Fractional Integrals and Morrey Spaces

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Outline

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The talk will consist of two parts:

Part I: A very brief survey on Hardy-Littlewood maximal operator and fractional integral operators, especially about their boundedness properties on (generalized) Morrey spaces.

Part II: Some basic, but important, properties of (generalized) Morrey spaces, namely their inclusion properties.

PART I

Fractional Integral Operator

Let $0 < \alpha < d$. The operator I_α which maps every locally integrable function f on $\mathbb{R}^d$ to

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy$$

is known as the **Riesz potential** or the **fractional integral operator**.[1]

Note that $I_\alpha = \kappa(-\Delta)^{-\alpha/2}$, where $-\Delta$ is the **Laplacian** on $\mathbb{R}^d$.

For $d \geq 3$ and $\alpha = 2$, I_2 is the **Newton potential** and $u := \kappa^{-1} I_2 f$ is a solution of the **Poisson equation** $(-\Delta)u = f$.

[1] **M. Riesz** (1938), "Intégrales de Riemann-Liouville et potentiels", *Acta Sci. Math. (Szeged)* **9**

An Example: Characteristic Function

For $R > 0$, consider the **characteristic function** $\chi_{B(0,R)}$.
Then, for $|x| \le 2R$,

$$I_\alpha \chi_{B(0,R)}(x) \le C_1 R^\alpha$$

and, for $2^k R < |x| \le 2^{k+1} R$ $(k = 1, 2, 3, \dots)$,

$$I_\alpha \chi_{B(0,R)}(x) \le \frac{C_2 R^\alpha}{2^{k(d-\alpha)}}.$$

Observe that $\|\chi_{B(0,R)}\|_{L^p} \sim R^{d/p}$ for $1 \le p < \infty$.
Accordingly $I_\alpha \chi_{B(0,R)} \in L^q$ with

$$\|I_\alpha \chi_{B(0,R)}\|_{L^q} \le C\, R^{d/p}$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, $1 < p < \frac{d}{\alpha}$.

Hardy-Littlewood-Sobolev Inequality

Theorem 2.1

Let $1 < p < \frac{d}{\alpha}$. *If* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, *then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^q} \leq C\|f\|_{L^p}, \quad f \in L^p,$$

i.e. I_α *is bounded from* L^p *to* L^q.

This result is known as the **Hardy-Littlewood**[2]**-Sobolev**[3] **Inequality**

[2]**G.H. Hardy and J.E. Littlewood** (1927 & 1932), "Some properties of fractional integrals. I & II", *Math. Zeit.* **27** & **34**

[3]**S.L. Sobolev** (1938), "On a theorem in functional analysis" (Russian), *Mat. Sob.* **46**

First Page of Hardy & Littlewood's 1927 Paper

Some properties of fractional integrals. I.

Von

G. H. Hardy in Oxford und J. E. Littlewood in Cambridge.

1. Introduction.

1.1. In this memoir we present the first systematic treatment of certain theorems some of which we enunciated in a short note published in 1924 [1]). Our theme is the properties of the "Riemann-Liouville" integrals and derivatives of arbitrary order of functions of certain standard classes, in particular the "Lebesgue classes L^p", the "Lipschitz classes Lip. k", and the more general classes of functions which satisfy "integrated Lipschitz conditions". We shall give the formal definitions of these classes in a moment.

Our arguments in this memoir are entirely "real" and direct. Our results have many interesting applications to the theory of analytic functions and the theory of Fourier series; but we make no use of these theories here, our only weapons being elementary inequalities and the ordinary methods of the theory of functions of a real variable. It seems clear indeed that the "right" proofs of all the theorems which we prove here are of this "elementary" character. All our variables and functions are accordingly real.

1.2. We say (following F. Riesz [2])) that $f(x)$ belongs to the class L^p, where $p \geqq 1$, in a finite interval (a, b), if $f(x)$ and $|f(x)|^p$ are integrable in (a, b) in the sense of Lebesgue. The class L^1 or L is the class of integrable functions.

We say that $f(x)$ belongs to the class Lip. k, where $0 \leqq k \leqq 1$, in (a, b) if

$$f(x) - f(x-h) = O(h^k)$$

[1]) Hardy and Littlewood, **4**.
[2]) F. Riesz, **11**.

Hardy-Littlewood Maximal Operator

One way to prove the H-L-S inequality is by using the boundedness of the **maximal operator** $\mathcal{M}$, given by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

This operator was introduced by Hardy & Littlewood in 1930.[4]

[4] **G.H. Hardy and J.E. Littlewood** (1930), "A maximal theorem with function-theoretic applications", *Acta Math.* **54**

Maximal Inequality

Theorem 2.2

For $1 < p \leq \infty$, we have

$$\|\mathcal{M}f\|_{L^p} \leq C\,\|f\|_{L^p}, \quad f \in L^p.$$

Note: Hardy & Littlewood proved it for $d = 1$; while Marcinkiewicz & Zygmund[5] and Wiener[6] proved the general result for $d \geq 1$.

[5] **J. Marcinkiewicz and A. Zygmund** (1939), "On the summability of double Fourier series", *Fund. Math.* **32**

[6] **N. Wiener** (1939), "The ergodic theorem", *Duke Math. J.* **5**

Proof of H-L-S Inequality via Hedberg's Inequality

For every $R > 0$, we can split the integral into two parts:

$$I_\alpha f(x) = \int_{|x-y|\le R} \frac{f(y)}{|x-y|^{d-\alpha}} dy + \int_{|x-y|>R} \frac{f(y)}{|x-y|^{d-\alpha}} dy = I_1 + I_2.$$

Then, by using **dyadic decomposition**, we obtain these estimates:

$$|I_1| \le C_1 R^\alpha \mathcal{M}f(x) \quad \text{and} \quad |I_2| \le C_2 R^{\alpha - d/p} \|f\|_{L^p}.$$

Now choose $R > 0$ s.t. $\mathcal{M}f(x) = R^{-d/p}\|f\|_{L^p}$. Then we get **Hedberg inequality**[7]:

$$|I_\alpha f(x)| \le C \mathcal{M}f(x)^{p/q} \|f\|_{L^p}^{1-p/q}.$$

The H-L-S inequality follows immediately. □

[7] **L.I. Hedberg** (1972), "On certain convolution inequalities", *Proc. Amer. Math. Soc.* **36**

Morrey Spaces

The inequalities for the maximal operator and the fractional integral operator are extendable to **Morrey spaces**.[8]

For $1 \le p \le q < \infty$ and $0 \le \lambda \le d$, the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^d)$ is the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^d)$ for which

$$\|f\|_{L^{p,\lambda}} := \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(y)|^p \, dy \right)^{\frac{1}{p}} < \infty$$

where $B(a,r)$ denotes the open ball in $\mathbb{R}^d$ centered at $a \in \mathbb{R}^d$ with radius $r > 0$, and $|B(a,r)|$ denotes its Lebesgue measure.

Note that $L^{p,0} = L^p$ and $L^{p,d} = L^\infty$.

[8] **C.B. Morrey** (1940), "Functions of several variables and absolute continuity", *Duke Math. J.* **6**

Spanne's & Adam's Results

In the 1960's, S. Spanne[9] proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < d/\alpha,\ 1/q = 1/p - \alpha/d,\ 0 \le \lambda < d - \alpha p$.

A stronger result was obtained by D. R. Adams[10]:

Theorem 3.1

For $1 < p < \frac{d}{\alpha}$ *and* $0 \le \lambda < d - \alpha p$, *we have*

$$\|I_\alpha f\|_{L^{q,\lambda}} \le C\,\|f\|_{L^{p,\lambda}}, \quad f \in L^{p,\lambda},$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d-\lambda}$.

[9] **J. Peetre** (1969), "On the theory of $\mathcal{L}_{p,\lambda}$ spaces", *J. Funct. Anal.* **4**

[10] **D. Adams** (1975), "A note on Riesz potentials", *Duke Math. J.* **42**

Chiarenza & Frasca's Results

In 1987, F. Chiarenza & M. Frasca[11] proved the boundedness of $\mathcal{M}$ on $L^{p,\lambda}$ (using **Feffermann-Stein inequality**[12]):

Theorem 3.2

For $1 < p < \infty$ *and* $0 \leq \lambda \leq d$, *we have*

$$\|\mathcal{M}f\|_{L^{p,\lambda}} \leq C\,\|f\|_{L^{p,\lambda}}, \quad f \in L^{p,\lambda}.$$

Using this theorem, they reproved Adam's result via a Hedberg type inequality.

[11] **F. Chiarenza and M. Frasca** (1987), "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7**

[12] **C. Feffermann and E.M. Stein** (1971), "Some maximal inequalities", *Amer. J. Math.* **93**

In 1994, E. Nakai[13] introduced some generalization of Morrey spaces and proved the boundedness of Hardy-Littlewood maximal operator $\mathcal{M}$ and the fractional integral operator I_α, which may be viewed as a generalization of Spanne's result.

The generalized Morrey spaces that we shall discuss now is a version which is widely used in the last ten years or so.

[13] **E. Nakai** (1994), "Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces", *Math. Nachr.* **166**

Generalized Morrey Spaces

Let $1 \le p < \infty$, $\phi : (0,\infty) \to (0,\infty)$ be nonincreasing and $t \mapsto \phi(t)^p t^d$ be nondecreasing. Consequently, ϕ satisfies the **doubling condition**: there exists $C > 0$ s.t.

$$\frac{1}{C} \le \frac{\phi(s)}{\phi(t)} \le C \quad \text{for } \frac{1}{2} \le \frac{s}{t} \le 2.$$

The **generalized Morrey space** $M^p_\phi = M^p_\phi(\mathbb{R}^d)$ is the space of all functions $f \in L^p_{\text{loc}}$ for which

$$\|f\|_{M^p_\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(x)|^p dx \right)^{1/p} < \infty.$$

Note that for $\phi(t) = t^{(\lambda-d)/p}$, $0 \le \lambda \le d$, we have $M^p_\phi = L^{p,\lambda}$.

Nakai's Result

Theorem 4.1

For $1 < p < \infty$, we have

$$\|\mathcal{M}f\|_{M^p_\phi} \leq C\,\|f\|_{M^p_\phi}, \quad f \in M^p_\phi.$$

Note: Nakai didn't use this result to prove the boundedness of I_α on Morrey spaces via a Hedberg type inequality. If he did, he would have obtained a generalization of Adam's result instead of Spanne's, as demonstrated by G. and Eridani.[14]

[14] **G. and Eridani** (2009), "Fractional integrals and generalized Olsen inequalities", *Kyungpook Math. J.* **49**

Generalized FIO

In 2001, E. Nakai[15] studied the **generalized FIO**:

$$I_\rho f(x) := \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where $\rho : (0,\infty) \to (0,\infty)$ satisfies the doubling condition.

Note: If $\rho(t) = t^\alpha,\ 0 < \alpha < d$, then $I_\rho = I_\alpha$ (the classical FIO).

Nakai and also Eridani[16] proved the boundedness of I_ρ from M^p_ϕ to M^q_ψ, assuming some conditions on ρ, ϕ and ψ.

[15] **E. Nakai** (2001), "On generalized fractional integrals", *Taiwanese J. Math.* **5**

[16] **Eridani** (2002), "On the boundedness of a generalized fractional integral on generalized Morrey spaces", *Tamkang J. Math.* **33**

The Bdd'ness of I_ρ on Generalized Morrey Spaces[17]

Theorem 4.2

Let $1 < p < q < \infty$. *Suppose that* ϕ *is surjective and* $\int_r^\infty \frac{\phi(t)^p}{t} dt \le C\phi(r)^p$ *for every* $r > 0$. *If* ρ *satisfies*

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le C\phi(r)^{p/q} =: C\psi(r)$$

for every $r > 0$, *then*

$$\|I_\rho f\|_{M_\psi^q} \le C\|f\|_{M_\phi^p}, \quad f \in M_\phi^p.$$

[17] **G.** (2003), "A note on the generalized fractional integral operators", *J. Indones. Math. Soc.* **9**

An Example

Let $1 < p < q < \infty$, $\alpha := \frac{d}{p} - \frac{d}{q}$, $\beta > 0$, and

$$\rho(r) := r^{\alpha} \ell(r)^{\beta}$$

where $\ell(r) = -1/\log r$ for small $r > 0$ and $\ell(r) = \log r$ for large r s.t. ρ satisfies the doubing condition.

Let $\phi(r) := r^{-d/p} \ell(r)^{\beta q/(p-q)}$.

Then $\phi(r)^{(p-q)/q} = \rho(r)$, and ρ and ϕ satisfy the hypotheses of the previous theorem.

Accordingly, I_{ρ} is bounded from M_{ϕ}^{p} to M_{ψ}^{q} with $\psi := \phi^{p/q}$.

Various Works on FIO

In the 2000's, there are many works on FIO; some of which are:

E. Nakai (2001), "On generalized fractional integrals in the Orlicz spaces on spaces of homogeneous type", *Sci. Math. Jpn.* **54**
J. Garcia-Cuerva and J.M. Martell (2001), "Two weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces", *Indiana Univ. Math. J.* **50**
J. Garcia-Cuerva and E. Gatto (2004), "Boundedness properties of fractional integral operators associated to non-doubling measures", *Studia Math.* **162**
Y. Sawano; T. Sobukawa and H. Tanaka (2006), "Limiting case of the boundedness of fractional integral operators on non-homogeneous space", *J. Inequal. Appl.* **2006**
G.; Y. Sawano and I. Sihwaningrum (2009), "Fractional integral operators in nonhomogeneous spaces", *Bull. Austral. Math. Soc.* **80**

Necessary & Sufficient Conditions

In 2014[18], we obtained that the previous condition on the function ρ is not only necessary but also sufficient for the boundedness of I_ρ on generalized Morrey spaces.

Theorem 4.3

Let $1 < p < q < \infty$. Suppose that $\int_0^r \frac{\phi(t)t^{d/p}}{t}dt \le C\phi(r)r^{d/p}$ for every $r > 0$. Then I_ρ is bounded from M_ϕ^p to M_ψ^q with $\psi = \phi^{p/q}$ iff

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q}$$

for every $r > 0$.

[18] **Eridani; G.; E. Nakai; and Y. Sawano** (2014), "Characterizations for the generalized fractional integral operators on Morrey spaces", *Math. Ineq. Appl.* **17**

Weak Type Inequalities for FIO[19]

Theorem 5.1

Let $1 \leq p < q < \infty$. *Suppose that* $\inf_{r>0} \phi(r) = 0$, $\sup_{r>0} \phi(r) = \infty$ *and* $\int_r^\infty \frac{\phi(t)^p}{t} dt \leq C\phi(r)^p$. *If*

$$r^\alpha \phi(r) + \int_r^\infty t^{\alpha-1}\phi(t)dt \leq C\phi(r)^{p/q}$$

for every $r > 0$, *then for* $f \in M_\phi^p$ *we have*

$$\Big|\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\Big| \leq C\, r^d \phi(r)^p \left(\frac{\|f\|_{M_\phi^p}}{\gamma}\right)^q,$$

for every $B(a,r)$ *and* $\gamma > 0$.

[19] **D.I. Hakim and G.** (2013), "Weak (p,q) inequalities for fractional integral operators on generalized Morrey spaces", *Math. Aeterna* **3**

Weak Type Inequalities for Generalized FIO[20]

Theorem 5.2

Let $1 \leq p < q < \infty$. Suppose that ϕ satisfies

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \leq C\,\phi(r)^{p/q}$$

for every $r > 0$. Then, for any function $f \in M^p_\phi$, we have

$$\Big|\{x \in B(a,r) : |I_\rho f(x)| > \gamma\}\Big| \leq C\,r^d\phi(r)^p\left(\frac{\|f\|_{M^p_\phi}}{\gamma}\right)^q,$$

for every ball $B(a,r) \subseteq \mathbb{R}^d$ and $\gamma > 0$.

[20] **G.; D.I. Hakim; Y. Sawano and I. Sihwaningrum** (2013), "Weak type inequalities for some integral operators on generalized nonhomogeneous Morrey spaces", *J. Funct. Spaces Appl.* **2013**

Related Works

M. Idris; G. and Eridani (2016), "The boundedness of Bessel-Riesz operators on generalized Morrey spaces", *AJMAA* **13**

M. Idris; G., J. Lindiarni and Eridani (2016), "The boundedness of Bessel-Riesz operators on Morrey spaces", *AIP Conf. Proc.* **1729**

PART II

M_q^p and M_ϕ^p Notations for Morrey Spaces

Morrey spaces seem to be the right ambient for integral operators, especially the maximal operator and fractional integral operators.

One notable difference with Lebesgue spaces is that they contain power functions $|x|^{-a}$ for $0 < a < d$.

Morover, Morrey spaces satisfy the inclusion properties – which we shall discuss now.

For this purpose, we shall use another notation for Morrey spaces:

$$f \in M_q^p \text{ iff } \sup_B |B|^{1/q-1/p} \left(\int_B |f|^p dy \right)^{1/p} < \infty.$$

Here $M_q^p = L^{p,d(1-p/q)}$. Note also that $M_q^p = M_\phi^p$ for $\phi(r) = r^{-d/q}$.

Bdd'ness of FIO on Morrey Spaces Reformulated

Theorem 6.1

Let $1 < p \le q < \dfrac{d}{\alpha}$ *and* $1 < s \le t < \infty$. *If*

$$\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{d} \quad \text{and} \quad \frac{p}{q} = \frac{s}{t},$$

then there exists a constant $C > 0$ *such that*

$$\|I_\alpha f\|_{M_t^s} \le C\|f\|_{M_q^p}, \quad f \in M_q^p,$$

i.e. I_α *is bounded from* M_q^p *to* M_t^s.

Inclusion Property of Morrey Spaces

In the proof of the boundedness of I_α, we use the fact that $M_q^p \subseteq M_q^1$, which is a special case of the following inclusion property:[21]

Theorem 6.2

For $1 \le p_1 \le p_2 \le q < \infty$, the following inclusion holds:

$$L^q = M_q^q \subseteq M_q^{p_2} \subseteq M_q^{p_1} \subseteq M_q^1.$$

For $1 \le p < q$, the inclusion $M_q^q \subseteq M_q^p$ is proper. If $f_q(x) := |x|^{-\frac{d}{q}}$, then $f_q \in M_q^p \setminus M_q^q$. ($\|f_q\|_{M_q^p} = c_q\left(\frac{1}{p} - \frac{1}{q}\right)^{-1/p}$.)
For $1 \le p_1 < p_2 < q$, the inclusion $M_q^{p_2} \subseteq M_q^{p_1}$ is also proper. (It is a good exercise to find the example.)

[21] **Y. Sawano and H. Tanaka** (2005), "Morrey spaces for non-doubling measures", *Acta Math. Sinica* **21**

Weak Morrey Spaces

The boundedness of fractional integral operators I_α from Morrey spaces M^p_q to M^s_t only holds for $1 < p \le q < \frac{d}{\alpha}$ (and suitable s and t). For $p = 1$, nevertheless, we have weak type inequalities.

Let $1 \le p \le q < \infty$. The **weak Morrey space** $wM^p_q = wM^p_q(\mathbb{R}^d)$ is the set of all measurable functions f for which

$$\begin{aligned}\|f\|_{wM^p_q} &:= \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\, \gamma \Big|\{x \in B(a,r) \colon |f(x)| > \gamma\}\Big|^{\frac{1}{p}} \\ &= \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B(a,r))} < \infty.\end{aligned}$$

Note that $\|\cdot\|_{wM^p_q}$ forms a **quasi-norm** in wM^p_q.
If $p = q$, then $\|\cdot\|_{wM^p_q} = \|\cdot\|_{wL^p}$.

The Weak Contains the Strong

For each $p \leq q$, the weak Morrey space wM_q^p contains the Morrey space M_q^p, as stated in the following proposition.

Theorem 6.3

Let $1 \leq p \leq q < \infty$. *Then,*

$$M_q^p \subset wM_q^p,$$

with

$$\|f\|_{wM_q^p} \leq \|f\|_{M_q^p}, \quad f \in M_q^p.$$

Inclusion Property of Weak Morrey Spaces [22]

Theorem 6.4

If $1 \le p_1 \le p_2 \le q < \infty$, *then*

$$wM_q^{p_2} \subset wM_q^{p_1}$$

with

$$\|f\|_{wM_q^{p_1}} \le \|f\|_{wM_q^{p_2}}, \quad f \in wM_q^{p_2}.$$

[22] **G.; D.I. Hakim; K.M. Limanta; and A.A. Masta** (2016), "Inclusion properties of generalized Morrey spaces", *Math. Nachr.*

A Sufficient Condition

A sufficient condition for the inclusion property of generalized Morrey spaces is discussed by Sihwaningrum[23]:

Theorem 7.1

Let $1 \le p_1 \le p_2 < \infty$. *If* $\phi_2 \le C\,\phi_1$, *then*

$$M^{p_2}_{\phi_2} \subseteq M^{p_1}_{\phi_1}$$

with

$$\|f\|_{M^{p_1}_{\phi_1}} \le C\,\|f\|_{M^{p_2}_{\phi_2}}, \quad f \in M^{p_2}_{\phi_2}.$$

[23] **I. Sihwaningrum** (2010), "Operator integral fraksional dan ruang Morrey tak homogen yang diperumum" (Indonesia), Doctoral Thesis (ITB)

A Necessary Condition[24]

Theorem 7.2

Let $1 \le p_1 \le p_2 < \infty$. *If*

$$M^{p_2}_{\phi_2} \subseteq M^{p_1}_{\phi_1}$$

with

$$\|f\|_{M^{p_1}_{\phi_1}} \le C\,\|f\|_{M^{p_2}_{\phi_2}}, \quad f \in M^{p_2}_{\phi_2},$$

then $\phi_2 \le C\,\phi_1$.

[24] **G.; D.I. Hakim; K.M. Limanta and A.A. Masta** (2016), "Inclusion properties of generalized Morrey spaces", *Math. Nachr.*

The Key Estimate

The proof of the necessary condition uses the estimate

$$\frac{1}{\phi(r)} \le \|\chi_{B(0,r)}\|_{M^p_\phi} \le \frac{C}{\phi(r)}$$

which holds for every $r > 0$.

Necessary & Sufficient Condition Put Together

Theorem 7.3

Let $1 \le p_1 \le p_2 < \infty$, ϕ_1 *and* ϕ_2 *satisfy the required condition (for the definition of generalized Morrey spaces). Then, the following statements are equivalent:*

(a) $\phi_2 \le C\phi_1$.

(b) $M^{p_2}_{\phi_2} \subset M^{p_1}_{\phi_1}$ *with*

$$\|f\|_{M^{p_1}_{\phi_1}} \le C\|f\|_{M^{p_2}_{\phi_2}},$$

for every $f \in M^{p_2}_{\phi_2}$.

Generalized Weak Morrey Spaces

For $1 \le p < \infty$, the **generalized weak Morrey space** $wM^p_\phi = wM^p_\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{wM^p_\phi} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} \frac{\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{1/p}} < \infty.$$

The relation between the generalized Morrey spaces and their weak type is given in the following proposition.

Theorem 7.4

For $1 \le p < \infty$, *we have*

$$M_\phi^p \subseteq wM_\phi^p$$

with

$$\|f\|_{wM_\phi^p} \le \|f\|_{M_\phi^p}, \quad f \in M_\phi^p.$$

The Estimate for the Characteristic Functions

Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Then there exists $C > 1$ such that

$$\frac{1}{\phi(r_0)} \le \|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \le \frac{C}{\phi(r_0)},$$

for every ball $B_0 := B(0, r_0)$.

Inclusion Property of Generalized Weak Morrey spaces[25]

Theorem 7.5

Let $1 \leq p_1 \leq p_2 < \infty$, ϕ_1 *and* ϕ_2 *satisfy the required condition. Then, the following statements are equivalent:*

(a) $\phi_2 \leq C\phi_1$.

(b) $wM^{p_2}_{\phi_2} \subset wM^{p_1}_{\phi_1}$ *with*

$$\|f\|_{wM^{p_1}_{\phi_1}} \leq C\|f\|_{wM^{p_2}_{\phi_2}},$$

for every $f \in wM^{p_2}_{\phi_2}$.

[25] **G.; D.I. Hakim; K.M. Limanta and A.A. Masta** (2016), "Inclusion properties of generalized Morrey spaces", *Math. Nachr.*

A Question

So far we have the following inclusion relations

$$\begin{array}{ccc} M^{p_2}_{\phi_2} & \rightarrow & M^{p_1}_{\phi_1} \\ \downarrow & \searrow & \downarrow \\ wM^{p_2}_{\phi_2} & \rightarrow & wM^{p_1}_{\phi_1} \end{array}$$

for $1 \le p_1 \le p_2 < \infty$ and $\phi_2 \le C\phi_1$, where the arrows mean 'contained in'.

One question remains: what is the relation between $wM^{p_2}_{\phi_2}$ and $M^{p_1}_{\phi_1}$ for $1 \le p_1 \le p_2 < \infty$ and $\phi_2 \le C\phi_1$?

Completing the Inclusion Relation

Theorem 7.6

Let $1 \leq p_1 < p_2 < \infty$, ϕ_1 *and* ϕ_2 *satisfy the required condition. If* $\phi_2 \leq C\phi_1$, *then* $w\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$ *with*

$$\|f\|_{M^{p_1}_{\phi_1}} \leq C \left(\frac{p_1}{p_2 - p_1} \right)^{1/p_2} \|f\|_{wM^{p_2}_{\phi_2}}, \quad f \in wM^{p_2}_{\phi_2}. \tag{1}$$

Conversely, if $wM^{p_2}_{\phi_2} \subseteq M^{p_1}_{\phi_1}$, *then it is necessary that* $\phi_2 \leq C\phi_1$.

Related Works

A.A. Masta, G. and W. Setya-Budhi (2016), "Inclusion properties of Orlicz and weak Orlicz spaces", submitted

A.A. Masta, G. and W. Setya-Budhi (2016), "An inclusion property of Orlicz-Morrey spaces", presented at AMC 2016

G. and E. Kikianty (2016), "Discrete Morrey spaces and their inclusion properties", preprint

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To the audience, thank you very much for your attention! I hope some of you get something from my talk.

TERIMA KASIH ...