EQUIVALENCE RELATIONS OF n-NORMS ON A VECTOR SPACE

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Abstract. A vector space can be equipped with more than one n-norms. In such a case, an equivalence relation of n-norms is usually studied. Here we discuss and present some results on several equivalence relations of n-norms which may be defined on a vector space. In particular, our results correct the error that we found in [9]. We also discuss an equivalence relation of n-norms on finite dimensional spaces.

1. INTRODUCTION

Let X be a (real) vector space of dimension at least n. An *n-norm* on X is a mapping $\|\cdot,\dots,\cdot\| : X^n \to \mathbb{R}$ which satisfies the following four conditions:

(N1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;

(N2) $\|x_1,\dots,x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1,\dots,x_n\| = |\alpha|\,\|x_1,\dots,x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2,\dots,x_n\| \le \|x_1,x_2,\dots,x_n\| + \|x_1',x_2,\dots,x_n\|$.

The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an *n-normed space*. Note that in this space, we have $\|x_1 + y, x_2,\dots,x_n\| = \|x_1,x_2,\dots,x_n\|$ for any $y = \alpha_2 x_2 + \cdots + \alpha_n x_n$.

The theory of n-normed spaces was developed by S. Gähler [1, 2, 3, 4] in the 1960's. See also [7] for many properties of n-normed spaces.

To give some examples, one may check that for $1 \le p \le \infty$, the following function

$$\|x,y\|_p := \left[\frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\left|\,\begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}\,\right|^p\right]^{1/p},$$

where $x = (x_1,\dots,x_d)$ and $y = (y_1,\dots,y_d)$, defines a 2-norm on $\mathbb{R}^d$ ($d \ge 2$). In general, if X is a normed space, then, according to Gähler, the following function

$$\|x_1,\dots,x_n\| := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i=1,\dots,n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X. (Here X' denotes the dual of X, which consists of bounded linear functionals on X.) Moreover, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then the following function

$$\|x_1,\dots,x_n\|_2 := \begin{vmatrix} \langle x_1,x_1\rangle & \cdots & \langle x_1,x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n,x_1\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix}^{1/2}$$

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defines an n-norm on X. Here $\|x_1,\ldots,x_n\|_2$ represents the volume of the n-dimensional parallelepiped spanned by $x_1,\ldots,x_n$ in X.

As in [5], a vector space can be equipped with several n-norms. In such a case, we may have an equivalence relation between them. We found that several equivalence relations of n-norms may be defined on a given vector space. The aim of this paper is to study the relationship between these equivalence relations. In particular, our results correct the error made in [9]. An equivalence relation of n-norms on finite dimensional spaces will also be discussed.

2. MAIN RESULTS

From now on, let X be a vector space (of dimension at least n) unless otherwise stated, and suppose that $\|\cdot,\ldots,\cdot\|_1$ and $\|\cdot,\ldots,\cdot\|_2$ are two n-norms on X.

Definition 2.1 $\|\cdot,\ldots,\cdot\|_1$ and $\|\cdot,\ldots,\cdot\|_2$ are *equivalent of type 1* (in short, E1) if there are constants $A < B$ such that

$$A\,\|x_1,x_2,\ldots,x_n\|_1 \le \|x_1,x_2,\ldots,x_n\|_2 \le B\,\|x_1,x_2\ldots,x_n\|_1$$

for every $x_1,x_2,\ldots,x_n \in X$.

For example, on the space ℓ^p of p-summable sequences (of real numbers), the following two n-norms:

$$\|x_1,\ldots,x_n\|_p' := \sup_{\substack{z_i\in\ell^{p'},\ \|z_i\|_{p'}\le 1\\ i=1,\ldots,n}} \begin{vmatrix} \sum x_{1j}z_{1j} & \cdots & \sum x_{1j}z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj}z_{1j} & \cdots & \sum x_{nj}z_{nj} \end{vmatrix},$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and

$$\|x_1,\ldots,x_n\|_p := \left[\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\mathrm{abs}\begin{vmatrix} x_{1j_1} & \cdots & x_{nj_1} \\ \vdots & \ddots & \vdots \\ x_{1j_n} & \cdots & x_{nj_n} \end{vmatrix}^p\right]^{1/p},$$

are equivalent of type 1 (see [6, 8, 10]).

Recall that for $n=1$, two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are equivalent if there constants $A < B$ such that for every $x \in X$ we have

$$A\,\|x\|_1 \le \|x\|_2 \le B\,\|x\|_1.$$

Consequently, if $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, then the convergence of a sequence in the norm $\|\cdot\|_1$ implies the convergence in $\|\cdot\|_2$, and vice versa — that is, for any sequence $\{x(k)\}$ and a vector x in X, we have

$$\lim_{k\to\infty}\|x(k)-x\|_1 = 0 \iff \lim_{k\to\infty}\|x(k)-x\|_2 = 0.$$

We say that two norms are *sequentially equivalent* if the convergence in one norm implies the convergence in another norm. The statement in the previous paragraph says that if two norms are equivalent, then they must be sequentially equivalent. One may verify that the converse of this statement is also true: if two norms are sequentially equivalent, then they must be equivalent.

Now, in an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$, a sequence $\{x(k)\}$ is said to *converge* to x in X if

$$\lim_{k\to\infty} \|x(k) - x, x_2, \ldots, x_n\| = 0$$

for every $x_2, \ldots, x_n \in X$. With this notion of convergence of a sequence in an n-normed space, we have the following equivalence relation.

Definition 2.2 $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are *sequentially equivalent of type 1* (in short, SE1) if the convergence of a sequence in $\|\cdot, \ldots, \cdot\|_1$ implies the convergence in $\|\cdot, \ldots, \cdot\|_2$, and vice versa.

It is easy to see that E1 implies SE1. Recently, B.S. Reddy and H. Dutta "proved" the converse, that is, SE1 implies E1 (see [9], Theorem 1). Their proof, however, contains a flaw, which we shall indicate below.

Suppose that $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are not E1. Then, without loss of generality, we may assume that one of the following holds: there is no constant $A > 0$ such that

$$A\,\|x_1, \ldots, x_n\|_1 \le \|x_1, \ldots, x_n\|_2$$

for every $x_1, \ldots, x_n \in X$; or there is no constant $B > 0$ such that

$$\|x_1, \ldots, x_n\|_1 \le B\,\|x_1, \ldots, x_n\|_2$$

for every $x_1, \ldots, x_n \in X$. Suppose that the former holds. Then, Reddy and Dutta argued that, for every $k \in \mathbb{N}$, there exists $x(k) \in X$ such that

$$\frac{1}{k}\,\|x(k), x_2, \ldots, x_n\|_1 > \|x(k), x_2, \ldots, x_n\|. \qquad (1)$$

They then defined $y(k) := \frac{1}{\sqrt{k}}\,\frac{x(k)}{\|x(k),x_2,\ldots,x_n\|_2}$, $k \in \mathbb{N}$, and claimed that $\{y(k)\}$ converges to 0 in $\|\cdot, \ldots, \cdot\|_2$ but at the same time it does not converge in $\|\cdot, \ldots, \cdot\|_1$. From this, they concluded that the two n-norms are not SE1.

As we clearly see, they missed the fact that the vectors $x_2, \ldots, x_n$ that satisfy (1) must also depend on k. Therefore, their claim about the convergence of $\{y(k)\}$ is not valid, and so their conclusion about the two norms is not justified.

To fix the situation, we introduce two more equivalence relations.

Definition 2.3 $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are *equivalent of type 2* (in short, E2) if for every $x_2, \ldots, x_n \in X$, there are constants $A < B$ such that

$$A\,\|x_1, x_2, \ldots, x_n\|_1 \le \|x_1, x_2, \ldots, x_n\|_2 \le B\,\|x_1, x_2, \ldots, x_n\|_1$$

for every $x_1 \in X$ (especially for $x_1 \in X \setminus \mathrm{span}\,\{x_2, \ldots, x_n\}$).

Definition 2.4 $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are *sequentially equivalent of type 2* (in short, SE2) if for every $x_2, \ldots, x_n \in X$, we have

$$\lim_{k\to\infty} \|x(k), x_2, \ldots, x_n\|_1 = 0 \iff \lim_{k\to\infty} \|x(k), x_2, \ldots, x_n\|_2 = 0.$$

Note that SE2 *is* different from SE1. Two n-norms $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are SE1 means that

$$\lim_{k\to\infty} \|x(k), x_2, \ldots, x_n\|_1 = 0 \text{ for every } x_2, \ldots, x_n \in X$$

if and only if

$$\lim_{k\to\infty} \|x(k), x_2, \ldots, x_n\|_2 = 0 \text{ for every } x_2, \ldots, x_n \in X.$$

One can verify that SE2 implies SE1, but the converse is not guaranteed.

As for E2, we see that it is weaker than E1 (that is, E1 implies E2). Moreover, we have the following theorem.

Theorem 2.5 *$\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are E2 if and only if they are SE2.*

Proof. Suppose that $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are E2: for every $x_2, \ldots, x_n \in X$, there are constants $A < B$ such that

$$A\,\|x_1, x_2, \ldots, x_n\|_1 \le \|x_1, x_2, \ldots, x_n\|_2 \le B\,\|x_1, x_2, \ldots, x_n\|_1$$

for every $x_1 \in X \setminus \text{span}\,\{x_2, \ldots, x_n\}$. It follows that $\lim\limits_{k\to\infty} \|x(k) - x, x_2, \ldots, x_n\|_1 = 0$ if and only if $\lim\limits_{k\to\infty} \|x(k) - x, x_2, \ldots, x_n\|_2 = 0$, that is, $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are SE2.

Next, suppose that $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are not E2, that is, there are $x_2, \ldots, x_n$ in X such that one of the following holds: there is no constant $A > 0$ such that

$$A\,\|x_1, x_2 \ldots, x_n\|_1 \le \|x_1, x_2, \ldots, x_n\|_2$$

for every $x_1 \in X \setminus \text{span}\,\{x_2, \ldots, x_n\}$; or there is no constant $B > 0$ such that

$$\|x_1, x_2, \ldots, x_n\|_1 \le B\,\|x_1, x_2, \ldots, x_n\|_2$$

for every $x_1 \in X \setminus \text{span}\,\{x_2, \ldots, x_n\}$. Without loss of generality, suppose that the former holds. Then, for every $k \in \mathbb{N}$, we can find $x(k) \in X \setminus \text{span}\,\{x_2, \ldots, x_n\}$ such that

$$\frac{1}{k}\,\|x(k), x_2, \ldots, x_n\|_1 > \|x(k), x_2, \ldots, x_n\|_2.$$

Define $y(k) := \frac{1}{\sqrt{k}}\,\frac{x(k)}{\|x(k), x_2, \ldots, x_n\|_2}$, $k \in \mathbb{N}$. Then we have

$$\|y(k), x_2, \ldots, x_n\|_2 = \frac{1}{\sqrt{k}} \to 0, \quad \text{as } k \to \infty,$$

while

$$\|y(k), x_2, \ldots, x_n\|_1 = \frac{1}{\sqrt{k}}\,\frac{\|x(k), x_2, \ldots, x_n\|_1}{\|x(k), x_2, \ldots, x_n\|_2} > \sqrt{k} \to \infty, \quad \text{as } k \to \infty.$$

This tells us that $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are not SE2. □

Corollary 2.6 *The following relationships between the four equivalence relations hold:*

$$\begin{array}{ccc} E1 & \Longrightarrow & SE1 \\ \Downarrow & & \Uparrow \\ E2 & \Longleftrightarrow & SE2 \end{array}$$

The following is a result on finite dimensional vector spaces.

Theorem 2.7 *Let X be a vector space of finite dimension. If $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are n-norms on X, then they must be E2.*

Proof. Suppose that $n \leq \dim(X) = d < \infty$. For every linearly independent set $\{x_1, \ldots, x_{n-1}\} \subset X$, choose $\{x_n, \ldots, x_d\}$ such that $\{x_1, x_2, \ldots, x_d\}$ is a basis for X. Now consider the subspace $S := \mathrm{span}\,\{x_n, \ldots, x_d\}$. Then one may observe that the following functions

$$\|x\|_1 := \|x, x_1, \ldots, x_{n-1}\|_1, \quad x \in S,$$

and

$$\|x\|_2 := \|x, x_1, \ldots, x_{n-1}\|_2, \quad x \in S,$$

define norms on S. Since S is of finite dimension, $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent: there are constants $A < B$ such that

$$A\,\|x\|_1 \leq \|x\|_2 \leq B\,\|x\|_1$$

for every $x \in S$. Accordingly, we have

$$A\,\|x, x_1, \ldots, x_{n-1}\|_1 \leq \|x, x_1, \ldots, x_{n-1}\|_2 \leq B\,\|x, x_1, \ldots, x_{n-1}\|_1$$

for every $x \in S$.

Now, every $z \in X$ can be written as $z := x + y$ with $x \in S$ and $y = \alpha_1 x_1 + \cdots + \alpha_{n-1} x_{n-1}$, so that $\|z, x_1, \ldots, x_{n-1}\|_i = \|x, x_1, \ldots, x_{n-1}\|_i$ for $i = 1$ and 2. It thus follows from the previous inequalities that

$$A\,\|z, x_1, \ldots, x_{n-1}\|_1 \leq \|z, x_1, \ldots, x_{n-1}\|_2 \leq B\,\|z, x_1, \ldots, x_{n-1}\|_1$$

for every $z \in X$. This proves that $\|\cdot, \ldots, \cdot\|_1$ and $\|\cdot, \ldots, \cdot\|_2$ are E2. $\square$

3. CONCLUDING REMARKS

We have introduced some equivalence relations of n-norms on a vector space and discussed some relationships between them, as in Corollary 2.6. At the present, we do not know if the remaining relationships hold (for instance, whether SE1 implies E1 or not), nor we have counterexamples which show that they do not hold.

In the previous section, we have also shown that all n-norms on a finite dimensional vector space are equivalent of type 2. We conjecture that all n-norms on a finite dimensional space are equivalent of type 1, but we have not been able to prove it up to now. Nevertheless, if X is a two-dimensional space, say $X := \mathrm{span}\{e_1, e_2\}$, and $\|\cdot,\cdot\|_1$, $\|\cdot,\cdot\|_2$ are two n-norms on X, then one may verify that the two n-norms are equivalent. In fact, one can show that $\|x, y\|_2 = A\|x, y\|_1$ with $A = \frac{\|e_1, e_2\|_2}{\|e_1, e_2\|_1}$.

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