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# A GENERALIZATION OF MAXIMAL FUNCTIONS ON COMPACT SEMISIMPLE LIE GROUPS

HENDRA GUNAWAN

**Let $G$ be a compact semisimple Lie group with finite centre. For each positive number $s$, let $\mu_{sH}$ denote the $\mathrm{Ad}(G)$-invariant probability measure carried on the conjugacy class of $\exp(sH)$ in $G$. With this one-parameter family of measures, we define the maximal operator $\mathscr{M}_H$ on $\mathscr{C}(G)$. We then estimate the Fourier transform of $\mu_{sH}$ and of some derived distributions. Our result leads to the boundedness of $\mathscr{M}_H$ on $L^p(G)$, for all $p$ greater than some index $p_0$ in $(1, 2)$. This generalizes a recent result of M. Cowling and C. Meaney [2].**

**Introduction.** Let $G$ be a compact semisimple Lie group of rank $l$ with finite centre, and with its Haar measure normalized to have total mass 1. Let $\mathfrak{g}$ denote its Lie algebra, and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. We denote by $\Phi$ the root system of $(\mathfrak{g}^c, \mathfrak{h}^c)$, and fix $\Delta = \{\alpha_j : j \in I\}$, where $I = \{1, \dots, l\}$, to be a base of $\Phi$ (as in [**3**, §10.1]). With respect to $\Delta$, we write $\Phi^+$ for the set of positive roots, whose members are of the form

$$\alpha = \sum_{j\in I} n_j(\alpha)\alpha_j,$$

with $n_j(\alpha) \in \mathbf{Z}^+ \cup \{0\}$ for all $j \in I$, and $\Lambda^+$ for the set of dominant weights, which parametrizes the dual object of $G$.

We equip the Lie algebra $\mathfrak{g}$ with the positive definite inner product $(\cdot, \cdot)$ derived from the Killing form. For each $\nu \in \mathfrak{h}^*$, we define $H_\nu \in \mathfrak{h}$ by

$$\nu(H) = (H_\nu, H) \quad \forall H \in \mathfrak{h}.$$

We also transfer the inner product to $\mathfrak{h}^*$ via

$$(\nu, \nu') = (H_\nu, H_{\nu'}) \quad \forall \nu, \nu' \in \mathfrak{h}^*.$$

The norm on $\mathfrak{h}^*$ and $\mathfrak{h}$, induced by these inner products, will then be denoted by $|\cdot|$.

We choose a regular element $H \in \mathfrak{h}$, for which $\alpha(H) \neq 0$ for all $\alpha \in \Phi^+$, and fix $R > 0$ such that $\exp(sH)$ is regular in $G$ for any

$s \in (0, R)$. For a continuous function $f$ on $G$, the maximal function $\mathscr{M}_H f$ is defined by

$$\mathscr{M}_H f(x) = \sup_{s \in (0, R)} |\mu_{sH} * f(x)| \quad \forall x \in G,$$

where $\mu_{sH}$ is the $\mathrm{Ad}(G)$-invariant probability measure carried on the conjugacy class of $\exp(sH)$ in $G$. This definition generalizes one in the paper of Cowling and Meaney [**2**], in which $H$ was a particular regular element of $\mathfrak{h}$. Our main results are the following.

THEOREM A. *For all* $k = 0, 1, 2, \ldots$, *there exist positive constants* $C_k = C_k(H)$ *such that*

$$\left| \left( \frac{\partial}{\partial s} \right)^k \hat{\mu}_{sH}(\lambda) \right| \leq C_k \frac{(1 + |\lambda|)^k}{(1 + s|\lambda|)^\gamma} \quad \forall s \in (0, R), \lambda \in \Lambda^+,$$

*where* $\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}|$.

It is clear that Theorem A, together with the arguments of [**2**], imply the boundedness of $\mathscr{M}_H$ on $L^p(G)$ for all $p > 1 + (2\gamma)^{-1}$. So we state

THEOREM B. *For all* $p > 1 + (2\gamma)^{-1}$, *with* $\gamma$ *as above, there exist positive constants* $C_p = C_p(H)$ *such that*

$$\|\mathscr{M}_H f\|_p \leq C_p \|f\|_p \quad \forall f \in \mathscr{C}(G).$$

We prove Theorem A by handling first the case when $G$ is simple, and then extend the result to the semisimple case. Our method is based on arguments of representation theory, involving formulae for characters and dimensions, a study of root systems, the theory of weights, and properties of the Weyl group, all developed in the first part of this note. The proof of Theorem A will be given in the second part. It is clear that Theorem A is sharp since the explicit expression used in [**2**] for the particular case in which $H = H_p$ shows no improvement is possible. In the third part of this note, we give an example which shows that Theorem B too is sharp at least in the case where $G = \mathbf{SU}(2)$.

Some related results can be found in M. Christ [**1**] and C. D. Sogge and E. M. Stein [**5**].

Throughout this note, the expressions $C$, $C_k$, and $C_{k_1, k_2, k_3}$ denote various positive constants which possibly vary from line to line. These constants may depend on $G$, and some may also depend on the choice of $H$. When a constant, $C$ say, depends on $H$, we write $C(H)$ in place of $C$.

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**1. Representation theoretic arguments.** We shall assume throughout this part that the Lie algebra $\mathfrak{g}$ is simple.

1.1. We start with some formulae for characters and dimensions of representations of $G$. To each $\lambda \in \Lambda^+$, we associate the representation $\pi_\lambda$, the set of weights $\varpi_\lambda$, the character $\chi_\lambda$, and the dimension $d_\lambda = \chi_\lambda(\mathbf{1})$. For all $\lambda \in \Lambda^+$, we have (see [**3**, §22])

$$\chi_\lambda(\exp(H)) = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(H)),$$

where $m_\lambda(\lambda') \in \mathbf{Z}^+$ is the multiplicity of $\lambda'$ in $\pi_\lambda$. Accordingly,

$$d_\lambda = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda').$$

Let $\mathscr{W}$ be the Weyl group of $(\mathfrak{g}^c, \mathfrak{h}^c)$, generated by the reflections $\sigma_\alpha$ corresponding to $\alpha \in \Delta$. Introduce the special element $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha$. For all $\lambda \in \Lambda^+$, the character and dimension formulae of Weyl read (see [**3**, §24.3])

$$\chi_\lambda(\exp(H)) = \frac{\sum_{\sigma\in\mathscr{W}} \det(\sigma)\exp(i\sigma(\lambda+\rho)(H))}{\prod_{\alpha\in\Phi^+} 2i\sin\frac{1}{2}\alpha(H)}$$

and

$$d_\lambda = \prod_{\alpha\in\Phi^+} \frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}.$$

1.2. It is well known that $\mathfrak{g}^c$ has the root space decomposition (see [7, p. 273])

$$\mathfrak{g}^c = \mathfrak{g}^c \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^c_\alpha,$$

where $\mathfrak{g}^c_\alpha$ denotes the root subspace of $\mathfrak{g}^c$ corresponding to $\alpha \in \Phi$.

Assuming $l \geq 2$, we choose $j_0 \in I$, and then remove $\alpha_{j_0}$ from $\Delta$ to obtain

$$\Delta_0 = \{\alpha_j : j \in I_0\}, \quad \text{where } I_0 = I\backslash\{j_0\}.$$

Set $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and put $\Phi_0 = \Phi_0^+ \cup -\Phi_0^+$. Clearly $\Phi_0 = -\Phi_0$ and $\sigma_\alpha\Phi_0 = \Phi_0$ for all $\sigma_\alpha$ $(\alpha \in \Delta_0)$. This shows that

$\Phi_0$ is a root system (see [7, p. 370]). Let $\mathfrak{h}_0$ be the subspace of $\mathfrak{h}$ spanned by $H_\alpha$ $(\alpha \in \Phi_0)$. Then one may verify that

$$\mathfrak{g}_0^{\mathrm{c}} = \mathfrak{h}_0^{\mathrm{c}} \oplus \bigoplus_{\alpha\in\Phi_0} \mathfrak{g}_\alpha^{\mathrm{c}}$$

is a semisimple subalgebra of $\mathfrak{g}^{\mathrm{c}}$, with maximal toral subalgebra $\mathfrak{h}_0^{\mathrm{c}}$ (see [7, Ex. 30 of Ch. 4]). Evidently $\Phi_0$ is the root system of $(\mathfrak{g}_0^{\mathrm{c}}, \mathfrak{h}_0^{\mathrm{c}})$, $\Delta_0$ is a base of $\Phi_0$, and $\Phi_0^+$ is the set of positive roots with respect to $\Delta_0$.

Write $\Phi_0$ as a disjoint union of irreducible root systems, say

$$\Phi_0 = \Phi_{01} \cup \cdots \cup \Phi_{0r}.$$

Let $q \in \{1, \ldots, r\}$. Denote by $\mathfrak{h}_{0q}$ the subspace of $\mathfrak{h}_0$ spanned by $H_\alpha$ $(\alpha \in \Phi_{0q})$. Then we find that

$$\mathfrak{g}_{0q}^{\mathrm{c}} = \mathfrak{h}_{0q}^{\mathrm{c}} \oplus \bigoplus_{\alpha\in\Phi_{0q}} \mathfrak{g}_\alpha^{\mathrm{c}}$$

is a simple ideal of $\mathfrak{g}_0^{\mathrm{c}}$, with maximal toral subalgebra $\mathfrak{h}_{0q}^{\mathrm{c}}$. We also note that

$$\mathfrak{h}_0^{\mathrm{c}} = \mathfrak{h}_{01}^{\mathrm{c}} \oplus \cdots \oplus \mathfrak{h}_{0r}^{\mathrm{c}}$$

and

$$\mathfrak{g}_0^{\mathrm{c}} = \mathfrak{g}_{01}^{\mathrm{c}} \oplus \cdots \oplus \mathfrak{g}_{0r}^{\mathrm{c}}.$$

Now denote by $(\cdot, \cdot)_0$ and $(\cdot, \cdot)_{0q}$ the inner products of $g_0$ and $g_{0q}$ respectively. Then we have (see [3, Lemma 5.1])

$$(\cdot, \cdot)_0|\mathfrak{g}_{0q} \times \mathfrak{g}_{0q} = (\cdot, \cdot)_{0q},$$

and so

$$(X, Y)_0 = (X_1, Y_1)_{01} + \cdots + (X_r, Y_r)_{0r}$$

for all $X = X_1 + \cdots + X_r$, $Y = Y_1 + \cdots + Y_r \in \mathfrak{g}_0$, with $X_q, Y_q \in \mathfrak{g}_{0q}$. Further, since $\mathfrak{g}$ and $\mathfrak{g}_{0q}$ are simple, there exists a positive constant $C_q$ satisfying (see [4, p. 242])

$$(X, Y)_{0q} = C_q(X, Y) \quad \forall X, Y \in \mathfrak{g}_{0q}.$$

We transfer these inner products to the corresponding dual spaces in the usual way.

Let $\Lambda_0^+$ denote the set of dominant weights with respect to $\Delta_0$. We need to determine the set of fundamental dominant weights in $\Lambda_0^+$. Suppose $\{\omega_j : j \in I\}$ is the set of fundamental dominant weights in $\Lambda^+$, for which (see [3, §13.1] for definition)

$$2\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk} \quad \forall j, k \in I.$$

If we now set

$$\tilde{\omega}_j = \omega_j - \mathrm{proj}_{\omega_{j_0}}(\omega_j) \quad \forall j \in I,$$

then we have the following facts.

*Fact* 1. For each $j \in I_0$, $\tilde{\omega}_j \in \mathfrak{h}^*_{0Q}$ whenever $\alpha_j \in \mathfrak{h}^*_{0Q}$.

*Proof*. For all $j, k \in I_0$, we have

$$\begin{aligned} 2\frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} &= 2\frac{\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2\frac{(\mathrm{proj}_{\omega_{j_0}}(\omega_j), \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2\frac{(\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})}\frac{(\omega_{j_0}, \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 0 = \delta_{jk}. \end{aligned}$$

Now take $j \in I_0$, and let $Q \in \{1, \dots, r\}$ such that $\alpha_j \in \mathfrak{h}^*_{0Q}$. Clearly

$$\tilde{\omega}_j \perp \mathfrak{h}^*_{0q} \quad \forall q \neq Q.$$

Writing $\tilde{\omega}_j = \tilde{\omega}_{j1} + \cdots + \tilde{\omega}_{jr}$, with $\tilde{\omega}_{jq} \in \mathfrak{h}^*_{0q}$ for all $q \in \{1, \dots, r\}$, we find that

$$\tilde{\omega}_{jq} = 0 \quad \forall q \neq Q.$$

We therefore have

$$\tilde{\omega}_j = \tilde{\omega}_{jQ} \in \mathfrak{h}^*_{0Q},$$

as stated. □

*Fact* 2. $\{\tilde{\omega}_j : j \in I_0\}$ is the set of fundamental dominant weights in $\Lambda^+_0$.

*Proof*. Take $j, k \in I_0$. Suppose $\tilde{\omega}_j \in \mathfrak{h}^*_{0q}$ and $\alpha_k \in \mathfrak{h}^*_{0q'}$ for some $q, q' \in \{1, \dots, r\}$. If $q \neq q'$, then clearly $(\tilde{\omega}_j, \alpha_k)_0 = 0$; otherwise we have

$$2\frac{(\tilde{\omega}_j, \alpha_k)_0}{(\alpha_k, \alpha_k)_0} = 2\frac{(\tilde{\omega}_j, \alpha_k)_{0q}}{(\alpha_k, \alpha_k)_{0q}} = 2\frac{(\tilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk}.$$

Using Fact 1, the assertion follows. □

*Fact* 3. Suppose $\lambda = \sum_{j\in I} n_j\omega_j \in \Lambda^+$. Then $\lambda$ can be rewritten as

$$\lambda = \lambda_0 + \lambda_1$$

where $\lambda_0 = \sum_{j\in I_0} n_j\tilde{\omega}_j \in \Lambda^+_0$ (with the same $n_j$'s) and $\lambda_1 = \mathrm{proj}_{\omega_{j_0}}(\lambda)$.

*Proof*. Noting that $\tilde{\omega}_{j_0} = 0$, we have

$$\begin{aligned}
\lambda &= \sum_{j\in I} n_j\omega_j = \sum_{j\in I} n_j\tilde{\omega}_j + \sum_{j\in I} n_j \operatorname{proj}_{\omega_{j_0}}(\omega_j) \\
&= \sum_{j\in I_0} n_j\tilde{\omega}_j + \sum_{j\in I} n_j \frac{(\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\tilde{\omega}_j + \frac{(\sum_{j\in I} n_j\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\tilde{\omega}_j + \frac{(\lambda, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\tilde{\omega}_j + \operatorname{proj}_{\omega_{j_0}}(\lambda) = \lambda_0 + \lambda_1,
\end{aligned}$$

as claimed. □

REMARK. It is well known that the special element $\rho$ is a dominant weight in $\Lambda^+$. Indeed, $\rho = \sum_{j\in I}\omega_j$ (see [**3**, Lemma 13.3A]). By Fact 3, we may rewrite $\rho = \rho_0 + \rho_1$ where $\rho_0 = \sum_{j\in I_0}\tilde{\omega}_j \in \Lambda_0^+$ and $\rho_1 = \operatorname{proj}_{\omega_{j_0}}(\rho)$. But then $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Phi_0^+}\alpha$, giving $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Phi_1^+}\alpha$ where $\Phi_1^+ = \Phi^+\backslash\Phi_0^+$. As another consequence, we also have $\rho_1 = c\omega_{j_0}$ for some $c > 0$. But we know that $2(\omega_{j_0}, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0}) = 1$, and so we find $c = 2(\rho_1, \alpha_{j_0})/(\alpha_{j_0}, \alpha_{j_0})$. Hence we determine $\omega_{j_0} = \frac{1}{2}((\alpha_{j_0}, \alpha_{j_0})/(\rho_1, \alpha_{j_0}))\rho_1$, with $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Phi_1^+}\alpha$. This offers a method of finding the fundamental dominant weight $\omega_{j_0}$ for any given $j_0 \in I$.

Introduce $\mathfrak{h}_1 = \{H \in \mathfrak{h}\colon \alpha(H) = 0 \ \ \forall\alpha \in \Delta_0\}$. Obviously $\mathfrak{h}_1$ is a subalgebra of $\mathfrak{h}$, which is spanned by $H_{\rho_1}$ (by the above remark). Moreover, we have (like Fact 3)

*Fact* 4. Every $H \in \mathfrak{h}$ can be written as

$$H = H_0 + H_1$$

where $H_0 \in \mathfrak{h}_0$ and $H_1 \in \mathfrak{h}_1$.

REMARK. $H_0 \in \mathfrak{h}_0$ means that $H_0 = H_{\nu_0}$, where $\nu_0 \in \operatorname{span}(\Delta_0)$, while $H_1 \in \mathfrak{h}_1$ means that $H_1 = H_{\nu_1}$, where $\nu_1 = r\rho_1$ for some $r \in \mathbf{R}$. Thus clearly $\mathfrak{h}_0 \perp \mathfrak{h}_1$, and so Fact 4 actually states that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$.

Suppose we are in $(\mathfrak{g}_0, \mathfrak{h}_0)$. To each $\lambda_0 \in \Lambda_0^+$, we associate the representation $\tilde{\pi}_{\lambda_0}$, the set of weights $\tilde{\varpi}_{\lambda_0}$, the character $\tilde{\chi}_{\lambda_0}$, and the dimension $\tilde{d}_{\lambda_0}$. For all $\lambda_0 \in \Lambda_0^+$ and $H_0 \in \mathfrak{h}_0$, we have

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \sum_{\lambda' \in \tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i\lambda'(H_0))$$

and

$$\tilde{d}_{\lambda_0} = \sum_{\lambda' \in \tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda'),$$

with $\tilde{m}_{\lambda_0}(\lambda') \in \mathbf{Z}^+$ being the multiplicity of $\lambda'$ in $\tilde{\pi}_{\lambda_0}$.

Let $\mathscr{W}_0$ (or $\mathscr{W}[\Delta_0]$ if necessary) denote the subgroup of $\mathscr{W}$ generated by $\sigma_\alpha$ $(\alpha \in \Delta_0)$. The Weyl formulae then read

$$\tilde{\chi}_{\lambda_0}(\exp(H_0)) = \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(H_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_0)}$$

and

$$\tilde{d}_{\lambda_0} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)}.$$

We should note that the inner product in the expression above is really the inner product of $\mathfrak{g}$. Indeed, we may calculate

$$\begin{aligned}
\tilde{d}_{\lambda_0} &= \lim_{s \to 0} \tilde{\chi}_{\lambda_0}(\exp(sH_{\rho_0})) \\
&= \lim_{s \to 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(sH_{\rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\
&= \lim_{s \to 0} \frac{\sum_{\tau \in \mathscr{W}_0} \det(\tau) \exp(i\tau\rho_0(sH_{\lambda_0 + \rho_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\
&= \lim_{s \to 0} \frac{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\lambda_0 + \rho_0})}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(sH_{\rho_0})} \\
&= \prod_{\alpha \in \Phi_0^+} \frac{\alpha(H_{\lambda_0 + \rho_0})}{\alpha(H_{\rho_0})} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)}
\end{aligned}$$

(see [**8**, p. 106] for clarification).

Allowing $\mathscr{W}$ to act, one may observe that all the above facts still hold for the system constituted by $\sigma\Phi_0$ $(\sigma \in \mathscr{W})$, as well as for that by $\Phi_0$. Moreover, the two facts below explain the connection between one system and another.

*Fact* 5. $\sigma\mathscr{W}[\Delta_0]\sigma^{-1} = \mathscr{W}[\sigma\Delta_0]$ for any $\sigma \in \mathscr{W}$.

*Proof*. Obvious (see [**3**, Lemma 9.2] for justification). □

*Fact* 6. $\tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) = \tilde{\chi}_{\lambda_0}(\exp(H_{\nu_0}))$ for any $\sigma \in \mathscr{W}$.

*Proof*. For any $\sigma \in \mathscr{W}$, we have (by Fact 5)

$$\begin{aligned}
\tilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) &= \frac{\sum_{\tau\in\mathscr{W}[\sigma\Delta_0]}\det(\tau)\exp(i\tau(\sigma\lambda_0+\sigma\rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha\in\sigma\Phi_0^+}2i\sin\frac{1}{2}\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau\in\sigma\mathscr{W}[\Delta_0]\sigma^{-1}}\det(\tau)\exp(i\tau\sigma(\lambda_0+\rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha\in\sigma\Phi_0^+}2i\sin\frac{1}{2}\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau\in\mathscr{W}[\Delta_0]}\det(\sigma\tau\sigma^{-1})\exp(i\sigma\tau(\lambda_0+\rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha\in\Phi_0^+}2i\sin\frac{1}{2}\sigma\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau\in\mathscr{W}[\Delta_0]}\det(\tau)\exp(i\tau(\lambda_0+\rho_0)(H_{\nu_0}))}{\prod_{\alpha\in\Phi_0^+}2i\sin\frac{1}{2}\alpha(H_{\nu_0})} \\
&= \tilde{\chi}\lambda_0(\exp(H_{\nu_0})),
\end{aligned}$$

as stated. □

**2. The proof of the theorem.** The outline of the proof is as follows. We first look for an estimate for all $s \in (0, R)$, then examine the decay for large $s$, and finally combine the results. The result obtained is valid under the assumption that $G$ is simple, but then it extends to every semisimple Lie group $G$.

2.1. For all $s \in (0, R)$, $\lambda \in \Lambda^+$, we have (see [**2**, p. 813])

$$\hat{\mu}_{sH}(\lambda) = \frac{\chi_\lambda(\exp(sH))}{d\lambda}.$$

Using the multiplicity formulae, we write

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\lambda'\in\varpi_\lambda} m_\lambda(\lambda')\exp(i\lambda'(sH))}{\sum_{\lambda'\in\varpi_\lambda} m_\lambda(\lambda')}.$$

Hence, we have

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial s}\right)^k \hat{\mu}_{sH}(\lambda)\right| &\le \frac{\sum_{\lambda'\in\varpi_\lambda} m_\lambda(\lambda')|(\frac{\partial}{\partial s})^k\exp(i\lambda'(sH))|}{\sum_{\lambda'\in\varpi_\lambda} m_\lambda(\lambda')} \\
&\le |H|^k|\lambda|^k = C_k(H)|\lambda|^k,
\end{aligned}$$

for all $k = 0, 1, 2, \ldots$.

2.2. By the Weyl formulae, for all $s \in (0, R)$, $\lambda \in \Lambda^+$, we have

$$\hat{\mu}_{sH}(\lambda) = \frac{\sum_{\sigma\in\mathscr{W}} \det(\sigma)\exp(i(\lambda+\rho)(sH))}{\prod_{\alpha\in\Phi^+} 2i\sin\frac{1}{2}\alpha(sH)} \prod_{\alpha\in\Phi^+} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)}.$$

In the case $l = 1$, one can easily obtain

$$\left|\left(\frac{\partial}{\partial s}\right)^k \hat{\mu}_{sH}(\lambda)\right| \le C_k(H)\frac{|\lambda+\rho|^k}{s|\lambda+\rho|},$$

for all $k = 0, 1, 2, \ldots$. So assume, hereafter, that $l \ge 2$.

For each $\lambda \in \Lambda^+$, choose $j_0 \in I$ for which $(\lambda+\rho, \alpha_{j_0})$ is maximal. As before, we write $\Delta_0 = \Delta\backslash\{\alpha_{j_0}\}$, $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and $\Phi_1^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) \ge 1\}$. (Note that $\Phi_1^+ = \Phi^+\backslash\Phi_0^+$, and that $\Phi_1^+$ depends on the choice of $j_0$, and so depends on $\lambda$.) Clearly, if $\alpha \in \Phi_0^+$, then

$$(\lambda+\rho, \alpha) \ge (\rho, \alpha) \ge C,$$

and if $\alpha \in \Phi_1^+$, then (by the choice of $j_0$)

$$(\lambda+\rho, \alpha) \ge n_{j_0}(\alpha)(\lambda+\rho, \alpha_{j_0}) \ge C|\lambda+\rho|.$$

Moreover,

$$\gamma = \min_{j\in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \ge 1\}| \le |\Phi_1^+|.$$

Recall that $\mathscr{W}_0$ is the subgroup of $\mathscr{W}$ generated by $\sigma_\alpha$ $(\alpha \in \Delta_0)$. For an appropriate $\mathscr{S} \subset \mathscr{W}$, we write $\mathscr{W} = \bigcup_{\sigma\in\mathscr{S}} \sigma\mathscr{W}_0$ (disjoint union). We then obtain

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma\in\mathscr{S}} \left(\frac{\sum_{\tau\in\mathscr{W}_0} \det(\sigma\tau)\exp(i\sigma\tau(\lambda+\rho)(sH))}{\prod_{\alpha\in\Phi^+} 2i\sin\frac{1}{2}\alpha(sH)} \prod_{\alpha\in\Phi^+} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)}\right).$$

For each reflection $\sigma_\alpha \in \mathscr{W}$, we know that $\det(\sigma_\alpha) = -1$, $\sigma_\alpha\alpha = -\alpha$, and $\sigma_\alpha(\Phi^+\backslash\{\alpha\}) = \Phi^+\backslash\{\alpha\}$ (see [3, Lemma 10.2B]). Thus, for any $\sigma \in \mathscr{W}$, we have

$$\prod_{\alpha\in\Phi^+} 2i\sin\frac{1}{2}\alpha(sH) = \det(\sigma)\prod_{\alpha\in\Phi^+} 2i\sin\frac{1}{2}\sigma\alpha(sH).$$

It follows that

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma\in\mathscr{S}} \left(\frac{\sum_{\tau\in\mathscr{W}_0} \det(\tau)\exp(i\sigma\tau(\lambda+\rho)(sH))}{\prod_{\alpha\in\Phi_0^+} 2i\sin\frac{1}{2}\sigma\alpha(sH)} \prod_{\alpha\in\Phi_0^+} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)}\right) \times \left(\prod_{\alpha\in\Phi_1^+} \frac{1}{2i\sin\frac{1}{2}\sigma\alpha(sH)} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)}\right).$$

Now fix $\sigma \in \mathscr{S}$. We write $H = H_{\sigma\nu}$, with $\nu = \nu_0 + \nu_1$, where $\nu_0 \in \operatorname{span}(\Delta_0)$ and $\nu_1 = r\rho_1$ for some $r \in \mathbf{R}$. Then put $H_0 = H_{\nu_0}$ and $H_1 = H_{\nu_1}$. Next recall that $\Lambda_0^+$ is the set of dominant weights corresponding to $\Phi_0^+$. For each $\lambda \in \Lambda^+$, we write $\lambda = \lambda_0 + \lambda_1$, where $\lambda_0 \in \Lambda_0^+$ and $\lambda_1 = c\rho_1$ for some $c \in \mathbf{R}^+$. Hence, for all $\alpha \in \Phi_0^+$, we have $(\rho, \alpha) = (\rho_0, \alpha)$ and $(\lambda + \rho, \alpha) = (\lambda_0 + \rho_0, \alpha)$. Further, for all $\alpha \in \Phi_0^+$,

$$\begin{aligned} \sigma\alpha(H) &= (\sigma\alpha, \sigma\nu) = (\alpha, \nu) \\ &= (\alpha, \nu_0 + \nu_1) = (\alpha, \nu_0) \quad (\text{as } \nu_1 \perp \alpha) \\ &= \alpha(H_{\nu_0}) = \alpha(H_0), \end{aligned}$$

and whenever $\tau \in \mathscr{W}_0$,

$$\begin{aligned} \sigma\tau(\lambda+\rho)(H) &= (\sigma\tau(\lambda+\rho), \sigma\nu) = (\tau(\lambda+\rho), \nu) \\ &= (\tau(\lambda_0+\rho_0) + \tau(\lambda_1+\rho_1), \nu_0+\nu_1) \\ &= (\tau(\lambda_0+\rho_0) + (\lambda_1+\rho_1), \nu_0+\nu_1) \quad (\text{as } \tau \in \mathscr{W}_0) \\ &= (\tau(\lambda_0+\rho_0), \nu_0) + (\lambda_1+\rho_1, \nu_1) \quad (\text{by orthogonality}) \\ &= \tau(\lambda_0+\rho_0)(H_{\nu_0}) + (\lambda_1+\rho_1)(H_{\nu_1}) \\ &= \tau(\lambda_0+\rho_0)(H_0) + (\lambda_1+\rho_1)(H_1). \end{aligned}$$

It turns out that

$$\begin{aligned} &\frac{\sum_{\tau\in\mathscr{W}_0} \det(\tau)\exp(i\sigma\tau(\lambda+\rho)(sH))}{\prod_{\alpha\in\Phi_0^+} 2i\sin\frac{1}{2}\sigma\alpha(sH)} \prod_{\alpha\in\Phi_0^+} \frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)} \\ &= \exp(i(\lambda_1+\rho_1)(sH_1)) \frac{\sum_{\tau\in\mathscr{W}_0} \det(\tau)\exp(i\tau(\lambda_0+\rho_0)(sH_0))}{\prod_{\alpha\in\Phi_0^+} 2i\sin\frac{1}{2}\alpha(sH_0)} \\ &\quad \times \prod_{\alpha\in\Phi_0^+} \frac{(\rho_0,\alpha)}{(\lambda_0+\rho_0,\alpha)} \\ &= \exp(i(\lambda_1+\rho_1)(sH_1)) \frac{\tilde{\chi}_{\lambda_0}(\exp(sH_0))}{\tilde{d}_{\lambda_0}} \\ &= \exp(i(\lambda_1+\rho_1)(sH_1)) \frac{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')\exp(i\lambda'(sH_0))}{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \\ &= \frac{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')\exp(i(\lambda'+\lambda_1+\rho_1)(sH))}{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \quad (\text{by orthogonality}). \end{aligned}$$

So we have

$$\hat{\mu}_{sH}(\lambda) = \sum_{\sigma\in\mathscr{S}} \left( \frac{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right) \times \left( \prod_{\alpha\in\Phi_1^+} \frac{1}{\sigma\alpha(sH)} \frac{\sigma\alpha(sH)}{2i\sin\frac{1}{2}\sigma\alpha(sH)} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)} \right).$$

For all $k = 0, 1, 2, \dots,$ we have the estimates

(1) $$\left| \left(\frac{\partial}{\partial s}\right)^k \frac{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda') \exp(i(\lambda' + \lambda_1 + \rho_1)(sH))}{\sum_{\lambda'\in\tilde{\varpi}_{\lambda_0}} \tilde{m}_{\lambda_0}(\lambda')} \right| \le |H|^k |\lambda+\rho|^k,$$

(2) $$\left| \left(\frac{\partial}{\partial s}\right)^k \prod_{\alpha\in\Phi_1^+} \sigma\alpha(sH)^{-1} \right| \le C_k(H) s^{-k-|\Phi_1^+|},$$

(3) $$\left| \left(\frac{\partial}{\partial s}\right)^k \prod_{\alpha\in\Phi_1^+} \frac{\sigma\alpha(sH)}{2i\sin\frac{1}{2}\sigma\alpha(sH)} \right| \le C_k \quad \text{(by Leibniz' rule)},$$

(4) $$\left| \prod_{\alpha\in\Phi_1^+} \frac{(\rho, \alpha)}{(\lambda+\rho, \alpha)} \right| \le C|\lambda+\rho|^{-|\Phi_1^+|}$$

(as $(\lambda+\rho, \alpha) \ge C|\lambda+\rho|$ for all $\alpha \in \Phi_1^+$).

Therefore, by Leibniz' rule for the derivatives of products, we obtain

$$\begin{aligned}
&\left| \left(\frac{\partial}{\partial s}\right)^k \hat{\mu}_{sH}(\lambda) \right| \\
&\le \sum_{\sigma\in\mathscr{S}} \sum_{k_1+k_2+k_3=k} C_{k_1,k_2,k_3} \left| \left(\frac{\partial}{\partial s}\right)^{k_1} (\text{1st term}) \right| \\
&\qquad \times \left| \left(\frac{\partial}{\partial s}\right)^{k_2} (\text{2nd term}) \right| \left| \left(\frac{\partial}{\partial s}\right)^{k_3} (\text{3rd term}) \right| |\text{4th term}| \\
&\le \sum_{\sigma\in\mathscr{S}} \sum_{k_1+k_2+k_3=k} C_{k_1,k_2,k_3}(H) |H|^{k_1} |\lambda+\rho|^{k_1} s^{-k_2} (s|\lambda+\rho|)^{-|\Phi_1^+|} \\
&\le C_k(H)(1+|H|)^k \frac{|\lambda+\rho|^k}{(s|\lambda+\rho|)^{|\Phi_1^+|}} \quad (\text{provided } s|\lambda+\rho| > 1) \\
&\le C_k(H) \frac{|\lambda+\rho|^k}{(s|\lambda+\rho|)^{\gamma}} \quad (\text{as } \gamma \le |\Phi_1^+|),
\end{aligned}$$

for all $k = 0, 1, 2, \dots,$ as desired.

Combining this with the previous estimate, we obtain the result.

2.3. We shall now extend our result to every semisimple Lie group $G$. The key is to prove that Fact 2 in §1.2 is still valid.

Let us write $\Phi$ as a disjoint union of irreducible root systems

$$\Phi = \Phi^{(1)} \cup \cdots \cup \Phi^{(n)},$$

and split $\Delta$ into

$$\Delta = \Delta^{(1)} \cup \cdots \cup \Delta^{(n)},$$

with $\Delta^{(m)} = \Delta \cap \Phi^{(m)}$ being a base of $\Phi^{(m)}$ for each $m \in \{1, \ldots, n\}$. The Lie algebra $\mathfrak{g}^{\mathbf{c}}$ is now a direct sum of simple ideals

$$\mathfrak{g}^{\mathbf{c}} = \mathfrak{g}^{(1)\mathbf{c}} \oplus \cdots \oplus \mathfrak{g}^{(n)\mathbf{c}}.$$

As before, we choose $j_0 \in I$ and remove $\alpha_{j_0}$ from $\Delta$ to obtain

$$\Delta_0 = \Delta \backslash \{\alpha_{j_0}\}.$$

But $\alpha_{j_0} \in \Delta^{(M)}$ for some $M \in \{1, \ldots, n\}$, and so

$$\Delta_0 = \Delta^{(1)} \cup \cdots \cup \Delta_0^{(M)} \cup \cdots \cup \Delta^{(n)},$$

with $\Delta_0^{(M)} = \Delta^{(M)} \backslash \{\alpha_{j_0}\}$. The Lie algebra $\mathfrak{g}_0$ (as in §1.2) then decomposes into

$$\mathfrak{g}_0^{\mathbf{c}} = \mathfrak{g}^{(1)\mathbf{c}} \oplus \cdots \oplus \mathfrak{g}_0^{(M)\mathbf{c}} \oplus \cdots \oplus \mathfrak{g}^{(n)\mathbf{c}},$$

where $\mathfrak{g}_0^{(M)\mathbf{c}}$ is the Lie subalgebra corresponding to $\Delta_0^{(M)}$. Now let $K$, $K_0$, $K^{(m)}$, and $K_0^{(M)}$ denote the Killing forms of $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{g}^{(m)}$, and $\mathfrak{g}_0^{(M)}$ respectively. Then, for each $m \in \{1, \ldots, n\}$, $m \neq M$, we have

$$K_0|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}} = K^{(m)} = K|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}};$$

while for $m = M$, the connection between $K^{(M)}$ and $K_0^{(M)}$ is explained in §1.2. We therefore find that Fact 2 still holds, and thus the extension is clear.

**3. An example: The sharpness of the estimate.** We shall here consider an example concerning the sharpness of the $L^p$-estimate.

Let $G = \mathbf{SU}(2)$, the Lie group consisting of $2 \times 2$ complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Its Lie algebra $\mathfrak{g}$ then contains all matrices of the form

$$\begin{pmatrix} ia & b \\ -\overline{b} & -ia \end{pmatrix}$$

with $a \in \mathbf{R}$, $b \in \mathbf{C}$. Here $\gamma = 1$ and the special element is

$$H_\rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In $\mathfrak{g}$, one may define the norm $|\cdot|$ by

$$\left| \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \right| = (a^2 + |b|^2)^{1/2} \quad \forall a \in \mathbf{R}, b \in \mathbf{C}.$$

For any $y \in G$, $X \in \mathfrak{g}$, one may observe that

$$X' = yXy^{-1} \in \mathfrak{g},$$

with $|X'| = |X|$. Conversely, for any $X$, $X' \in \mathfrak{g}$ with $|X| = |X'|$, one can find $y \in G$ such that $X' = yXy^{-1}$.

Denote by $B_0(\pi)$ the ball in $\mathfrak{g}$ which has centre 0 and radius $\pi$. It is then evident that the map $\exp: B_0(\pi) \to G$ is injective. Indeed, for each $x \in G$, there exists a unique $X \in B_0(\pi)$ for which $x = \exp(X)$. Diagonalizing such an $X$, one has

$$x = y \exp(\omega H_\rho) y^{-1}, \quad \text{where } \omega = |X|,$$

for some $y \in G$. It is seen here that $\operatorname{trace}(x) = 2\cos\omega$.

As suggested in **[6]**, let us consider the function $f: G \to \mathbf{R}^+$ given by

$$f(\exp(X)) = \begin{cases} \dfrac{|X|^{-2}}{\log|X|^{-1}}, & \text{if } 0 < |X| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

One may observe that $f \in L^p(G)$, whenever $1 \le p \le \frac{3}{2}$. On the other hand, regarding the maximal function $\mathscr{M}f = \mathscr{M}_{H_\rho} f$, we claim that $\mathscr{M}f(x) = \infty$ for all $x \in G$.

Before verifying our claim, we remark that

$$f(-\exp(X)) = f(\exp(X'))$$

where $|X'| = \pi - |X|$. Moreover, $f(yxy^{-1}) = f(x)$ for all $x, y \in G$. In fact, for all $x, y \in G$, we have

$$\begin{aligned} f(yxy^{-1}) &= f(y\exp(X)y^{-1}) \qquad (\text{for some } X \in \mathfrak{g}) \\ &= f(\exp(yXy^{-1})) \\ &= f(\exp(X')) \qquad (\text{where } |X'| = |X|) \\ &= f(\exp(X)) = f(x). \end{aligned}$$

Similarly, we observe that $\mathscr{M}f(yxy^{-1}) = \mathscr{M}f(x)$ for all $x, y \in G$. To be precise, for all $x, y \in G$, we have

$$\begin{aligned}\mathscr{M}f(yxy^{-1}) &= \sup_{s\in(0,\pi)} \int_G f(y \times y^{-1} g \exp(sH_\rho) g^{-1})\, dg \\ &= \sup_{s\in(0,\pi)} \int_G f(xy^{-1} g \exp(sH_\rho) g^{-1} y)\, dg \\ &= \sup_{s\in(0,\pi)} \int_G f(xg' \exp(sH_\rho) g'^{-1})\, dg' = \mathscr{M}f(x).\end{aligned}$$

We shall now verify our claim. First, for $x = \pm\mathbf{1}$, we have

$$\begin{aligned}\mathscr{M}f(\pm\mathbf{1}) &= \sup_{s\in(0,\pi)} \int_G f(\pm g \exp(sH_\rho) g^{-1})\, dg \\ &= \sup_{s\in(0,\pi)} \int_G f(\pm \exp(sH_\rho))\, dg = \sup_{s\in(0,\pi)} f(\pm \exp(sH_\rho)) \\ &= \sup_{s\in(0,\frac{1}{2})} \frac{s^{-2}}{\log s^{-1}} = \infty.\end{aligned}$$

Next, for $x \neq \pm\mathbf{1}$, we may assume that $x = \exp(\frac{t}{2}H_\rho)$ for some $0 < t < 2\pi$, and hence

$$\begin{aligned}\mathscr{M}f(x) &= \mathscr{M}f(\exp(\tfrac{t}{2}H_\rho)) \\ &= \sup_{s\in(0,\pi)} \int_G f(\exp(\tfrac{t}{2}H_\rho) g \exp(sH_\rho) g^{-1})\, dg \\ &\geq \int_G f(\exp(\tfrac{t}{2}H_\rho) g \exp(\tfrac{t}{2}H_\rho) g^{-1})\, dg.\end{aligned}$$

Writing each $g \in G$ as $g = h_\theta k_\phi h_{\theta'}$, where $h_\theta = \exp(\frac{\theta}{2}H_\rho)$ and $k_\phi$ is the matrix of rotation with angle $\frac{\phi}{2}$, we have (see [**9**, pp. 99–100])

$$\begin{aligned}\mathscr{M}f(x) &\geq \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^{\pi} f(h_t h_\theta k_\phi h_{\theta'} h_t h_{-\theta'} k_{-\phi} h_{-\theta}) \sin\phi\, d\phi\, d\theta\, d\theta' \\ &= \frac{1}{16\pi^2} \int_{-2\pi}^{2\pi} \int_0^{2\pi} \int_0^{\pi} f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin\phi\, d\phi\, d\theta\, d\theta' \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta}) \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(h_{-\theta} h_t h_\theta k_\phi h_t k_{-\phi}) \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(h_t k_\phi h_t k_{-\phi}) \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{2} \int_0^{\pi} f(h_t k_\phi h_t k_{-\phi}) \sin\phi\, d\phi.\end{aligned}$$

Let us now investigate the integrand. Multiplying out, we get

$$h_t k_\phi h_t k_{-\phi} = \begin{pmatrix} e^{it}\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} & \cos\frac{\phi}{2}\sin\frac{\phi}{2}(1-e^{it}) \\ -\cos\frac{\phi}{2}\sin\frac{\phi}{2}(1-e^{-it}) & e^{-it}\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} \end{pmatrix}.$$

As seen before, this matrix is similar to $\exp(\omega H_\rho)$, where

$$\omega = \cos^{-1}\left(\sin^2\tfrac{\phi}{2} + \cos^2\tfrac{\phi}{2}\cos t\right).$$

By observation (thanks to John Cornwall for making it easier), there exists a constant $C = C_t \in (0, 1)$ such that

$$\cos(\pi - \phi) \leq \sin^2\frac{\phi}{2} + \cos^2\frac{\phi}{2}\cos t \leq \cos C(\pi - \phi) \quad \forall \phi \in (\pi - \tfrac{1}{2}, \pi),$$

and accordingly

$$0 < C(\pi - \phi) \leq \omega \leq \pi - \phi < \tfrac{1}{2} \quad \forall \phi \in (\pi - \tfrac{1}{2}, \pi).$$

Hence we find that

$$f(h_t k_\phi h_t k_{-\phi}) = f(\exp(\omega H_\rho)) = \frac{\omega^{-2}}{\log \omega^{-1}} \geq \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}},$$

for all $\phi \in (\pi - \frac{1}{2}, \pi)$. It therefore follows that

$$\begin{aligned} \mathscr{M} f(x) &\geq \frac{1}{2}\int_{\pi-1/2}^{\pi} \frac{(\pi - \phi)^{-2}}{\log\{C(\pi - \phi)\}^{-1}} \sin\phi \, d\phi \\ &\geq \frac{1}{4}\int_{\pi-1/2}^{\pi} \frac{(\pi - \phi)^{-1}}{\log\{C(\pi - \phi)\}^{-1}} \, d\phi \\ &= \frac{1}{4}\int_0^{C/2} \frac{\varphi^{-1}}{\log \varphi^{-1}} \, d\varphi = \frac{1}{4}\int_{\log(2/C)}^{\infty} \frac{d\psi}{\psi} = \infty, \end{aligned}$$

as claimed.

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THE UNIVERSITY OF NEW SOUTH WALES
KENSINGTON, N.S.W. 2033
AUSTRALIA

*Correspondence address*: Jurusan Matematika
Institut Teknologi Bandung
Jalan Ganesha 10 Bandung
Indonesia