An Interpolation Method that Minimizes an Energy Functional

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In [3], Alghofari discussed the following interpolation problem that minimizes an energy functional:

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \dots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$, find a continuously differentiable function u on $[0, 1]$ that passes through these $N+1$ points and minimizes the energy functional

$$E_2(u) := \int_0^1 |u''(x)|^2 dx.$$

The functional $E_2(u)$ here represents the curvature (or the potential energy) of u.

Thus, in this problem, Alghofari was interested in finding an interpolant whose graph "curves as little as possible". To solve the problem, he used a Fourier series approach as well as functional analysis arguments. In particular, he showed that the problem has a unique solution and gave a hint to approximate the solution.

In this note, we shall generalize Alghofari's results by replacing the energy functional $E_2(u)$ with

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx,$$

where $u^{(\alpha)}$ denotes the fractional derivative of u of order $\alpha \geq 0$.

We show that for $\alpha > \frac{1}{2}$, the problem has a unique solution u which is continuous on $[0,1]$. An iterative procedure to obtain the solution will be presented and some examples will be given.

Let $u : [0,1] \to \mathbb{R}$ be a continuous function with $u(0) = u(1) = 0$. If, for instance, u is piecewise smooth, then u may be expressed as a Fourier sine series

$$u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad x \in [0,1],$$

where

$$a_n = 2 \int_0^1 u(x) \sin n\pi x \, dx, \quad n = 1, 2, 3, \ldots.$$

Parseval's identity states that $2 \int_0^1 |u(x)|^2 dx = \sum_{n=1}^{\infty} a_n^2$.

If u is of class $C^{(k-1)}$ and $u^{(k-1)}$ is piecewise smooth (so that $u^{(k)}$ exists except at finitely many points and is piecewise continuous), then the Fourier sine coefficients a_n's satisfy the condition

$$\sum_{n=1}^{\infty} n^{2k} a_n^2 < \infty \qquad (1)$$

(see [2]). Conversely, if the coefficients a_n's satisfy the condition (1), then $u, \ldots, u^{(k-1)}$ are absolutely continuous and $u^{(k)}$ is square integrable with

$$\|u^{(k)}\|_2^2 := 2 \int_0^1 |u^{(k)}(x)|^2 dx = \pi^{2k} \sum_{n=1}^{\infty} n^{2k} a_n^2$$

(see, for instance, [5]). All these results tell us that we may identify $u^{(k)}$ with the square summable sequence $(n^k a_n)$.

Inspired by the above facts, we may define the fractional derivative of u of order $\alpha \geq 0$, denoted by $u^{(\alpha)}$, by the following formula

$$u^{(\alpha)}(x) = \pi^\alpha \sum_{n=1}^{\infty} n^\alpha a_n \sin(n\pi x + \alpha\tfrac{\pi}{2}),$$

provided that $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$. One may check that the family $\{\sin(n\pi x + \alpha\frac{\pi}{2})\} : n \in \mathbb{N}\}$ forms an orthogonal system and that

$$2\int_0^1 |u^{(\alpha)}(x)|^2 dx = \pi^{2\alpha} \sum_{n=1}^{\infty} n^{2\alpha} a_n^2.$$

Accordingly, $u^{(\alpha)}$ is a square integrable function on $[0,1]$, which may be identified with the square summable sequence $(n^\alpha a_n)$.

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \dots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$, we wish to find a continuous function u on $[0, 1]$ that passes through these $N+1$ points and minimizes the energy functional

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx. \tag{2}$$

[For $\alpha = 1$, $E_1(u)$ represents the tension (or the potential energy of axial load) of u. The problem will be presented specifically by Ferry PRANOLO in a parallel session.]

To solve this problem, we consider the space W consisting all functions u on $[0,1]$ of the form $u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ with $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$ for some $\alpha > \frac{1}{2}$. On W, we define the inner product

$$\langle u, v \rangle := \sum_{n=1}^{\infty} n^{2\alpha} a_n b_n,$$

where a_n's and b_n's are the coefficients of u and v, respectively. Here minimizing the integral $\int_0^1 |u^{(\alpha)}(x)|^2 dx$ in W is equivalent to minimizing the sum $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 =: \|u\|^2$. Notice that with respect to the above inner product, W is a Hilbert space.

Lemma 1. *If* $\|u_m - u\| \to 0$ *as* $m \to \infty$, *then* (u_m) *converges uniformly to* u *on* $[0, 1]$.

Proof. For $m \in \mathbb{N}$, let $a_{m,n}$'s and a_n's be the coefficients of u_m and u. Then, for each $x \in [0, 1]$, we have

$$\begin{aligned}|u_m(x) - u(x)| &= \left|\sum_{n=1}^{\infty}(a_{m,n} - a_n)\sin n\pi x\right| \\ &\leq \left[\sum_{n=1}^{\infty} n^{2\alpha}(a_{m,n} - a_n)^2\right]^{1/2}\left[\sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{n^{2\alpha}}\right]^{1/2} \\ &\leq C\|u_m - u\|,\end{aligned}$$

with C independent of x. Thus (u_m) converges uniformly to u. □

Corollary 2. *If $u \in W$, then u is continuous.*

Proof. Each $u \in W$ is a limit, and hence a uniform limit, of its partial sums. Now since the partial sums are continuous, u too must be continuous. □

Now consider the subspace V of W consisting of all functions u that vanish at x_i, $i = 1, ..., N-1$; that is,

$$V := \{u \in W \ : \ u(x_i) = 0, \ i = 1, \dots, N-1\}.$$

Meanwhile, let U be the subset of W given by

$$U := \{u \in W \ : \ u(x_i) = c_i, \ i = 1, \dots, N-1\}.$$

Then, as stated in [3], we have:

Lemma 3 [3]. *V is closed, while U is nonempty, closed and convex.*

Theorem 4. *The minimization problem (2) has a unique solution in W, and the solution is given by*

$$u = u_0 - \mathrm{proj}_V(u_0),$$

where u_0 is an arbitrary element of U and $\mathrm{proj}_V(u_0)$ is the orthogonal projection of u_0 on V.

Proof. Let u_0 be an element in U. Then, for any $v \in V$, $u_0 - v$ is also in U. Since U is a convex subset of W, there must exist a unique element $v_0 \in V$ such that $\|u_0 - v_0\|$ is of smallest norm [1]. Thus $u := u_0 - v_0$ is the unique solution in W for our minimization problem (2). By the theory of best approximation in Hilbert spaces, the element $v_0 \in V$ for which $\|u_0 - v_0\|$ is minimized is the orthogonal projection of u_0 on V, that is, $v_0 = \mathrm{proj}_V(u_0)$. □

As we have indicated before, to find an element in U is easy. What is rather difficult is to find an orthonormal basis for V. In the next section, we develop a procedure to find an initial element in U and an orthonormal basis for V, and to obtain the minimum solution iteratively through finite computations.

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \dots, (x_N, c_N)$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, we can obtain (or approximate) the solution to (2) in W through the following steps.

Step 1. To obtain an initial element in U, we solve the system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \dots, N-1,$$

for the coefficients b_j's. The $(N-1) \times (N-1)$ matrix $[\sin j\pi x_i]_{i,j}$ is always nonsingular (see [1]), and so the above system has a solution. Having found b_j's, we put $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$.

Step 2. To obtain a basis for V, we consider the system of equations

$$\sum_{n=1}^{\infty} a_n \sin n\pi x_i = 0, \quad i = 1, \ldots, N-1,$$

each of which contains infinitely many unknowns a_n's. However, we can tackle this system by writing it as

$$\sum_{j=1}^{N-1} a_j \sin j\pi x_i = -\sum_{n=N}^{\infty} a_n \sin n\pi x_i, \quad i = 1, \ldots, N-1.$$

From this we can express $a_1, \ldots, a_{N-1}$ in terms of $a_n,\ n \geq N$.

Now if $(a_1, \ldots, a_{N-1}, a_N,\ a_{N+1}, \ldots)$ stands for $\sum_{n=1}^{\infty} a_n \sin n\pi x$, then by expressing $a_1, \ldots, a_{N-1}$ in terms of a_n with $n \geq N$, every element in V can be expressed as

$$a_N(*, \ldots, *, 1, 0, 0, \ldots) + a_{N+1}(*, \ldots, *, 0, 1, 0, \ldots) + \\ + a_{N+2}(*, \ldots, *, 0, 0, 1, \ldots) + \cdots ,$$

where the first $N-1$ terms marked by asterisks come from a_1, $a_2, \ldots, a_{N-1}$. Clearly the sequence $v_1 := (*, \ldots, *, 1, 0, 0, \ldots)$, $v_2 := (*, \ldots, *, 0, 1, 0, \ldots)$, $v_3 := (*, \ldots, *, 0, 0, 1, \ldots)$, ... here form a basis for V.

Step 3. The minimum solution u is given by $u = u_0 - \mathrm{proj}_V(u_0)$. To find (or approximate) it, we compute the orthogonal projection of u_0 on the subspace $V_m := \mathrm{span}\{v_1, \dots, v_m\}$ for $m = 1, 2, 3, \dots$ iteratively. (But since v_n's may not be orthogonal, we might need to orthogonalize them first.) Now if $u_m := u_0 - \mathrm{proj}_{V_m}(u_0)$, then the sequence (u_m) approximates the minimum solution u. Indeed, $\|u_m\|$ gets smaller and $\|u_m - u\| \to 0$ as $m \to \infty$. In practice, we may stop the iteration process at u_M basically when $\|u_M - u_{M-1}\| < \epsilon$ for a given value of ϵ.

To illustrate how our procedure works, we present a few examples.

Example 1. Suppose that we wish to find a continuous, piecewise smooth functions u on $[0, 1]$ that minimizes the integral

$$E_1(u) := \int_0^1 |u'(x)|^2 \, dx, \tag{3}$$

subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$. For this, consider the subspace V of W consisting of all functions u that vanish at $\frac{1}{2}$; that is,

$$V := \{u \in W \, : \, u\left(\tfrac{1}{2}\right) = 0\},$$

and the subset U of W given by

$$U := \{u \in W \, : \, u\left(\tfrac{1}{2}\right) = 1\}.$$

For our initial approximation, we have $u_0(x) = \sin n\pi x$. Next, if $v(x) := \sum_{n=1}^{\infty} a_n \sin n\pi x$ is in V, then $v\left(\frac{1}{2}\right) = 0$ is equivalent to $a_1 - a_3 + a_5 - a_7 + - \cdots = 0$, for which we get

$$a_1 = a_3 - a_5 + a_7 - + \cdots .$$

Hence, every element $(a_1, a_2, a_3, a_4, a_5, \dots)$ in V can be expressed as

$$a_2(0,1,0,0,0,\dots) + a_3(1,0,1,0,0,\dots) + \\ + a_4(0,0,0,1,0,\dots) + a_5(-1,0,0,0,1,\dots) + \cdots .$$

From this we get the following basis for V:

$$\begin{aligned} v_1 &:= (0, 1, 0, 0, 0, \dots) \\ v_2 &:= (1, 0, 1, 0, 0, \dots) \\ v_3 &:= (0, 0, 0, 1, 0, \dots) \\ v_4 &:= (-1, 0, 0, 0, 1, \dots) \\ &\vdots \end{aligned}$$

Instead of computing the orthogonal projection u_0 on V as described in Step 3, we can compute its orthogonal complement directly as follows.

If $u = (b_1, b_2, b_3, \dots)$ is orthogonal to V, then $u \perp v_m$ for each $m \in \mathbb{N}$, and so $b_2, b_4, b_6, \dots$ must be equal to 0 and

$$b_1 = -3^2 b_3 = 5^2 b_5 = -7^2 b_7 = \cdots .$$

Hence $u = b_1(1, 0, -\frac{1}{3^2}, 0, \frac{1}{5^2}, \dots)$, that is,

$$u(x) = b_1\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - + \cdots\Big).$$

But $u(\frac{1}{2}) = 1$ gives

$$b_1 = \Big(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big)^{-1} = \frac{8}{\pi^2},$$

and therefore

$$u(x) = \frac{8}{\pi^2}\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - + \cdots\Big).$$

As one would expect, this is nothing but the Fourier sine series of the piecewise linear function f given by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1. \end{cases}$$

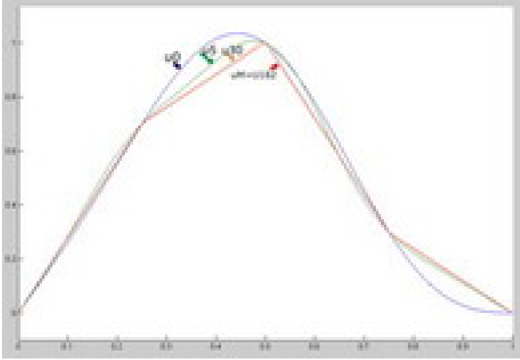
Note that if one carries out Step 3 as prescribed, one will get a sequence (u_m) that approximates f in the norm (in W) and uniformly. Starting from $u_0 = (1, 0, 0, 0, 0, \dots)$, one will get $u_1 = (1, 0, 0, 0, 0, \dots)$, $u_2 = u_3 = \frac{9}{10}(1, 0, -\frac{1}{3^2}, 0, 0, \dots)$, and so on.

The difference between the sequence (u_m) and the Fourier partial sums is that each u_m passes through the point $\left(\frac{1}{2}, 1\right)$ while the Fourier partial sums do not.

Example 2. In general, given $N+1$ points (x_0, c_0), $(x_1, c_1), \dots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$, the solution to the minimization problem (2) for $\alpha = 1$ is the Fourier sine series of the piecewise linear function f for which $f(x_i) = c_i$ and f is linear on each subinterval $[x_{i-1}, x_i]$. [See F. PRANOLO's presentation.]

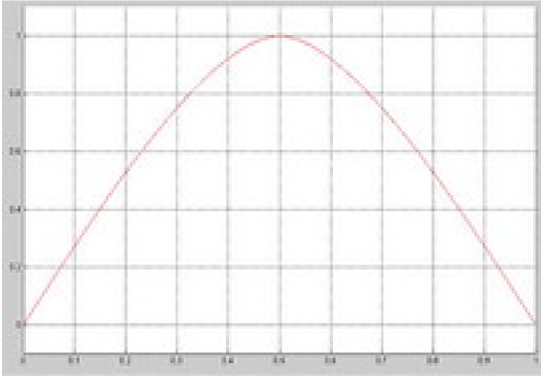
For example, let $x_i = \frac{i}{4},\ i = 0, \dots, 4$, and $c_0 = 0,\ c_1 = \frac{7}{10}$, $c_2 = 1$, $c_3 = \frac{3}{10},\ c_4 = 0$. With a computer program, we apply our procedure and get a sequence (u_m) that approximates the solution in W. We stop the iterations at u_M basically when $\|u_M - u_{M-1}\| < \epsilon$.

For $\epsilon = 0.01$, the iterations stop at u_{182}. The following figure shows the graphs of u_0, u_5, u_{30}, and $u_M = u_{182}$, which clearly indicate that the limiting series must be that of the piecewise linear function passing through the points (x_i, c_i), $i = 0, \ldots, 4$.



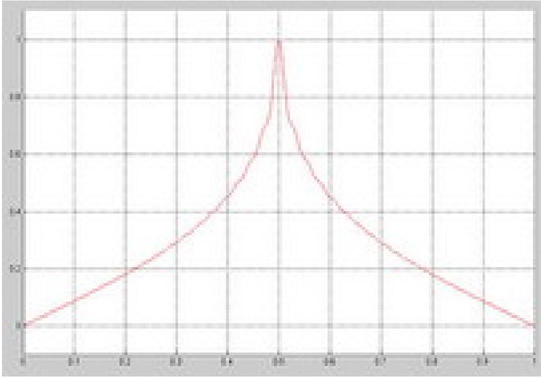
Example 3. Suppose that $\alpha = 1.5$ and we wish to find a smooth function u on $[0, 1]$ that minimizes the integral $E_\alpha(u)$ subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

With a computer program, we apply our procedure and stop the iterations at u_M basically when $\|u_M - u_{M-1}\| < 0.01$. The following figure shows the graph of the approximate solution.



Example 4. Suppose that $\alpha = 0.6$ and we wish to find a continuous function u on $[0,1]$ that minimizes the integral $E_\alpha(u)$ subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

Again, with a computer program, we apply our procedure and stop the iterations at u_M when $\|u_M - u_{M-1}\| < 0.05$ (we use a large value of ϵ because the rate of convergence of the approximate solutions is low for small α). The following figure shows the graph of the approximate solution.



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