FOURIER SERIES OF PIECEWISE LINEAR FUNCTIONS

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Abstract. In this note the Fourier sine series of piecewise linear functions on $[0,1]$ will be discussed. The problem of finding such Fourier series may be viewed as an interpolation problem that minimizes some kind of energy, as indicated in [3]. A procedure to find such series together with its implementation and results will be presented.

1. Introduction

Let $u : [0,1] \to \mathbb{R}$ be a continuous function with $u(0) = u(1) = 0$. If, for instance, u is piecewise smooth, then u may be expressed as a Fourier sine series

$$u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad x \in [0,1],$$

where

$$a_n = 2\int_0^1 u(x) \sin n\pi x \, dx, \quad n = 1, 2, 3, \ldots.$$

In this note, we shall be interested in finding the Fourier sine series when u is continuous and piecewise linear.

It turns out that the problem can be viewed as an interpolation problem that minimizes an energy integral, as indicated in [3]. Indeed, suppose $f : [0,1] \to \mathbb{R}$ is continuous, $0 = x_0 < x_1 < \cdots < x_N = 1$, $f(x_i) = c_i$, $i = 0, \ldots, N$, with $c_0 = c_N = 0$, and f is linear on each $[x_{i-1}, x_i]$, $i = 1, \ldots, N$. Then the Fourier sine series of f can be found by solving the following minimization problem:

Among continuous, piecewise smooth functions u on $[0,1]$, minimize the integral

$$E(u) := \int_0^1 |u'(x)|^2 \, dx, \tag{1}$$

subject to the condition that $u(x_i) = c_i$, $i = 0, \ldots, N$.

The integral $E(u) = \int_0^1 |u'(x)|^2 dx$ represents the *tension* or the *potential energy of axial load* of the curve $y = u(x)$ [4]. It is not hard to see that the piecewise linear function f solves the problem. What our

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observation shows in the next section is that the solution can be expressed as a sine series and that the series can be obtained iteratively through finite computations. By the uniqueness of the solution, we thus find the sine series for f — which must be the Fourier sine series of f. A procedure to obtain the series together with its implementation and results will be presented in this paper.

2. Analytical Results

The fact that the piecewise linear function f solves the minimization problem (1) follows from the following lemma.

Lemma 1. *On every $[a, b]$ where $u(a)$ and $u(b)$ are fixed, the integral $\int_a^b |u'(x)|^2 dx$ is minimized (among continuously differentiable function u) if and only if u is linear.*

Proof. Let $m := \frac{u(b)-u(a)}{b-a}$. Then, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned}\int_a^b |u'(x) - m|^2 dx &= \int_a^b |u'(x)|^2 dx - 2m \int_a^b u'(x)\, dx + m^2(b-a) \\ &= \int_a^b |u'(x)|^2 dx - 2m[u(b) - u(a)] + m^2(b-a) \\ &= \int_a^b |u'(x)|^2 dx - m^2(b-a).\end{aligned}$$

Hence

$$\int_a^b |u'(x)|^2 dx \geq \int_a^b m^2\, dx,$$

and $\int_a^b |u'(x)|^2 dx$ is minimized if and only if $u'(x) = m$ for every $x \in [a, b]$; that is, if and only if u is linear. □

Now we move to the Fourier series discussion. We know that for a continuous, piecewise smooth function u, the Fourier sine coefficients a_n's satisfy the condition

$$\sum_{n=1}^{\infty} n^2 a_n^2 < \infty \tag{2}$$

(see [2]). Conversely, if the coefficients a_n's satisfy the condition (2), then u is absolutely continuous and u' is square integrable with

$$\|u'\|_2^2 := 2 \int_0^1 |u'(x)|^2 dx = \pi^2 \sum_{n=1}^{\infty} n^2 a_n^2$$

(see, for instance, [5]).

From here on, we consider the minimization problem (1) in the space W consisting all functions u on $[0,1]$ of the form $u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ with $\sum_{n=1}^{\infty} n^2 a_n^2 < \infty$. On W, we define the inner product

$$\langle u, v \rangle := \sum_{n=1}^{\infty} n^2 a_n b_n,$$

where a_n's and b_n's are the coefficients of u and v, respectively. Here minimizing the integral $\int_0^1 |u'(x)|^2 dx$ in W is equivalent to minimizing the sum $\sum_{n=1}^{\infty} n^2 a_n^2 =: \|u\|^2$.

Notice that with respect to the above inner product, W is a Hilbert space. Further, we have the following lemma.

Lemma 2. *If $\|u_k - u\| \to 0$ as $k \to \infty$, then (u_k) converges uniformly to u on $[0,1]$.*

Proof. For $k \in \mathbb{N}$, let $a_{k,n}$'s and a_n's be the coefficients of u_k and u. Then, for each $x \in [0,1]$, we have

$$\begin{aligned} |u_k(x) - u(x)| &= \Big|\sum_{n=1}^{\infty} (a_{k,n} - a_n) \sin n\pi x\Big| \\ &\le \Big[\sum_{n=1}^{\infty} n^2 (a_{k,n} - a_n)^2\Big]^{1/2} \Big[\sum_{n=1}^{\infty} \frac{\sin^2 n\pi x}{n^2}\Big]^{1/2} \\ &\le C\|u_k - u\|, \end{aligned}$$

where C is independent of x. Hence (u_k) converges uniformly to u. □

Now consider the subspace V of W consisting of all functions u that vanish at $x_i,\ i = 1, ..., N-1$; that is,

$$V := \{u \in W \,:\, u(x_i) = 0,\ i = 1, \dots, N-1\}.$$

Meanwhile, let U be the subset of W given by

$$U := \{u \in W \,:\, u(x_i) = c_i,\ i = 1, \dots, N-1\}.$$

Then, as stated in [3], we have:

Lemma 3 [3]. *V is closed, while U is nonempty, closed and convex.*

Proof. Let u be the limit of a convergent sequence (u_k) in V. Then, for each $i = 1, \dots, N-1$, it follows from Lemma 2 that $u(x_i) = 0$ because $u_k(x_i) = 0$ for every $k \in \mathbb{N}$. Therefore V is closed. Similarly, U is closed. Next, it is nonempty because one can easily find a function $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$ satisfying the following system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \dots, N-1.$$

Finally, if u_1 and u_2 in U, then $\alpha u_1 + \beta u_2 \in U$ provided that $\alpha + \beta = 1$. This tells us particularly that U is convex. □

The following result is inspired by Alghofari's [3].

Theorem 4. *The minimization problem (1) has a unique solution in W, and the solution is given by*

$$u = u_0 - \mathrm{proj}_V(u_0),$$

where u_0 is an arbitrary element of U and $\mathrm{proj}_V(u_0)$ is the orthogonal projection of u_0 on V. Moreover, u represents the Fourier sine series of the piecewise linear function that passes through the given points $(x_i, c_i),\ i = 1, \ldots, N-1$.

Proof. Let u_0 be an element in U. Then, for any $v \in V$, $u_0 - v$ is also in U. Since U is a convex subset of W, there must exist a unique element $v_0 \in V$ such that $\|u_0 - v_0\|$ is of smallest norm [1]. Thus $u := u_0 - v_0$ is the unique solution in W for our minimization problem (1). By the theory of best approximation in Hilbert spaces, the element $v_0 \in V$ for which $\|u_0 - v_0\|$ is minimized is the orthogonal projection of u_0 on V, that is, $v_0 = \mathrm{proj}_V(u_0)$.

By Lemma 1 and the uniqueness of the solution, the series u must be the Fourier sine series of the piecewise linear function passing through the points $(x_i, c_i),\ i = 1, \ldots, N-1$. □

As we have indicated before, to find an element in U is easy. What is rather difficult is to find an orthonormal basis for V. In the next section, we develop an algorithm to find an initial element in U and an orthonormal basis for V, and obtain the minimum solution iteratively through finite computations.

3. Numerical Results

Given the partition $0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$ and the values of $c_i,\ i = 1, \ldots, N-1$, we obtain (or approximate) the solution to (1) in W through the following steps.

Step 1. To obtain an initial element in U, we solve the system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \ldots, N-1,$$

for the coefficients b_j's. The $(N-1) \times (N-1)$ matrix $[\sin j\pi x_i]_{i,j}$ is always nonsingular (see [1]), and so the above system has a solution. Having found b_j's, we put $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$.

Step 2. To obtain a basis for V, we consider the system of equations

$$\sum_{n=1}^{\infty} a_n \sin n\pi x_i = 0, \quad i = 1, \dots, N-1,$$

each of which contains infinitely many unknowns a_n's. However, we can tackle this system by writing it as

$$\sum_{j=1}^{N-1} a_j \sin j\pi x_i = -\sum_{n=N}^{\infty} a_n \sin n\pi x_i, \quad i = 1, \dots, N-1.$$

From this we can express $a_1, \dots, a_{N-1}$ in terms of a_n, $n \geq N$. Now if $(a_1, \dots, a_{N-1}, a_N, a_{N+1}, \dots)$ stands for $\sum_{n=1}^{\infty} a_n \sin n\pi x$, then by expressing $a_1, \dots, a_{N-1}$ in terms of a_n, $n \geq N$, every element in V can be expressed as

$$a_N(*, \dots, *, 1, 0, 0, \dots) + a_{N+1}(*, \dots, *, 0, 1, 0, \dots) + \\ + a_{N+2}(*, \dots, *, 0, 0, 1, \dots) + \cdots,$$

where the first $N-1$ terms marked by asterisks come from $a_1, \dots, a_{N-1}$. The sequence $v_1 := (*, \dots, *, 1, 0, 0, \dots)$, $v_2 := (*, \dots, *, 0, 1, 0, \dots)$, $v_3 := (*, \dots, *, 0, 0, 1, \dots)$, $\dots$ clearly form a basis for V.

Step 3. The minimum solution u is given by $u = u_0 - \mathrm{proj}_V(u_0)$. To find (or approximate) it, we compute the orthogonal projection of u_0 on the subspace $V_k := \mathrm{span}\{v_1, \dots, v_k\}$ for $k = 1, 2, 3, \dots$ iteratively. (But since v_n's may not be orthogonal, we might need to orthogonalize them first.) Now if $u_k := u_0 - \mathrm{proj}_{V_k}(u_0)$, then the sequence (u_k) approximates the minimum solution u. Indeed, $\|u_k\|$ gets smaller and $\|u_k - u\| \to 0$ as $k \to \infty$.

To illustrate how our procedure works, we present two examples. The first one is simple; the reader can follow the computations in details.

Example 1. Let f be given by

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

By computing the Fourier sine coefficients directly, one gets

$$f(x) = \frac{8}{\pi^2}\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - + \cdots\Big), \quad x \in [0,1].$$

We shall here show that we can obtain this series by applying our procedure to the following minimization problem:

Among continuous, piecewise smooth functions u on $[0,1]$, minimize the integral

$$E(u) := \int_0^1 |u'(x)|^2\, dx, \tag{3}$$

subject to the condition that $u(0) = u(1) = 0$ and $u\big(\frac{1}{2}\big) = 1$.

For this, consider the subspace V of W consisting of all functions u that vanish at $\frac{1}{2}$; that is,

$$V := \{u \in W \,:\, u\big(\tfrac{1}{2}\big) = 0\},$$

and the subset U of W given by

$$U := \{u \in W \,:\, u\big(\tfrac{1}{2}\big) = 1\}.$$

For our initial approximation, we have $u_0(x) = \sin n\pi x$. Next, if $v(x) := \sum_{n=1}^{\infty} a_n \sin n\pi x$ is in V, then $v\big(\frac{1}{2}\big) = 0$ is equivalent to $a_1 - a_3 + a_5 - a_7 + - \cdots = 0$, for which we get

$$a_1 = a_3 - a_5 + a_7 - + \cdots .$$

Hence, every element $(a_1, a_2, a_3, a_4, a_5, \dots)$ in V can be expressed as

$$a_2(0,1,0,0,0,\dots) + a_3(1,0,1,0,0,\dots) + \\ + a_4(0,0,0,1,0,\dots) + a_5(-1,0,0,0,1,\dots) + \cdots .$$

From this we get the following basis for V:

$$\begin{aligned} v_1 &:= (0,1,0,0,0,\dots) \\ v_2 &:= (1,0,1,0,0,\dots) \\ v_3 &:= (0,0,0,1,0,\dots) \\ v_4 &:= (-1,0,0,0,1,\dots) \\ &\ \vdots \end{aligned}$$

Instead of computing the orthogonal projection of u_0 on V as described in Step 3, we can compute its orthogonal complement directly as follows. If $u = (b_1, b_2, b_3, \dots)$ is orthogonal to V, then $u \perp v_m$ for each $m \in \mathbb{N}$, and so $b_2, b_4, b_6, \dots$ must be equal to 0 and

$$b_1 = -3^2 b_3 = 5^2 b_5 = -7^2 b_7 = \cdots .$$

Hence $u = b_1(1, 0, -\frac{1}{3^2}, 0, \frac{1}{5^2}, \dots)$, that is,

$$u(x) = b_1\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - + \cdots\Big).$$

But $u\big(\frac{1}{2}\big) = 1$ gives

$$b_1 = \Big(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big)^{-1} = \frac{8}{\pi^2},$$

and therefore

$$u(x) = \frac{8}{\pi^2}\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - + \cdots\Big).$$

We have thus showed that the Fourier sine series of f can be obtained by solving its associated minimization problem. Note that if one carries out Step 3 as prescribed, one will get a sequence (u_k) that approximates f in the norm (in W) and uniformly. (To be more precise, starting from $u_0 = (1,0,0,0,0,\dots)$, one will get $u_1 = (1,0,0,0,0,\dots)$, $u_2 = u_3 =$

$\frac{9}{10}(1, 0, -\frac{1}{3^2}, 0, 0, \dots)$, and so on.) The difference between the sequence (u_k) and the Fourier partial sums is that each u_k passes through the point $\left(\frac{1}{2}, 1\right)$ while the Fourier partial sums do not. Both converge to u with the rate of $O(1/k)$.

The next example is processed by a computer program. We present the graphs of some approximate functions, which clearly indicates that the resulting series gets closer to the related piecewise linear function.

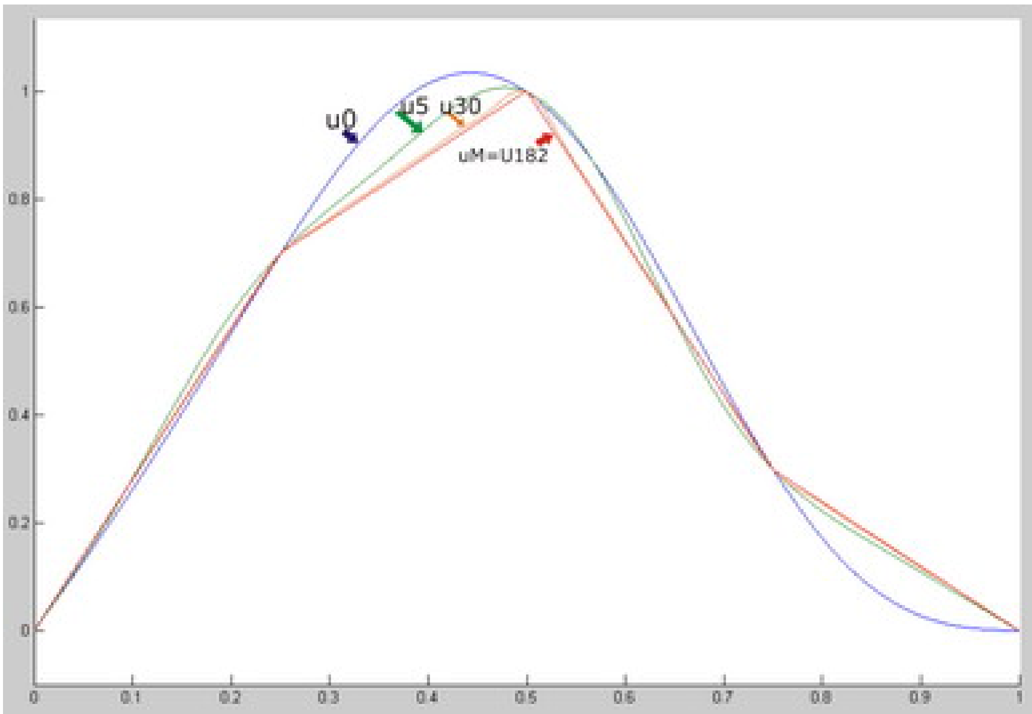
Example 2. Let $x_i = \frac{i}{4},\ i = 0, \dots, 4$, and $c_0 = 0,\ c_1 = \frac{7}{10},\ c_2 = 1,\ c_3 = \frac{3}{10},\ c_4 = 0$. Consider the following minimization problem:

Among continuous, piecewise smooth functions u on $[0, 1]$, minimize the integral

$$E(u) := \int_0^1 |u'(x)|^2\, dx, \tag{4}$$

subject to the condition that $u(x_i) = c_i,\ i = 0, \dots, 4$.

With a computer program, we apply our procedure and get a sequence (u_k) of finite sums of sine functions that approximates the solution in W. We stop the iterations at u_M basically when $\|u_M - u_{M-1}\| < \epsilon$. For $\epsilon = 0.01$, the iterations stop at u_{182}. The following graphs of u_0, u_5, u_{30}, and $u_M = u_{182}$ confirm that the limiting series must be that of the piecewise linear function passing through the points $(x_i, c_i),\ i = 0, \dots, 4$.



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