn}(\sigma)=-1$ if σ is an odd permutation.]

Proof. We give the proof for the case where $n=2$ and leave the other case to the reader. Here, F is antisymmetric if and only if $F(x_1,x_2)=-F(x_2,x_1)$ for every $x_1,x_2\in X$. To see this, we observe that

$$F(x_1+x_2,x_1+x_2) = F(x_1,x_1)+F(x_1,x_2)+F(x_2,x_1)+F(x_2,x_2).$$

But $F(x,x)=0$ for every $x\in X$, and so we are done. □

We note that the set X^* of all bounded multilinear n-functionals on $(X,\|\cdot,\dots,\cdot\|)$ forms a vector space. Next, for a bounded multilinear n-functional F, we may define

$$\|F\| := \inf\{K>0 \,:\, (3.4)\text{ holds}\},$$

or equivalently

$$\|F\| := \sup\{|F(x_1,\dots,x_n)| \,:\, \|x_1,\dots,x_n\| \le 1\}.$$

This formula defines a norm on X^*.

We shall now discuss some multilinear n-functionals on ℓ^p (where $1 \le p < \infty$). Let $Y := \{y_1,\dots,y_n\}$ in $\ell^{p'}$, where p' is the dual exponent of p. We define

$$F_Y(x_1,\dots,x_n) := \frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix}, \tag{3.5}$$

for $x_1,\dots,x_n \in \ell^p$. Clearly F_Y is linear in each variable. Further, we have

$$|F_Y(x_1,\dots,x_n)| \le \|x_1,\dots,x_n\|_p^H \|y_1,\dots,y_n\|_{p'}^H,$$

and so F_Y is bounded on $(\ell^p, \|\cdot,\dots,\cdot\|_p^H)$ with $\|F_Y\| \le \|y_1,\dots,y_n\|_{p'}^H$.

For $p = 2$, we have the following fact.

Fact 3.3 [6] *Consider the n-normed space $(\ell^2, \|\cdot,\dots,\cdot\|_2^H)$. For fixed $Y := \{y_1,\dots,y_n\}$ in ℓ^2, let F_Y be the multilinear n-functional defined as in (3.5). Then F_Y is bounded on $(\ell^2, \|\cdot,\dots,\cdot\|_2^H)$ with*

$$\|F_Y\| = \|y_1,\dots,y_n\|_2^H.$$

Proof. From the inequality

$$|F_Y(x_1,\dots,x_n)| \le \|x_1,\dots,x_n\|_2^H \;\|y_1,\dots,y_n\|_2^H,$$

we see that F_Y is bounded with $\|F_Y\| \le \|y_1,\dots,y_n\|_2^H$. Next, if we take

$$x_i := \frac{y_i}{\sqrt[n]{\|y_1,\dots,y_n\|_2^H}}, \quad i = 1,\dots,n,$$

then $\|x_1,\dots,x_n\|_2^H = 1$ and $F_Y(x_1,\dots,x_n) = \|y_1,\dots,y_n\|_2^H$. Hence we conclude that $\|F_Y\| = \|y_1,\dots,y_n\|_2^H$. □

Regarding the n-functional F_Y on $(\ell^p, \|\cdot,\dots,\cdot\|_p^H)$, we have an open problem.

`Open Problem 1`. Compute the exact norm of F_Y in (3.5), especially for $p \neq 2$.

`Open Problem 2`. Can every bounded multilinear n-functional on ℓ^p be identified by $(y_1,\dots,y_n)$ where $y_i \in \ell^{p'},\ i = 1,\dots,n$?

Note that the multilinear n-functional F_Y may be reformulated as

$$F_Y(x_1,\dots,x_n) = \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{vmatrix}.$$

From this expression, we get the following result.

Fact 3.3 *Let* $e_j := (0, \dots, 0, 1, 0, \dots)$ *where the* j*-th term is the only term with value 1. Then, for* $j_1, \dots, j_n \in \mathbb{N}$*, we have*

$$F_Y(e_{j_1}, \dots, e_{j_n}) = \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix}^T .$$

Accordingly, the multiindex sequence $\{F_Y(e_{j_1}, \dots, e_{j_n})\}$ *is* p'*-summable, in the sense that*

$$\left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \left| \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix} \right|^{p'} \right]^{\frac{1}{p'}} < \infty.$$

Proof. The first part is straightforward, while the second part follows from the fact that $y_1, \dots, y_n \in \ell^{p'}$ and that the sum is actually equal to $\|y_1, \dots, y_n\|^H_{p'}$. □

The following problem is still open.

`Open Problem 3`. Let F be a bounded multilinear n-functional on ℓ^p. Must the multiindex sequence $\{F(e_{j_1}, \dots, e_{j_n})\}$ be p'-summable?

In general, the converse of Fact 3.3 holds, as follows. (We leave the proof to the reader.)

Proposition 3.4 *Let* $c := \{c_{j_1 \cdots j_n}\}$ *be a multiindex sequence which is antisymmetric and* p'*-summable. Then, the* n*-functional* F_c *given by*

$$F_c(x_1, \dots, x_n) := \sum_{j_1} \cdots \sum_{j_n} x_{1j_1} \cdots x_{nj_n} c_{j_1 \cdots j_n}, \tag{3.6}$$

where $x_i := (x_{ij_i}) \in \ell^p$ $(i = 1, \dots, n)$*, is linear in each variable, and is bounded on* $(\ell^p, \|\cdot, \dots, \cdot\|^H_p)$ *with*

$$\|F_c\| \le \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |c_{j_1 \cdots j_n}|^{p'} \right]^{1/p'} .$$

`Remark`. Similar to Open Problem 1, we do not know the exact norm of the n-functional F_c in (3.6).

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References

[1] S. Gähler. *Lineare 2-normierte räume.* Math. Nachr. **28** (1964), 1–43.

[2] S. Gähler. *Untersuchungen über verallgemeinerte m-metrische räume. I.* Math. Nachr. **40** (1969), 165–189.

[3] S. Gähler. *Untersuchungen über verallgemeinerte m-metrische räume. II.* Math. Nachr. **40** (1969), 229–264.

[4] S. Gähler. *Untersuchungen über verallgemeinerte m-metrische räume. III.* Math. Nachr. **41** (1970), 23–36.

[5] F.R. Gantmacher. *The Theory of Matrices.* AMS Chelsea Publishing **Vol. 1** (2000), 252-253.

[6] S. G. Gozali, H. Gunawan and O. Neswan. *On n-norms and bounded n-linear functionals in a Hilbert space.* Ann. Funct. Anal. **1** (2010), 72–79.

[7] H. Gunawan. *On n-inner products, n-norms, and the Cauchy-Schwarz inequality.* Sci. Math. Jpn. **55** (2002), 53–60.

[8] H. Gunawan. *Inner products on n-inner product spaces.* Soochow J. Math. **28** (2002), 389–398.

[9] H. Gunawan and Mashadi. *On n-normed spaces.* Int. J. Math. Math. Sci. **27** (2001), 631–639.

[10] H. Gunawan. *The space of p-summable sequences and its natural n-norm.* Bull. Austral. Math. Soc. **64** (2001), 137–147.

[11] A. Misiak. *n-inner product spaces.* Math. Nachr. **140** (1989), 299–319.

[12] R.A. Wibawa-Kusumah and H. Gunawan. *Two equivalent n-norms on the space of p-summable sequences.* Preprint.

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