Boundedness Properties of Bessel-Riesz Operators on Morrey Spaces

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Abstract

In this talk, we discuss the boundedness properties of Bessel-Riesz operators on (generalized) Morrey spaces.

We show that the norm of such an operator is dominated by the norm of its kernel in an associated Morrey space.

The proof uses the usual dyadic decomposition, a Hedberg-type inequality, and the boundedness of Hardy-Littlewood maximal operator on Morrey spaces.

Previous Works, Among Others . . .

1975: D. Adams, "A note on Riesz potentials", *Duke Math. J.* **42**

1987: F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7**

1994: E. Nakai, "Hardy–Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces", *Math. Nachr.* **166**

1999: K. Kurata, S. Nishigaki, and S. Sugano, "Boundedness of integral operator on generalized Morrey space and its application to Schrödinger operator", *Proc. Amer. Math. Soc.* **128**

2009: H. Gunawan and Eridani, "Fractional integrals and generalized Olsen inequalities", *Kyungpook Math. J.* **1**

2016: M. Idris, H. Gunawan, J. Lindiarni, and Eridani, "The boundedness of Bessel-Riesz operators on Morrey spaces", *AIP Conference Proceedings* **1729**

Bessel-Riesz Operators

For $\gamma \geq 0$ and $0 < \alpha < n$, define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y) f(y)\, dy,$$

for every $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n),\ p \geq 1$, where

$$K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^{\gamma}}, \quad x \in \mathbb{R}^n.$$

We call $I_{\alpha,\gamma}$ a *Bessel-Riesz operator*, and $K_{\alpha,\gamma}$ a *Bessel-Riesz kernel*.

Hardy-Littlewood-Sobolev Inequality for I_α

For $\gamma = 0$, $I_{\alpha,0} =: I_\alpha$ is known as the *Riesz potential* or *fractional integral operator*.

For $0 < \alpha < n$, the fractional integral operator I_α is **bounded** from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, i.e. there exists a constant $C = C_{p,q} > 0$ such that

$$\|I_\alpha f\|_{L^q} \leq C \, \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),$$

provided that $1 < p < \frac{\alpha}{n}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

This result is due to Hardy & Littlewood and Sobolev, and the above inequality is known as *Hardy-Littlewood-Sobolev inequality*.

Morrey Spaces

For $1 \leq p \leq q$, the (classical) *Morrey space* $M_q^p := M_q^p(\mathbb{R}^n)$ is defined as the set of all locally integrable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{M_q^p} := \sup_{a\in\mathbb{R}^n,\, r>0} |B(a,r)|^{1/q}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(y)|^p dy\right)^{1/p} < \infty.$$

Here $B(a,r)$ denotes the ball in $\mathbb{R}^n$ centered at a with radius r, and $|B(a,r)| = c \cdot r^n$ denotes its volume.

Note: If $p = q$, then $M_q^p = L^q$.

Moreover, M_q^p equipped with $\|\cdot\|_{M_q^p}$ is a Banach space.

Inclusion Property of Morrey Spaces

For $1 \leq p_1 \leq p_2 \leq q < \infty$, the following inclusions hold:

$$L^q = M_q^q \subseteq M_q^{p_2} \subseteq M_q^{p_1} \subseteq M_q^1.$$

Moreover, if $1 < p_1 < p_2 < q$, then the inclusions are strict.

The Boundedness of FIO on Morrey Spaces

For $0 < \alpha < n$, the fractional integral operator I_α is **bounded** from M_q^p to M_t^s, i.e. there exists $C = C_{p,q,s,t}$ such that

$$\|I_\alpha f\|_{M_t^s} \leq C\,\|f\|_{M_q^p},$$

provided that $1 < p \leq q < \frac{n}{\alpha}$, $1 < s \leq t < \infty$, $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}$, and $\frac{p}{q} = \frac{s}{t}$.

This result is due to D. Adams (1975) and F. Chiarenza & M. Frasca (1987).

Generalized Morrey Spaces

For $1 \le p < \infty$ and a certain function $\phi : (0,\infty) \to (0,\infty)$, the *generalized Morrey space* $M^p_\phi = M^p_\phi(\mathbb{R}^d)$ consists of f for which

$$\|f\|_{M^p_\phi} := \sup_{a\in\mathbb{R}^n,\, r>0} \frac{1}{\phi(r)} \Big(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx \Big)^{1/p} < \infty.$$

Here ϕ is nonincreasing and $r \mapsto \phi(r)^p r^n$ is nondecreasing on $(0,\infty)$. Consequently, ϕ satisfies the *doubling condition*: $\exists C > 0$ s.t.

$$\frac{1}{C} \le \frac{\phi(r)}{\phi(u)} \le C \text{ whenever } \frac{1}{2} \le \frac{r}{u} \le 2.$$

Note that if $\phi(r) := r^{-n/q}$ for some $q > p$, then $M^p_\phi = M^p_q$ — the classical Morrey spaces.

The Boundedness of FIO on Generalized Morrey Spaces

For $0 < \alpha < n$, the fractional integral operator I_α is **bounded** from M_ϕ^p to M_ψ^q, i.e. there exists $C = C_{p,q,\phi,\psi} > 0$ such that

$$\|I_\alpha f\|_{M_\phi^p} \leq C\,\|f\|_{M_\psi^q}, \quad f \in M_\phi^p(\mathbb{R}^n),$$

provided that $1 < p < \frac{n}{\alpha}$, $\int_r^\infty \frac{\phi(u)}{u}\,du \leq C\,\phi(r)$, and $\phi(r) \leq C\,r^\beta$ for every $r > 0$, $q = \frac{\beta p}{\alpha+\beta}$, and $\psi(r) = \phi(r)^{\frac{p}{q}}$ for every $r > 0$.

This result can be found in G. & Eridani (2009).

The proof of the previous inequality for I_α uses a *Hedberg-type inequality* and the boundedness of *Hardy-Littlewood maximal operator* $\mathcal{M}$ on generalized Morrey spaces.

Theorem 3.1

(Nakai) *Let* $\mathcal{M}$ *be given by* $\mathcal{M}f(x) := \sup\limits_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy$. *Then, for* $1 < p \leq \infty$ *, we have*

$$\|\mathcal{M}f\|_{M^p_\phi} \leq C\, \|f\|_{M^p_\phi}, \quad f \in M^p_\phi(\mathbb{R}^n).$$

Properties of the Kernel

Hereafter, let $0 < \alpha < n$ and $\gamma > 0$. Then

$$K_{\alpha,\gamma} \in L^t \text{ for } \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha},$$

with

$$\|K_{\alpha,\gamma}\|_{L^t} \sim \left(\sum_{k\in\mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \right)^{\frac{1}{t}}$$

where $R > 0$ is fixed but arbitrary.

Further, $K_{\alpha,\gamma} \in M_t^s$ for $1 \le s \le t$, and there is a similar estimate for $\|K_{\alpha,\gamma}\|_{M_t^s}$.

By Young's inequality, we have

$$\|I_{\alpha,\gamma}f\|_{L^q} \leq \|K_{\alpha,\gamma}\|_{L^t}\|f\|_{L^p}$$

for $1 \leq p < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1$.

Hence $\|I_{\alpha,\gamma}\|_{L^p \to L^q} \leq \|K_{\alpha,\gamma}\|_{L^t}$ for those p, q, and t.

On Morrey spaces, however, we do not have Young's inequality (except for $t = 1$).

Using the fact the $K_{\alpha,\gamma} \leq K_\alpha$, the boundedness of $I_{\alpha,\gamma}$ on Morrey spaces follows immediately, but we would like to have an estimate for its norm (similar to Young's inequality).

The Boundedness of $I_{\alpha,\gamma}$ on Morrey Spaces

Using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on (classical) Morrey spaces, we obtain:

Theorem 4.1

(Idris, G., Lindiarni, Eridani) *For* $0 < \alpha < n$ *and* $\gamma > 0$, *we have*

$$\|I_{\alpha,\gamma} f\|_{M^{p_2}_{q_2}} \leq C \, \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^{p_1}_{q_1}}, \quad f \in M^{p_1}_{q_1}(\mathbb{R}^n),$$

for $1 < p_1 \leq q_1 < t', \; 1 \leq s \leq t, \; \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{q_1}{p_1 t'}, \; \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{t'}$.

Remark. For $\gamma > \alpha$, we can have $t = 1$ (so that $t' = \infty$, $s = 1$, $p_2 = p_1$ and $q_2 = q_1$), for which the inequality reduces to

$$\|I_{\alpha,\gamma} f\|_{M^{p_1}_{q_1}} \leq C \, \|K_{\alpha,\gamma}\|_{L^1} \|f\|_{M^{p_1}_{q_1}}$$

where $1 < p_1 \leq q_1 < \infty$.

The Boundedness of $I_{\alpha,\gamma}$ on Generalized Morrey Spaces

The result is extended to generalized Morrey spaces as follows:

Theorem 4.2

If $\phi(r) \le Cr^{\beta}$ *for every* $r > 0$, $-\frac{\alpha t'}{p_1} \le \beta < -\alpha$, $1 < p_1 < t'$, *and* $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, *then we have*

$$\|I_{\alpha,\gamma} f\|_{M_{\psi}^{p_2}} \le C \|K_{\alpha,\gamma}\|_{M_t^s} \|f\|_{M_{\phi}^{p_1}}, \quad f \in M_{\phi}^{p_1}(\mathbb{R}^n)$$

for $1 \le s \le t$, $p_2 = \frac{\beta p_1}{\alpha+\beta}$, *and* $\psi(r) = \phi(r)^{p_1/p_2}$.

The Proof I

Write $I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$.

We estimate I_1 using dyadic decomposition as follow:

$$\begin{aligned}|I_1(x)| &\leq \sum_{k=-\infty}^{-1} \int_{2^kR\leq|x-y|<2^{k+1}R} \frac{|x-y|^{\alpha-n}|f(y)|}{(1+|x-y|)^{\gamma}}dy \\ &\leq C_1 \sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{\alpha-n}}{(1+2^kR)^{\gamma}} \int_{2^kR\leq|x-y|<2^{k+1}R} |f(y)|\,dy \\ &= C_2\mathcal{M}f(x) \sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{\alpha-n+n/s}\left(2^kR\right)^{n/s'}}{(1+2^kR)^{\gamma}},\end{aligned}$$

where $1 \leq s \leq t$.

The Proof II

By Hölder's inequality, we have

$$|I_1(x)| \le C_2 \mathcal{M}f(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-\infty}^{-1} \left(2^k R\right)^n \right)^{1/s'} .$$

But $\sum_{k=-\infty}^{-1} \frac{\left(2^k R\right)^{(\alpha-n)s+n}}{\left(1+2^k R\right)^{\gamma s}} \lesssim \int_{0<|x|<R} K_{\alpha,\gamma}^s(x)dx$, and so

$$\begin{aligned} |I_1(x)| &\le C_3 \mathcal{M}f(x) \left(\int_{0<|x|<R} K_{\alpha,\gamma}^s(x)\, dx \right)^{\frac{1}{s}} R^{n/s'} \\ &\le C_3 \left\| K_{\alpha,\gamma} \right\|_{M_t^s} \mathcal{M}f(x)\, R^{n/t'} . \end{aligned}$$

The Proof III

Next, we estimate I_2. By using Hölder's inequality, we obtain

$$|I_2(x)| \le C_4 \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha-n+n/p_1'}}{(1+2^k R)^{\gamma}} \left(\int_{2^k R \le |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}.$$

It thus follows that

$$|I_2(x)|$$

$$\le C_5 \|f\|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha} \phi(2^k R)}{(1+2^k R)^{\gamma}} \frac{\left(\int_{2^k R \le |x-y| < 2^{k+1} R} dy \right)^{\frac{1}{s}}}{\left(2^k R\right)^{n/s}}$$

$$\le C_6 \|f\|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \phi(2^k R) \left(2^k R\right)^{n/t'} \frac{\left(\int_{2^k R \le |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s (x-y) dy \right)^{\frac{1}{s}}}{\left(2^k R\right)^{n/s - n/t}}$$

The Proof IV

Because $\phi(r) \le Cr^\beta$ and $\frac{\left(\int_{2^kR\le|x-y|<2^{k+1}R} K^s_{\alpha,\gamma}(x-y)dy\right)^{1/s}}{\left(2^kR\right)^{n/s-n/t}} \lesssim \|K_{\alpha,\gamma}\|_{M^s_t}$ for every $k = 0, 1, 2, \dots,$ we get

$$|I_2(x)| \le C_7 \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^{p_1}_\phi} \sum_{k=0}^{\infty} \left(2^kR\right)^{\beta+n/t'}$$
$$\le C_8 \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^{p_1}_\phi} R^\beta R^{n/t'}.$$

From the two estimates, we obtain

$$|I_{\alpha,\gamma}f(x)| \le C_9 \|K_{\alpha,\gamma}\|_{M^s_t} \left(\mathcal{M}f(x)R^{n/t'} + \|f\|_{M^{p_1}_\phi} R^{n/t'+\beta}\right),$$

for every $x \in \mathbb{R}^n$.
Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^\beta = \frac{\mathcal{M}f(x)}{\|f\|_{M^{p_1}_\phi}}$.

The Proof V

Hence we get a Hedberg-type inequality

$$|I_{\alpha,\gamma}f(x)| \leq C_9 \|K_{\alpha,\gamma}\|_{M_t^s} \|f\|_{M_\phi^{p_1}}^{-\alpha/\beta} \mathcal{M}f(x)^{1+\alpha/\beta},$$

which holds for every $x \in \mathbb{R}^n$.

Put $p_2 := \frac{\beta p_1}{\alpha+\beta}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma}f(x)|^{p_2} dx\right)^{1/p_2}$$
$$\leq C_9 \|K_{\alpha,\gamma}\|_{M_t^s} \|f\|_{M_\phi^{p_1}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |\mathcal{M}f(x)|^{p_1} dx\right)^{1/p_2}.$$

The Proof VI

Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get the following inequality:

$$\|I_{\alpha,\gamma} f\|_{M^{p_2}_\psi} \leq C_{10} \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|^{1-p_1/p_2}_{M^{p_1}_\phi} \|\mathcal{M} f\|^{p_1/p_2}_{M^{p_1}_\phi},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$.

With the boundedness of the maximal operator on generalized Morrey spaces (Nakai's Theorem), we obtain the desired result:

$$\|I_{\alpha,\gamma} f\|_{M^{p_2}_\psi} \leq C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^{p_1}_\phi}.$$

The proof is complete. □

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Our recent results include **lower estimates** for the norm of Bessel-Riesz operators on (generalized) Morrey estimates.

THANK YOU VERY MUCH FOR YOUR ATTENTION!