SOME WEIGHTED ESTIMATES FOR IMAGINARY POWERS OF LAPLACE OPERATORS

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ABSTRACT. We study the boundedness of singular integral operators that are imaginary powers of the Laplace operator in $\mathbf{R}^n$, especially from weighted Hardy spaces $H^p_w(\mathbf{R}^n)$ to weighted Lebesgue spaces $L^p_w(\mathbf{R}^n)$ where $0 < p \le 1$. In particular, we prove some $H^p_w - L^p_w$ estimates for these operators when $0 < p \le 1$ and w is in the Muckenhoupt's class A_q, for some $q > 1$.

1. INTRODUCTION

We shall study a class of singular integral operators that are imaginary powers of the Laplace operator in $\mathbf{R}^n$. But first let us review some basic properties of singular integral operators in general.

It is well-known that every singular integral operator T defined on $\mathcal{S}(\mathbf{R}^n)$ by

$$Tf = K * f,$$

where K is a tempered distribution on $\mathbf{R}^n$ with $\widehat{K} \in L^\infty(\mathbf{R}^n)$, extends to a bounded operator on $L^2(\mathbf{R}^n)$. Provided that the kernel K is locally integrable away from 0 and satisfies the Hörmander condition

$$\int_{|x|>2|y|} |K(x-y) - K(x)|\, dx \le C_K,$$

such an operator will also extend to a bounded operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$. For $p = 1$ and $p = \infty$, weaker results are available.

Moreover, for $p = 1$, one may also show that T extends to a bounded operator from $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. In fact, with some extra conditions on K, the operator

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T can be extended to a bounded operator from $H^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $0 < p \leq 1$. Here, for each $0 < p \leq 1$, $H^p(\mathbf{R}^n)$ denotes the Hardy space, whose members can be written as $\sum_j \lambda_j a_j$ where the a_j's are p-atoms and λ_j's are real numbers such that $\sum_j |\lambda_j|^p < C\|f\|_{H^p}^p$. [A p-atom in $\mathbf{R}^n$ is a function a supported in a finite cube $Q \subseteq \mathbf{R}^n$ such that $\|a\|_\infty \leq |Q|^{-1/p}$ and $\int_{\mathbf{R}^n} a(x)x^\alpha dx = 0$ for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leq n(1/p - 1)$.] See [3], Ch. II–III.

Now let w be a nonnegative measurable function or a *weight* on $\mathbf{R}^n$ and $L_w^p(\mathbf{R}^n)$ be the space of all functions $f : \mathbf{R}^n \to \mathbf{C}$ for which $\|f\|_{L_w^p} = \left[\int_{\mathbf{R}^n} |f(x)|^p w(x)dx\right]^{1/p} < \infty$. Then one may show that T extends to a bounded operator on $L_w^p(\mathbf{R}^n)$ for $1 < p < \infty$ provided that $w \in A_p$, that is, w satisfies the A_p condition

$$\left[\frac{1}{|Q|}\int_Q w(x)dx\right]\left[\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right]^{p-1} \leq C$$

for all cubes Q in $\mathbf{R}^n$. [For example, $|\cdot|^a \in A_p,\ 1 < p < \infty$, if and only if $-n < a < n(p-1)$.] As in the unweighted case, there is also a weaker result for $p = 1$ and $w \in A_1$, satisfying the A_1 condition

$$\frac{1}{|Q|}\int_Q w(x)dx \leq C\, w(y), \quad \text{a.e. } y \in Q,$$

for all cubes Q in $\mathbf{R}^n$, which can be viewed as the limit of A_p conditions for $p \to 1^+$. (For example, $|\cdot|^a \in A_1$ if and only if $-n < a \leq 0$.) See [3], Ch. IV.

In this note, we shall study the boundedness of singular integral operators that are imaginary powers of the Laplace operator in $\mathbf{R}^n$, especially from $H_w^p(\mathbf{R}^n)$ to $L_w^p(\mathbf{R}^n)$ where $0 < p \leq 1$. Here $H_w^p(\mathbf{R}^n)$ denotes the weighted Hardy space, defined just as $H^p(\mathbf{R}^n)$ but with measure $w(x)dx$ replacing the usual Lebesgue measure dx. These operators were studied by B. Muckenhoupt [6] in 1960, and used by Cowling and Mauceri [1] in 1978 to prove the boundedness of E.M. Stein's spherical maximal operator [9]. What we are interested in here is how their norms, especially from $H_w^p(\mathbf{R}^n)$ to $L_w^p(\mathbf{R}^n)$ where $0 < p \leq 1$, actually depends on the imaginary power.

Recent works indicate that the study of imaginary powers of operators in general have some applications in the theory of spectral multipliers. See, e.g., [2] and [7].

2. Main Results

For each $u \in \mathbf{R} \setminus \{0\}$, let K_u be the tempered distribution on $\mathbf{R}^n$ such that $\widehat{K_u}(\xi) = |\xi|^{-iu}$. Here $\widehat{K_u}$ is defined via $\langle \widehat{K_u}, f\rangle = \langle K_u, \widehat{f}\rangle \ \forall f \in \mathcal{S}(\mathbf{R}^n)$, with $\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{2\pi i x\cdot\xi}dx$ being the usual Fourier transform in $\mathbf{R}^n$. Explicitly, K_u may be given by

$$K_u(x) = C(u)|x|^{-n+iu}$$

where $C(u) = \pi^{-n/2+iu}\Gamma\left(\frac{n-iu}{2}\right)/\Gamma\left(\frac{iu}{2}\right)$ (see [8], p. 117). Then define the singular integral operator I_u on $\mathcal{S}(\mathbf{R}^n)$ by

$$I_u f = K_u * f.$$

By Plancherel's theorem, we see that I_u extends to an isometry on $L^2(\mathbf{R}^n)$, and that

$$(I_u f)\widehat{\ }(\xi) = |\xi|^{-iu}\widehat{f}(\xi) = (2\pi)^{iu}(\Delta^{-iu/2}f)\widehat{\ }(\xi),$$

that is, I_u is an imaginary power of the Laplace operator $\Delta = -\sum_{j=1}^n \partial_j^2$.

To be able to say more about I_u, let us examine its kernel K_u. Clearly K_u is locally integrable away from 0 and, since $C(u) = \mathrm{O}((1+|u|)^{n/2})$, we see that K_u satisfies

(a) $\quad |K_u(x)| \le C(1+|u|)^{n/2}|x|^{-n}, \quad x \neq 0$, and

(b) $\quad |K_u(x-y) - K_u(x)| \le C(1+|u|)^{n/2+1}|y||x|^{-n-1}, \quad |x| > 2|y| > 0,$

whence (by interpolation)

(c) $\quad |K_u(x-y) - K_u(x)| \le C(1+|u|)^{n/2+\delta}|y|^{\delta}|x|^{-n-\delta}, \quad |x| > 2|y| > 0,$

for any $0 \le \delta \le 1$. From (c), one may check that K_u satisfies the Hörmander condition

$$\int_{|x|>2|y|} |K_u(x-y) - K_u(x)|\, dx \le C_\delta(1+|u|)^{n/2+\delta}$$

whenever $0 < \delta \le 1$. Hence, our operator I_u extends to a bounded operator on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, and also from $H^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $n/(n+1) < p \le 1$. Indeed, one may verify that

$$\|I_u f\|_{L^p} \le C_{p,\delta}(1+|u|)^{|n/p-n/2|+\delta}\|f\|_{L^p}, \quad f \in L^p(\mathbf{R}^n),$$

for $1 < p < \infty$, and

$$\|I_u f\|_{L^p} \le C_{p,\delta}(1+|u|)^{n/p-n/2+\delta}\|f\|_{H^p}, \quad f \in H^p(\mathbf{R}^n),$$

for $n/(n+1) < p \leq 1$ and $0 < \delta \leq 1$. By observing that, for every $k \in \mathbf{N} = \{1, 2, 3, \dots\}$, K_u is of class C^k away from the origin, and satisfies

(d) $\qquad |D^\beta K_u(x)| \leq C(1+|u|)^{n/2+k}|x|^{-n-|\beta|}, \quad x \neq 0,$

for every multi-index β with $|\beta| \leq k$, one can show that I_u extends to a bounded operator from $H^p(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$ for $n/(n+k) < p \leq 1$, and hence for every $0 < p \leq 1$ (see [3], pp. 320–322).

2.1. **Unweighted $H^p - L^p$ estimates.** As recently shown in [5], we can actually get rid of δ in the $H^1 - L^1$ estimate for I_u (and hence, by interpolation with the L^2 result and duality arguments, we can also get rid of δ in the $L^p - L^p$ estimate for $1 < p < \infty$). This is the best we can achieve in the sense that we cannot have the exponent of $(1+|u|)$ less than $|n/p - n/2|$. See also [7] for similar results.

The following theorem states that the same is also true for $0 < p \leq 1$.

Theorem 1. *The $H^p - L^p$ inequality*

$$\|I_u f\|_{L^p} \leq C_p(1+|u|)^{n/p-n/2}\|f\|_{H^p}, \quad f \in H^p(\mathbf{R}^n),$$

holds for $0 < p \leq 1$.

Proof. Suppose first that $n/(n+1) < p \leq 1$. By the atomic decomposition, it suffices to show that

$$\|I_u a\|_{L^p} \leq C(1+|u|)^{n/p-n/2}$$

for any p-atom a. So, let a be a p-atom, supported in a cube Q, such that $\|a\|_\infty \leq |Q|^{-1/p}$ and $\int_{\mathbf{R}^n} a(y)\,dy = 0$, and assume that $|u| > 2$ [so that (b) holds for $|x| > |u||y| > 0$]. By translation-invariance, we may assume that Q is centered at the origin, say $Q = [-R, R]^n$. Now, to estimate $\|I_u a\|_{L^p}$, write

$$\int_{\mathbf{R}^n} |I_u a(x)|^p\,dx = \int_{|x|<|u|R} |I_u a(x)|^p dx + \int_{|x|>|u|R} |I_u a(x)|^p dx = \mathrm{I} + \mathrm{II}.$$

(Note the difference from the usual trick: instead of splitting the integral at $2R$, we split it at $|u|R$, just as in [5].) For the first integral, we use the fact that I_u is an isometry on $L^2(\mathbf{R}^n)$ and apply the Cauchy-Schwarz inequality to get

$$\mathrm{I} \le \Big[\int_{|x|<|u|R} dx\Big]^{1-p/2} \Big[\int_{|x|<|u|R} |I_u a(x)|^2 dx\Big]^{p/2} \le (|u|R)^{n-np/2} \|a\|_2 \le |u|^{n-np/2}.$$

For the second integral, we first observe that by using (b) we have

$$\begin{aligned}
|I_u a(x)| &= \Big|\int_{|y|<R} K_u(x-y) a(y)\, dy\Big| \\
&= \Big|\int_{|y|<R} [K_u(x-y) - K_u(x)] a(y)\, dy\Big| \\
&\le \int_{|y|<R} \Big|K_u(x-y) - K_u(x)\big|\, |a(y)|\, dy \\
&\le C(1+|u|)^{n/2+1} |x|^{-n-1} \int_{|y|<R} |y||a(y)| dy \\
&\le C(1+|u|)^{n/2+1} R^{n+1-n/p} |x|^{-n-1},
\end{aligned}$$

whenever $|x| > |u|R$. Hence, since $p > n/(n+1)$, we get

$$\mathrm{II} \le C^p (1+|u|)^{np/2+p} R^{np+p-n} \int_{|x|>|u|R} |x|^{-np-p} dx \le C^p (1+|u|)^{n-np/2}.$$

Combining with the previous estimate and then taking the p-th root, we obtain

$$\|I_u a\|_{L^p} \le C(1+|u|)^{n/p-n/2},$$

as desired.

Suppose now that $n/(n+k) < p \le n/(n+k-1)$ for some $k \in \mathbf{N}$. Take a p-atom a which is supported in $Q = [-R, R]^n$. We wish to show that

$$\|I_u a\|_{L^p} \le C(1+|u|)^{n/p-n/2}.$$

For this, we split again the integral at $|u|R$, with $|u| > 2$. The estimate for the first integral will be exactly the same as before. For the second integral, we use the fact that K_u is of class C^k away from the origin and satisfies (d). We subtract from $K_u(x-y)$ the Taylor polynomial of K_u at x of degree $[n(1/p-1)] = k-1$, to obtain

$$|I_u a(x)| \le C(1+|u|)^{n/2+k} R^{n+k-n/p} |x|^{-n-k}, \quad |x| > |u|R.$$

The estimate for the second integral will then follow immediately from this. $\square$

2.2. **Weighted $H^p - L^p$ estimates.** As for the unweighted case, one may easily verify that the weighted inequality

$$\|I_u f\|_{L^p_w} \le C_{p,w,\delta}(1+|u|)^{n/2+\delta}\|f\|_{L^p_w}, \quad f \in L^p_w(\mathbf{R}^n),$$

holds whenever $w \in A_p$, $1 < p < \infty$, and $\delta > 0$ sufficiently small. The proof reduces to establishing the pointwise estimate

$$(I_u f)^{\#}(x) \le C_{q,\delta}(1+|u|)^{n/2+\delta}[M(|f|^q)]^{1/q}(x)$$

for some $q > 1$ such that $w \in A_{p/q}$. [Here $f \mapsto f^{\#}$ denotes the sharp maximal operator (see [10], p. 146), while M is the standard Hardy-Littlewood maximal operator.] See [3], p. 411, for why, and [10], pp. 157–158, for how. See also [4] for an alternative proof. Further, as in the unweighted case, we can also get rid of δ here to obtain the sharp estimate (see [5]).

We shall now show that our operator I_u can also be extended to a bounded operator from $H^p_w(\mathbf{R}^n)$ to $L^p_w(\mathbf{R}^n)$ where $0 < p \le 1$ and $w \in A_q$, for some $q > 1$. More precisely, we have the following result.

Theorem 2. *Let $0 < p \le 1$. Suppose that $n/(n+k) < p \le n/(n+k-1)$ for some $k \in \mathbf{N}$ and let $0 < \epsilon < n+k-n/p$. Then, the inequality*

$$\|I_u f\|_{L^p_w} \le C_{p,w,\epsilon}(1+|u|)^{n/p-n/2+\epsilon}\|f\|_{H^p_w}, \quad f \in H^p_w(\mathbf{R}^n),$$

holds whenever $w \in A_{1+\epsilon p/n}$.

Proof. We shall only prove the case where $k = 1$. As usual, we shall use the atomic decomposition, which will reduce our task to showing that

$$\|I_u a\|_{L^p_w} \le C_{p,w,\epsilon}(1+|u|)^{n/p-n/2+\epsilon}$$

for any p-atom a with respect to $w \in A_q$, where $q = 1 + \epsilon p/n$. Let a be a p-atom with respect to w, supported in a finite cube Q, such that $\|a\|_\infty \le w(Q)^{-1/p}$ and $\int_{\mathbf{R}^n} a(y)w(y)dy = 0$, and assume that $|u| > 2$. We may also assume that Q is centered at the origin, say $Q = [-R, R]^n$. Now write

$$\int_{\mathbf{R}^n} |I_u a(x)|^p w(x)dx = \int_{|x|\le|u|R} |I_u a(x)|^p w(x)dx + \int_{|x|>|u|R} |I_u a(x)|^p w(x)dx = \mathrm{I} + \mathrm{II}.$$

To estimate I, we use Hölder's inequality and the fact that I_u is bounded on $L^q_w(\mathbf{R}^n)$, with norm $\le C_{q,w}(1+|u|)^{n/2}$. Precisely, we have

$$\begin{aligned}\mathrm{I} &\le \Big[\int_{|x|\le|u|R} w(x)dx\Big]^{1-p/q}\Big[\int_{|x|\le|u|R} |I_u a(x)|^q w(x)dx\Big]^{p/q}\\ &\le C^p_{q,w} w(Q^{|u|})^{1-p/q}(1+|u|)^{np/2}\|a\|^p_{q,w}\\ &\le C_{p,q,w}(1+|u|)^{nq-np/2}\|a\|^p_{q,w} w(Q)^{1-p/q},\end{aligned}$$

by Lemma 2.2 of [3], p. 396 (applied to $Q^{|u|}$, the $|u|$-dilate of Q). But

$$\|a\|^q_{q,w} = \int_{\mathbf{R}^n} |a(x)|^q w(x)dx \le \int_Q w(Q)^{-q/p} w(x)dx = w(Q)^{1-q/p},$$

and so $\|a\|^p_{q,w} \le w(Q)^{p/q-1}$, whence

$$\mathrm{I} \le C_{p,q,w}(1+|u|)^{nq-np/2}.$$

To estimate II, we first observe that

$$|I_u a(x)| \le C(1+|u|)^{n/2+1} R^{n+1} w(Q)^{-1/p}|x|^{-n-1},$$

so that

$$\mathrm{II} \le C(1+|u|)^{np/2+p} R^{np+p} w(Q)^{-1}\int_{|x|>|u|R} \frac{w(x)}{|x|^{np+p}}dx.$$

But, since $q < p(n+1)/n$, we have

$$\begin{aligned}\int_{|x|>|u|R} \frac{w(x)}{|x|^{np+p}}dx &= \sum_{j=1}^{\infty}\int_{2^{j-1}|u|R<|x|\le 2^j|u|R} \frac{w(x)}{|x|^{np+p}}dx\\ &\le C\sum_{j=1}^{\infty}(2^j|u|R)^{-np-p} w(Q^{2^j|u|})\\ &\le C(1+|u|)^{nq-np-p} R^{-np-p} w(Q)\sum_{j=1}^{\infty} 2^{j(nq-np-p)}\\ &< C_p(1+|u|)^{nq-np-p} R^{-np-p} w(Q),\end{aligned}$$

and hence

$$\mathrm{II} \le C_p(1+|u|)^{nq-np/2}.$$

Combining the two estimates, we get

$$\int_{\mathbf{R}^n} |I_u a(x)|^p w(x) dx \leq C_{p,q,w}(1+|u|)^{nq-np/2}.$$

Finally, substituting $q = 1 + \epsilon p/n$ and then taking the p-th root, we obtain

$$\|I_u a\|_{L^p_w} \leq C_{p,w,\epsilon}(1+|u|)^{n/p-n/2+\epsilon}.$$

This completes the proof. □

`Remark`. Notice that as ϵ tends to 0, the exponent of $(1+|u|)$ tends to $n/p - n/2$ and the set of weights w for which the inequality holds tends to the Muckenhoupt class A_1. However, we do not know whether the estimate

$$\|I_u\|_{L^p_w} \leq C_{p,w}(1+|u|)^{n/p-n/2} \|f\|_{H^p_w}, \quad f \in H^p_w(\mathbf{R}^n),$$

holds for $0 < p \leq 1$ and $w \in A_1$.

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