SOME WEIGHTED ESTIMATES FOR IMAGINARY POWERS OF LAPLACE OPERATORS

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Abstract. We study the boundedness of singular integral operators that are imaginary powers of the Laplace operator in $\mathbb{R}^n$, especially from weighted Hardy spaces $H^p_w(\mathbb{R}^n)$ to weighted Lebesgue spaces $L^p_w(\mathbb{R}^n)$ where $0 < p \leq 1$. In particular, we prove some $H^p_w - L^p_w$ estimates for these operators when $0 < p \leq 1$ and $w$ is in the Muckenhoupt's class $A_q$, for some $q > 1$.

1. Introduction

We shall study a class of singular integral operators that are imaginary powers of the Laplace operator in $\mathbb{R}^n$. But first let us review some basic properties of singular integral operators in general.

It is well-known that every singular integral operator $T$ defined on $\mathcal{S}(\mathbb{R}^n)$ by

$$Tf = K * f,$$

where $K$ is a tempered distribution on $\mathbb{R}^n$ with $\hat{K} \in L^\infty(\mathbb{R}^n)$, extends to a bounded operator on $L^2(\mathbb{R}^n)$. Provided that the kernel $K$ is locally integrable away from 0 and satisfies the Hörmander condition

$$\int_{|x|>2|y|} |K(x - y) - K(x)| \, dx \leq C_K,$$

such an operator will also extend to a bounded operator on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. For $p = 1$ and $p = \infty$, weaker results are available.

Moreover, for $p = 1$, one may also show that $T$ extends to a bounded operator from $H^1(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$. In fact, with some extra conditions on $K$, the operator

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$T$ can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $0 < p \leq 1$. Here, for each $0 < p \leq 1$, $H^p(\mathbb{R}^n)$ denotes the Hardy space, whose members can be written as $\sum_j \lambda_j a_j$ where the $a_j$'s are $p$-atoms and $\lambda_j$'s are real numbers such that $\sum_j |\lambda_j|^p < C\|f\|^p_{H^p}$. [A $p$-atom in $\mathbb{R}^n$ is a function $a$ supported in a finite cube $Q \subseteq \mathbb{R}^n$ such that $\|a\|_1 \leq |Q|^{1/p}$ and $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n(1/p - 1)$.] See [3], Ch. II–III.

Now let $w$ be a nonnegative measurable function or a weight on $\mathbb{R}^n$ and $L^p_w(\mathbb{R}^n)$ be the space of all functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ for which $\|f\|_{L^p_w} = \left[\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right]^{1/p} < \infty$. Then one may show that $T$ extends to a bounded operator on $L^p_w(\mathbb{R}^n)$ for $1 < p < \infty$ provided that $w \in A_p$, that is, $w$ satisfies the $A_p$ condition

$$\left[\frac{1}{|Q|} \int_Q w(x)dx\right] \left[\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx\right]^{p-1} \leq C$$

for all cubes $Q$ in $\mathbb{R}^n$. [For example, $| \cdot |^a \in A_p$, $1 < p < \infty$, if and only if $-n < a < n(p-1)$.] As in the unweighted case, there is also a weaker result for $p = 1$ and $w \in A_1$, satisfying the $A_1$ condition

$$\frac{1}{|Q|} \int_Q w(x)dx \leq C w(y), \quad \text{a.e. } y \in Q,$$

for all cubes $Q$ in $\mathbb{R}^n$, which can be viewed as the limit of $A_p$ conditions for $p \rightarrow 1^+$. (For example, $| \cdot |^a \in A_1$ if and only if $-n < a \leq 0$.) See [3], Ch. IV.

In this note, we shall study the boundedness of singular integral operators that are imaginary powers of the Laplace operator in $\mathbb{R}^n$, especially from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ where $0 < p \leq 1$. Here $H^p_w(\mathbb{R}^n)$ denotes the weighted Hardy space, defined just as $H^p(\mathbb{R}^n)$ but with measure $w(x)dx$ replacing the usual Lebesgue measure $dx$. These operators were studied by B. Muckenhoupt [6] in 1960, and used by Cowling and Mauceri [1] in 1978 to prove the boundedness of E.M. Stein’s spherical maximal operator [9]. What we are interested in here is how their norms, especially from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ where $0 < p \leq 1$, actually depends on the imaginary power.

Recent works indicate that the study of imaginary powers of operators in general have some applications in the theory of spectral multipliers. See, e.g., [2] and [7].
2. Main Results

For each \( u \in \mathbb{R} \setminus \{0\} \), let \( K_u \) be the tempered distribution on \( \mathbb{R}^n \) such that \( \hat{K}_u(\xi) = |\xi|^{-iu} \). Here \( \hat{K}_u \) is defined via \( \langle \hat{K}_u, f \rangle = \langle K_u, \hat{f} \rangle \) for \( f \in \mathcal{S}(\mathbb{R}^n) \), with \( \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{2\pi i x \cdot \xi} \, dx \) being the usual Fourier transform in \( \mathbb{R}^n \). Explicitly, \( K_u \) may be given by

\[
K_u(x) = C(u)|x|^{-n+iu}
\]

where \( C(u) = \pi^{-n/2+iu} \Gamma\left(\frac{n-in}{2}\right) / \Gamma\left(\frac{n}{2}\right) \) (see [8], p. 117). Then define the singular integral operator \( I_u \) on \( \mathcal{S}(\mathbb{R}^n) \) by

\[
I_u f = K_u \ast f.
\]

By Plancherel’s theorem, we see that \( I_u \) extends to an isometry on \( L^2(\mathbb{R}^n) \), and that

\[
(I_u f)\hat{}(\xi) = |\xi|^{-iu} \hat{f}(\xi) = (2\pi)^{iu}(\Delta^{-iu/2} f)\hat{}(\xi),
\]

that is, \( I_u \) is an imaginary power of the Laplace operator \( \Delta = -\sum_{j=1}^n \partial_j^2 \).

To be able to say more about \( I_u \), let us examine its kernel \( K_u \). Clearly \( K_u \) is locally integrable away from 0 and, since \( C(u) = O((1 + |u|)^{n/2}) \), we see that \( K_u \) satisfies

(a) \( |K_u(x)| \leq C(1 + |u|)^{n/2}|x|^{-n}, \quad x \neq 0, \) and

(b) \( |K_u(x - y) - K_u(x)| \leq C(1 + |u|)^{n/2+1}|y||x|^{-n-1}, \quad |x| > 2|y| > 0, \)

whence (by interpolation)

(c) \( |K_u(x - y) - K_u(x)| \leq C(1 + |u|)^{n/2+\delta}|y|^{\delta}|x|^{-n-\delta}, \quad |x| > 2|y| > 0, \)

for any \( 0 \leq \delta \leq 1 \). From (c), one may check that \( K_u \) satisfies the Hörmander condition

\[
\int_{|x| > 2|y|} |K_u(x - y) - K_u(x)| \, dx \leq C_\delta(1 + |u|)^{n/2+\delta}
\]

whenever \( 0 < \delta \leq 1 \). Hence, our operator \( I_u \) extends to a bounded operator on \( L^p(\mathbb{R}^n) \) for \( 1 < p < \infty \), and also from \( H^p(\mathbb{R}^n) \) to \( L^p(\mathbb{R}^n) \) for \( n/(n+1) < p \leq 1 \). Indeed, one may verify that

\[
\|I_u f\|_{L^p} \leq C_{p,\delta}(1 + |u|)^{n/p-n/2+\delta}\|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),
\]

for \( 1 < p < \infty \), and

\[
\|I_u f\|_{L^p} \leq C_{p,\delta}(1 + |u|)^{n/p-n/2+\delta}\|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n),
\]

for \( 1 < p < \infty \), and
for $n/(n + 1) < p \leq 1$ and $0 < \delta \leq 1$. By observing that, for every $k \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $K_u$ is of class $C^k$ away from the origin, and satisfies

\[(d) \quad |D^\beta K_u(x)| \leq C(1 + |u|)^{n/2 + k}|x|^{-n-|\beta|}, \quad x \neq 0,
\]

for every multi-index $\beta$ with $|\beta| \leq k$, one can show that $I_u$ extends to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $n/(n + k) < p \leq 1$, and hence for every $0 < p \leq 1$ (see [3], pp. 320–322).

2.1. **Unweighted $H^p - L^p$ estimates.** As recently shown in [5], we can actually get rid of $\delta$ in the $H^1 - L^1$ estimate for $I_u$ (and hence, by interpolation with the $L^2$ result and duality arguments, we can also get rid of $\delta$ in the $L^p - L^p$ estimate for $1 < p < \infty$). This is the best we can achieve in the sense that we cannot have the exponent of $(1 + |u|)$ less than $|n/p - n/2|$. See also [7] for similar results.

The following theorem states that the same is also true for $0 < p \leq 1$.

**Theorem 1.** The $H^p - L^p$ inequality

\[\|I_u f\|_{L^p} \leq C_p (1 + |u|)^{n/p - n/2}\|f\|_{H^p}, \quad f \in H^p(\mathbb{R}^n),\]

holds for $0 < p \leq 1$.

**Proof.** Suppose first that $n/(n + 1) < p \leq 1$. By the atomic decomposition, it suffices to show that

\[\|I_u a\|_{L^p} \leq C(1 + |u|)^{n/p - n/2}\]

for any $p$-atom $a$. So, let $a$ be a $p$-atom, supported in a cube $Q$, such that $\|a\|_{\infty} \leq |Q|^{-1/p}$ and $\int_{\mathbb{R}^n} a(y) dy = 0$, and assume that $|u| > 2$ so that (b) holds for $|x| > |u||y| > 0$. By translation-invariance, we may assume that $Q$ is centered at the origin, say $Q = [-R, R]^n$. Now, to estimate $\|I_u a\|_{L^p}$, write

\[\int_{\mathbb{R}^n} |I_u a(x)|^p \, dx = \int_{|x| < |u|R} |I_u a(x)|^p \, dx + \int_{|x| > |u|R} |I_u a(x)|^p \, dx = I + II.
\]

(Note the difference from the usual trick: instead of splitting the integral at $2R$, we split it at $|u|R$, just as in [5]...) For the first integral, we use the fact that $I_u$ is an
isometry on $L^2(\mathbb{R}^n)$ and apply the Cauchy-Schwarz inequality to get
\[
I \leq \left[ \int_{|x|<|u|R} dx \right]^{1-p/2} \left[ \int_{|x|<|u|R} |I_u a(x)|^2 dx \right]^{p/2} \leq (|u|R)^{n-np/2} \|a\|_2 \leq |u|^{n-np/2}.
\]

For the second integral, we first observe that by using (b) we have
\[
|I_u a(x)| = \left| \int_{|y|<R} K_u(x - y) a(y) \, dy \right|
= \left| \int_{|y|<R} [K_u(x - y) - K_u(x)] a(y) \, dy \right|
\leq \int_{|y|<R} \left| K_u(x - y) - K_u(x) \right| |a(y)| \, dy
\leq C(1 + |u|)^{n/2+1} |x|^{-n-1} \int_{|y|<R} |y| |a(y)| \, dy
\leq C(1 + |u|)^{n/2+1} R^{n+1-n/p} |x|^{-n-1},
\]
whenever $|x| > |u|R$. Hence, since $p > n/(n + 1)$, we get
\[
II \leq C^p (1 + |u|)^{np/2+p} R^{np+p-n} \int_{|x|>|u|R} |x|^{-np-p} \, dx \leq C^p (1 + |u|)^{n-np/2}.
\]

Combining with the previous estimate and then taking the $p$-th root, we obtain
\[
\|I_u a\|_{L^p} \leq C(1 + |u|)^{n/p-n/2},
\]
as desired.

Suppose now that $n/(n + k) < p \leq n/(n + k - 1)$ for some $k \in \mathbb{N}$. Take a $p$-atom $a$ which is supported in $Q = [-R, R]^n$. We wish to show that
\[
\|I_u a\|_{L^p} \leq C(1 + |u|)^{n/p-n/2}.
\]

For this, we split again the integral at $|u|R$, with $|u| > 2$. The estimate for the first integral will be exactly the same as before. For the second integral, we use the fact that $K_u$ is of class $C^k$ away from the origin and satisfies (d). We subtract from $K_u(x - y)$ the Taylor polynomial of $K_u$ at $x$ of degree $[n(1/p - 1)] = k - 1$, to obtain
\[
|I_u a(x)| \leq C(1 + |u|)^{n/2+k} R^{n+k-n/p} |x|^{-n-k}, \quad |x| > |u|R.
\]

The estimate for the second integral will then follow immediately from this. \qed
2.2. **Weighted $H^p - L^p$ estimates.** As for the unweighted case, one may easily verify that the weighted inequality

$$\|I_u f\|_{L^p_w} \leq C_{p,w,\delta}(1 + |u|)^{n/2+\delta}\|f\|_{L^p_w}, \quad f \in L^p_w(\mathbb{R}^n),$$

holds whenever $w \in A_p$, $1 < p < \infty$, and $\delta > 0$ sufficiently small. The proof reduces to establishing the pointwise estimate

$$(I_u f)^\#(x) \leq C_{q,\delta}(1 + |u|)^{n/2+\delta}[M(|f|^q)]^{1/q}(x)$$

for some $q > 1$ such that $w \in A_{p/q}$. [Here $f \mapsto f^\#$ denotes the sharp maximal operator (see [10], p. 146), while $M$ is the standard Hardy-Littlewood maximal operator.] See [3], p. 411, for why, and [10], pp. 157–158, for how. See also [4] for an alternative proof. Further, as in the unweighted case, we can also get rid of $\delta$ here to obtain the sharp estimate (see [5]).

We shall now show that our operator $I_u$ can also be extended to a bounded operator from $H^p_w(\mathbb{R}^n)$ to $L^p_w(\mathbb{R}^n)$ where $0 < p \leq 1$ and $w \in A_q$, for some $q > 1$. More precisely, we have the following result.

**Theorem 2.** Let $0 < p \leq 1$. Suppose that $n/(n+k) < p \leq n/(n+k-1)$ for some $k \in \mathbb{N}$ and let $0 < \epsilon < n+k-n/p$. Then, the inequality

$$\|I_u f\|_{L^p_w} \leq C_{p,w,\epsilon}(1 + |u|)^{n/p-n/2+\epsilon}\|f\|_{H^p_w}, \quad f \in H^p_w(\mathbb{R}^n),$$

holds whenever $w \in A_{1+\epsilon p/n}$.

**Proof.** We shall only prove the case where $k = 1$. As usual, we shall use the atomic decomposition, which will reduce our task to showing that

$$\|I_u a\|_{L^p_w} \leq C_{p,w,\epsilon}(1 + |u|)^{n/p-n/2+\epsilon}$$

for any $p$-atom $a$ with respect to $w \in A_q$, where $q = 1 + \epsilon p/n$. Let $a$ be a $p$-atom with respect to $w$, supported in a finite cube $Q$, such that $\|a\|_\infty \leq w(Q)^{-1/p}$ and $\int_{\mathbb{R}^n} a(y)w(y)dy = 0$, and assume that $|u| > 2$. We may also assume that $Q$ is centered
at the origin, say \( Q = [-R, R]^n \). Now write
\[
\int_{\mathbb{R}^n} |I_a a(x)|^p w(x) dx = \int_{|x| \leq |u|R} |I_a a(x)|^p w(x) dx + \int_{|x| > |u|R} |I_a a(x)|^p w(x) dx = I + II.
\]
To estimate I, we use Hölder’s inequality and the fact that \( I_a \) is bounded on \( L^q_w(\mathbb{R}^n) \), with norm \( \leq C_{q,w}(1 + |u|)^{n/2} \). Precisely, we have
\[
I \leq \left[ \int_{|x| \leq |u|R} w(x) dx \right]^{1-p/q} \left[ \int_{|x| \leq |u|R} |I_a a(x)|^q w(x) dx \right]^{p/q} \\
\leq C_{p,q,w}^p (1 + |u|)^{np/2} \|a\|_{q,w}^p \\
\leq C_{p,q,w} (1 + |u|)^{nq-np/2} \|a\|_{q,w}^p w(Q)^{1-p/q},
\]
by Lemma 2.2 of [3], p. 396 (applied to \( Q^{|u|} \), the \( |u| \)-dilate of \( Q \)). But
\[
\|a\|_{q,w}^q = \int_{\mathbb{R}^n} |a(x)|^q w(x) dx \leq \int_Q w(Q)^{-q/p} w(x) dx = w(Q)^{1-q/p},
\]
and so \( \|a\|_{p,q,w} \leq w(Q)^{p/q-1} \), whence
\[
I \leq C_{p,q,w} (1 + |u|)^{nq-np/2}.
\]
To estimate II, we first observe that
\[
|I_a a(x)| \leq C(1 + |u|)^{n/2+1} R^{n+1} w(Q)^{-1/p} |x|^{-n-1},
\]
so that
\[
II \leq C (1 + |u|)^{np/2+np} R^{np+p} w(Q)^{-1} \int_{|x| > |u|R} \frac{w(x)}{|x|^{np+p}} dx.
\]
But, since \( q < p(n+1)/n \), we have
\[
\int_{|x| > |u|R} \frac{w(x)}{|x|^{np+p}} dx = \sum_{j=1}^\infty \int_{2^{j-1} |u| < |x| \leq 2^j |u|R} \frac{w(x)}{|x|^{np+p}} dx \\
\leq C \sum_{j=1}^\infty (2^j |u|R)^{-np-p} w(Q^2 |u|) \\
\leq C (1 + |u|)^{nq-np-p} R^{-np-p} w(Q) \sum_{j=1}^\infty 2^{j(nq-np-p)} \\
< C_p (1 + |u|)^{nq-np-p} R^{-np-p} w(Q),
\]
and hence
\[
II \leq C_p (1 + |u|)^{nq-np/2}.
\]
Combining the two estimates, we get
\[
\int_{\mathbb{R}^n} |I_u a(x)|^p w(x) \, dx \leq C_{p,q,w} (1 + |u|)^{nq - np/2}.
\]
Finally, substituting \( q = 1 + \epsilon p/n \) and then taking the \( p \)-th root, we obtain
\[
\|I_u a\|_{L^p_w} \leq C_{p,w,\epsilon} (1 + |u|)^{n/p - n/2 + \epsilon}.
\]
This completes the proof. \( \square \)

Remark. Notice that as \( \epsilon \) tends to 0, the exponent of \( (1 + |u|) \) tends to \( n/p - n/2 \) and the set of weights \( w \) for which the inequality holds tends to the Muckenhoupt class \( A_1 \). However, we do not know whether the estimate
\[
\|I_u\|_{L^p_w} \leq C_{p,w}(1 + |u|)^{n/p - n/2} \|f\|_{H^p_w}, \quad f \in H^p_w(\mathbb{R}^n),
\]
holds for \( 0 < p \leq 1 \) and \( w \in A_1 \).

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8