ON THE TRIANGLE INEQUALITY FOR THE STANDARD 2-NORM

Hendra Gunawan and Wono Setya-Budhi

Abstract. We shall show here that the triangle inequality for the standard 2-norm is equivalent to a generalized Cauchy-Schwarz inequality, as is that for norms to the Cauchy-Schwarz inequality. Besides its direct proof, we also present two alternative proofs through its equivalent inequality.

1. Introduction. Let X be a real-vector space, equipped with an inner-product $\langle\cdot,\cdot\rangle$ together with its induced norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. Define the function $\|\cdot,\cdot\| : X \times X \to \mathbf{R}$ by

$$\|x, y\| := \left\{\|x\|^2\|y\|^2 - \langle x, y\rangle^2\right\}^{\frac{1}{2}},$$

which is equal to twice the area of the triangle having vertices 0, x, and y (or the area of the parallelogram spanned by the vectors x and y) in X.

It can be shown that the above function defines a *2-norm* on X, which satisfies the following four properties:

(i) $\|x, y\| = 0$ iff x and y are linearly dependent;

(ii) $\|x, y\| = \|y, x\|$;

(iii) $\|x, ay\| = |a|\,\|x, y\|,\ a \in \mathbf{R}$;

(iv) $\|x, y + z\| \le \|x, y\| + \|x, z\|$.

The space X, being equipped with the 2-norm, is thus a *2-normed space.*

The concepts of 2-normed spaces (and 2-metric spaces) were intially introduced by Gahler [**G1**], [**G2**], [**G3**] in 1960's. A standard example of a 2-normed space is $\mathbf{R}^2$ equipped

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with the following 2-norm

$$\|x, y\| \; := \; \text{the area of the triangle having vertices } 0,\ x,\ \text{and } y.$$

The 2-norm above on X is a just generalization of this standard example. For recent results on 2-normed spaces, see for example [**GM**].

As it usually happens with norms, given a candidate for a 2-norm, the hardest part is to check the property (iv) — better known as the triangle inequality. In this note, we shall show that the triangle inequality for the 2-norm above is equivalent to a generalized Cauchy-Schwarz inequality, just as is that for norms to the Cauchy-Schwarz inequality. Besides its direct proof, we also present two alternative proofs through its equivalent inequality.

2. The Inequality. For our 2-norm above on X, we have the following fact:

FACT. The triangle inequality is equivalent to

$$\|x\|^2\langle y,z\rangle^2 + \|y\|^2\langle x,z\rangle^2 + \|z\|^2\langle x,y\rangle^2 \le \|x\|^2\|y\|^2\|z\|^2 + 2\langle x,y\rangle\langle x,z\rangle\langle y,z\rangle.$$

PROOF. Observe that

$$\begin{aligned}\|x, y+z\|^2 &= \|x\|^2\|y+z\|^2 - \langle x, y+z\rangle^2 \\ &= \|x\|^2\big(\|y\|^2 + 2\langle y,z\rangle + \|z\|^2\big) - \big(\langle x,y\rangle^2 + 2\langle x,y\rangle\langle x,z\rangle + \langle x,z\rangle^2\big) \\ &= \|x,y\|^2 + 2\big(\|x\|^2\langle y,z\rangle - \langle x,y\rangle\langle x,z\rangle\big) + \|x,z\|^2.\end{aligned}$$

The triangle inequality is thus equivalent to

$$\|x\|^2\langle y,z\rangle - \langle x,y\rangle\langle x,z\rangle \le \|x,y\|\,\|x,z\|.$$

Replacing $y+z$ by $y-z$, we find that the triangle inequality is equivalent to

$$\big|\|x\|^2\langle y,z\rangle - \langle x,y\rangle\langle x,z\rangle\big| \le \|x,y\|\,\|x,z\|.$$

Squaring both sides, we get

$$\begin{aligned}&\|x\|^4\langle y,z\rangle^2 - 2\|x\|^2\langle x,y\rangle\langle x,z\rangle\langle y,z\rangle + \langle x,y\rangle^2\langle x,z\rangle^2\\ &\qquad\le \|x\|^4\|y\|^2\|z\|^2 - \|x\|^2\big(\|y\|^2\langle x,z\rangle^2 + \|z\|^2\langle x,y\rangle^2\big) + \langle x,y\rangle^2\langle x,z\rangle^2.\end{aligned}$$

Canceling $\langle x,y\rangle^2\langle x,z\rangle^2$ and then dividing both sides by $\|x\|^2$ (assuming that $x \neq 0$), we obtain the desired inequality.

Note. It is clear from the proof that the equality $\|x\|^2\langle y,z\rangle^2 + \|y\|^2\langle x,z\rangle^2 + \|z\|^2\langle x,y\rangle^2 = \|x\|^2\|y\|^2\|z\|^2 + 2\langle x,y\rangle\langle x,z\rangle\langle y,z\rangle$ holds iff $\|x, y+z\| = \|x,y\| + \|x,z\|$ or $\|x, y-z\| = \|x,y\| + \|x,z\|$ holds.

Let us take a closer look at the inequality. First note that the equality holds when x or y or z equals 0. One may also observe that the equality holds when $x = \pm y$ or $x = \pm z$ or $y = \pm z$. Further, if $z \perp \operatorname{span}\{x,y\}$ and $z \neq 0$, then the inequality becomes

$$\langle x,y\rangle^2 \le \|x\|^2\|y\|^2,$$

which is the Cauchy-Schwarz inequality. Hence the inequality may be viewed as a generalized Cauchy-Schwarz inequality.

For $X = \mathbf{R}$, the equality obviously holds. For $X = \mathbf{R}^2$, the equality also holds. To see this, assume $x, y, z \neq 0$. Dividing both sides by $\|x\|^2\|y\|^2\|z\|^2$, we obtain

$$\frac{\langle x,y\rangle^2}{\|x\|^2\|y\|^2} + \frac{\langle x,z\rangle^2}{\|x\|^2\|z\|^2} + \frac{\langle y,z\rangle^2}{\|y\|^2\|z\|^2} \le 1 + 2\frac{\langle x,y\rangle}{\|x\|\|y\|}\frac{\langle x,z\rangle}{\|x\|\|z\|}\frac{\langle y,z\rangle}{\|y\|\|z\|}.$$

Next, assuming that $\|x\| = \|y\| = \|z\| = 1$, the inequality becomes

$$\langle x,y\rangle^2 + \langle x,z\rangle^2 + \langle y,z\rangle^2 \le 1 + 2\langle x,y\rangle\langle x,z\rangle\langle y,z\rangle.$$

Now put $\alpha = \angle(x,y)$, $\beta = \angle(x,z)$, and $\gamma = \angle(y,z)$. Then $\alpha + \beta + \gamma = 2\pi$, and we have the equality

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma.$$

Alternatively, assume $x \neq \pm y$, so that $\text{span}\{x, y\} = \mathbf{R}^2$. Writing $z = ax + by$ for some $a, b \in \mathbf{R}$, one may check that both sides are equal to

$$1 + 2ab\langle x, y\rangle + (1 + a^2 + b^2)\langle x, y\rangle^2.$$

In general, it can be shown that the equality holds iff $\text{span}\{x, y, z\}$ is at most two dimensional.

3. The Proof. We shall now prove the inequality. There are at least three ways to do it. First, if X is separable, then we can verify the triangle inequality for the 2-norm directly. Let (e_i) be an orthonormal basis for X (indexed by a countable set). Then, by Parseval's formula and polarization identity, we have

$$\begin{aligned}\|x, y\| &= \Big\{\Big(\sum_i \langle x, e_i\rangle^2\Big)\Big(\sum_j \langle y, e_j\rangle^2\Big) - \Big(\sum_i \langle x, e_i\rangle\langle y, e_i\rangle\Big)^2\Big\}^{\frac{1}{2}} \\ &= \Big\{\frac{1}{2}\sum_i\sum_j \big(\langle x, e_i\rangle\langle y, e_j\rangle - \langle x, e_j\rangle\langle y, e_i\rangle\big)^2\Big\}^{\frac{1}{2}}.\end{aligned}$$

The triangle inequality then follows easily.

Second, whether or not X is separable, we can always prove its equivalent inequality as follows. As argued earlier, under the assumption $\|x\| = \|y\| = \|z\| = 1$, we only need to show

$$\langle x, y\rangle^2 + \langle x, z\rangle^2 + \langle y, z\rangle^2 \leq 1 + 2\langle x, y\rangle\langle x, z\rangle\langle y, z\rangle.$$

Assuming $x \neq \pm y$ and z is not perpendicular to $\text{span}\{x, y\}$, we may write $z = z_1 + z_2$ where $z_1 \in \text{span}\{x, y\}$, that is $z_1 = ax + by$ for some $a, b \in \mathbf{R}$, and $z_2 \perp \text{span}\{x, y\}$. As in $\mathbf{R}^2$, we then have the equality

$$\langle x, y\rangle^2 + \langle x, n_1\rangle^2 + \langle y, n_1\rangle^2 = 1 + 2\langle x, y\rangle\langle x, n_1\rangle\langle y, n_1\rangle$$

where $n_1 = z_1/\|z_1\|$. Multiplying both sides by $\|z_1\|^2$, we get

$$\|z_1\|^2\langle x, y\rangle^2 + \langle x, z\rangle^2 + \langle y, z\rangle^2 = \|z_1\|^2 + 2\langle x, y\rangle\langle x, z\rangle\langle y, z\rangle,$$

since $\langle x, z_1\rangle = \langle x, z\rangle$ and $\langle y, z_1\rangle = \langle y, z\rangle$. Hence

$$\langle x, z\rangle^2 + \langle y, z\rangle^2 - 2\langle x, y\rangle\langle x, z\rangle\langle y, z\rangle = \|z_1\|^2\big(1 - \langle x, y\rangle^2\big) \le 1 - \langle x, y\rangle^2,$$

since $\|z_1\| \le \|z\| = 1$ and $1 - \langle x, y\rangle^2 \ge 0$; and the equality holds iff $\|z_1\| = 1$ (and consequently $z_2 = 0$), that is iff $z \in \operatorname{span}\{x, y\}$.

Third, one may observe that our inequality is actually equivalent to

$$\det(M) \ge 0$$

where M is the Gram matrix

$$M = \begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle & \langle x, z\rangle \\ \langle y, x\rangle & \langle y, y\rangle & \langle y, z\rangle \\ \langle z, x\rangle & \langle z, y\rangle & \langle z, z\rangle \end{pmatrix}.$$

Since M is positive semidefinite, the inequality follows immediately. It is also easy to see here that the equality holds iff span $\{x, y, z\}$ is at most two dimensional (see [**HJ**, pp. 407-408).

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Department of Mathematics
Bandung Institute of Technology
Jl. Ganesa 10 Bandung 40132
Indonesia
Email hgunawan@dns.math.itb.ac.id *wono@dns.math.itb.ac.id*