Boundedness Properties of Bessel-Riesz Operators on Morrey Spaces

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Abstract & Previous Works

Introduction

Boundedness of $I_\alpha$ on Morrey Spaces

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Acknowledgement
Abstract

In this talk, we discuss the boundedness properties of Bessel-Riesz operators on (generalized) Morrey spaces.

We show that the norm of such an operator is dominated by the norm of its kernel in an associated Morrey space.

The proof uses the usual dyadic decomposition, a Hedberg-type inequality, and the boundedness of Hardy-Littlewood maximal operator on Morrey spaces.
Previous Works, Among Others . . .


For $\gamma \geq 0$ and $0 < \alpha < n$, define

$$I_{\alpha, \gamma}f(x) := \int_{\mathbb{R}^n} K_{\alpha, \gamma}(x - y)f(y) \, dy,$$

for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, $p \geq 1$, where

$$K_{\alpha, \gamma}(x) := \frac{|x|^{\alpha-n}}{(1 + |x|)^\gamma}, \quad x \in \mathbb{R}^n.$$

We call $I_{\alpha, \gamma}$ a Bessel-Riesz operator, and $K_{\alpha, \gamma}$ a Bessel-Riesz kernel.
Introduction

Hardy-Littlewood-Sobolev Inequality for $I_\alpha$

For $\gamma = 0$, $I_{\alpha,0} =: I_\alpha$ is known as the *Riesz potential* or *fractional integral operator*.

For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is **bounded** from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, i.e. there exists a constant $C = C_{p,q} > 0$ such that

$$\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),$$

provided that $1 < p < \frac{\alpha}{n}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

This result is due to Hardy & Littlewood and Sobolev, and the above inequality is known as *Hardy-Littlewood-Sobolev inequality*. 
Morrey Spaces

For $1 \leq p \leq q$, the (classical) Morrey space $M^p_q := M^p_q(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ on $\mathbb{R}^n$ for which

$$\|f\|_{M^p_q} := \sup_{a \in \mathbb{R}^n, r > 0} |B(a, r)|^{1/q}( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^p dy )^{1/p} < \infty.$$ 

Here $B(a, r)$ denotes the ball in $\mathbb{R}^n$ centered at $a$ with radius $r$, and $|B(a, r)| = c \cdot r^n$ denotes its volume.

**Note:** If $p = q$, then $M^p_q = L^q$.

Moreover, $M^p_q$ equipped with $\| \cdot \|_{M^p_q}$ is a Banach space.
For $1 \leq p_1 \leq p_2 \leq q < \infty$, the following inclusions hold:

$$L^q = M_q^q \subseteq M_q^{p_2} \subseteq M_q^{p_1} \subseteq M_q^1.$$ 

Moreover, if $1 < p_1 < p_2 < q$, then the inclusions are strict.
For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is **bounded** from $M^p_q$ to $M^s_t$, i.e. there exists $C = C_{p,q,s,t}$ such that

$$\|I_\alpha f\|_{M^s_t} \leq C \|f\|_{M^p_q},$$

provided that $1 < p \leq q < \frac{n}{\alpha}$, $1 < s \leq t < \infty$, $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{n}$, and $\frac{p}{q} = \frac{s}{t}$.

This result is due to D. Adams (1975) and F. Chiarenza & M. Frasca (1987).
Generalized Morrey Spaces

For $1 \leq p < \infty$ and a certain function $\phi : (0, \infty) \to (0, \infty)$, the generalized Morrey space $M^p_\phi = M^p_\phi(\mathbb{R}^d)$ consists of $f$ for which

$$\|f\|_{M^p_\phi} := \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$  

Here $\phi$ is nonincreasing and $r \mapsto \phi(r)^p r^n$ is nondecreasing on $(0, \infty)$. Consequently, $\phi$ satisfies the doubling condition: $\exists C > 0$ s.t.

$$\frac{1}{C} \leq \frac{\phi(r)}{\phi(u)} \leq C \text{ whenever } \frac{1}{2} \leq \frac{r}{u} \leq 2.$$  

Note that if $\phi(r) := r^{-n/q}$ for some $q > p$, then $M^p_\phi = M^p_q$ — the classical Morrey spaces.
For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is bounded from $M^p_\phi$ to $M^q_\psi$, i.e. there exists $C = C_{p,q,\phi,\psi} > 0$ such that

$$\|I_\alpha f\|_{M^p_\phi} \leq C \|f\|_{M^q_\psi}, \quad f \in M^p_\phi(\mathbb{R}^n),$$

provided that $1 < p < \frac{n}{\alpha}$, $\int_r^\infty \frac{\phi(u)}{u} \, du \leq C \phi(r)$, and $\phi(r) \leq C r^\beta$ for every $r > 0$, $q = \frac{\beta p}{\alpha + \beta}$, and $\psi(r) = \phi(r)^\frac{p}{q}$ for every $r > 0$.

This result can be found in G. & Eridani (2009).
The proof of the previous inequality for $I_\alpha$ uses a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator $\mathcal{M}$ on generalized Morrey spaces.

**Theorem 3.1**

(Nakai) Let $\mathcal{M}$ be given by $\mathcal{M}f(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy$. Then, for $1 < p \leq \infty$, we have

$$\|\mathcal{M}f\|_{M^p_\phi} \leq C \|f\|_{M^p_\phi}, \quad f \in M^p_\phi(\mathbb{R}^n).$$
Hereafter, let $0 < \alpha < n$ and $\gamma > 0$. Then

$$K_{\alpha, \gamma} \in L^t \text{ for } \frac{n}{n + \gamma - \alpha} < t < \frac{n}{n - \alpha},$$

with

$$\|K_{\alpha, \gamma}\|_{L^t} \sim \left( \sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1 + 2^k R)^{\gamma t}} \right)^{\frac{1}{t}},$$

where $R > 0$ is fixed but arbitrary.

Further, $K_{\alpha, \gamma} \in M^s_t$ for $1 \leq s \leq t$, and there is a similar estimate for $\|K_{\alpha, \gamma}\|_{M^s_t}$. 

By Young’s inequality, we have

\[ \| I_{\alpha, \gamma} f \|_{L^q} \leq \| K_{\alpha, \gamma} \|_{L^t} \| f \|_{L^p} \]

for \( 1 \leq p < t' \), \( \frac{n}{n + \gamma - \alpha} < t < \frac{n}{n - \alpha} \), and \( \frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1 \).

Hence \( \| I_{\alpha, \gamma} \|_{L^p \to L^q} \leq \| K_{\alpha, \gamma} \|_{L^t} \) for those \( p, q, \) and \( t \).

On Morrey spaces, however, we do not have Young’s inequality (except for \( t = 1 \)).

Using the fact the \( K_{\alpha, \gamma} \leq K_{\alpha} \), the boundedness of \( I_{\alpha, \gamma} \) on Morrey spaces follows immediately, but we would like to have an estimate for its norm (similar to Young’s inequality).
The Boundedness of $I_{\alpha, \gamma}$ on Morrey Spaces

Using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on (classical) Morrey spaces, we obtain:

**Theorem 4.1**

(Idris, G., Lindiarni, Eridani) For $0 < \alpha < n$ and $\gamma > 0$, we have

$$
\| I_{\alpha, \gamma} f \|_{M_p^2} \leq C \| K_{\alpha, \gamma} \|_{M_s^t} \| f \|_{M_q^{p_1}} , \quad f \in M_q^{p_1}(\mathbb{R}^n),
$$

for $1 < p_1 \leq q_1 < t'$, $1 \leq s \leq t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$,

$$
\frac{1}{p_2} = \frac{1}{p_1} - \frac{q_1}{p_1 t'}, \quad \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{t'}.
$$
**Remark.** For $\gamma > \alpha$, we can have $t = 1$ (so that $t' = \infty$, $s = 1$, $p_2 = p_1$ and $q_2 = q_1$), for which the inequality reduces to

$$\| I_{\alpha, \gamma} f \|_{M_{q_1}^{p_1}} \leq C \| K_{\alpha, \gamma} \|_{L^1} \| f \|_{M_{q_1}^{p_1}}$$

where $1 < p_1 \leq q_1 < \infty$. 
The result is extended to generalized Morrey spaces as follows:

**Theorem 4.2**

If \( \phi(r) \leq C r^\beta \) for every \( r > 0 \), \( -\frac{\alpha t'}{p_1} \leq \beta < -\alpha \), \( 1 < p_1 < t' \), and \( \frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha} \), then we have

\[
\| I_{\alpha, \gamma} f \|_{M_\psi^{p_2}} \leq C \| K_{\alpha, \gamma} \|_{M_t^s} \| f \|_{M_\phi^{p_1}}, \quad f \in M_\phi^{p_1}(\mathbb{R}^n)
\]

for \( 1 \leq s \leq t \), \( p_2 = \frac{\beta p_1}{\alpha+\beta} \), and \( \psi(r) = \phi(r)^{p_1/p_2} \).
Write $I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$.

We estimate $I_1$ using dyadic decomposition as follow:

$$|I_1(x)| \leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^\alpha f(y)}{(1+|x-y|)^\gamma} \, dy$$

$$\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1+2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| \, dy$$

$$= C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/s}}{(1+2^k R)^\gamma} \frac{(2^k R)^{n/s'}}$$

where $1 \leq s \leq t$. 
By Hölder’s inequality, we have

\[ |I_1(x)| \leq C_2 Mf(x) \left( \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1 + 2^k R)^\gamma s} \right)^{1/s} \left( \sum_{k=-\infty}^{-1} (2^k R)^n \right)^{1/s'} \]

But

\[ \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1 + 2^k R)^\gamma s} \lesssim \int_{0 < |x| < R} K_{\alpha, \gamma}^s(x) \, dx, \] and so

\[ |I_1(x)| \leq C_3 Mf(x) \left( \int_{0 < |x| < R} K_{\alpha, \gamma}^s(x) \, dx \right)^{1/s} R^{n/s'} \]

\[ \leq C_3 \| K_{\alpha, \gamma} \|_{M_t^s} Mf(x) R^{n/t'} . \]
The Proof III

Next, we estimate $I_2$. By using Hölder’s inequality, we obtain

$$|I_2(x)| \leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/p_1'}}{(1 + 2^k R)^{\gamma}} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} \, dy \right)^{1/p_1}.$$

It thus follows that

$$|I_2(x)| \leq C_5 \| f \|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha} \phi(2^k R)}{(1 + 2^k R)^{\gamma}} \frac{\left( \int_{2^k R \leq |x-y| < 2^{k+1} R} \, dy \right)^{1/s}}{(2^k R)^{n/s}} \frac{\left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s (x-y) \, dy \right)^{1/s}}{(2^k R)^{n/s-n/t}}.$$

$$\leq C_6 \| f \|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \frac{\phi(2^k R)}{\phi(2^k R)} (2^k R)^{n/t'} \frac{\left( \int_{2^k R \leq |x-y| < 2^{k+1} R} \, dy \right)^{1/s}}{(2^k R)^{n/s-n/t}} \frac{\left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s (x-y) \, dy \right)^{1/s}}{(2^k R)^{n/s-n/t}}.$$
Because \( \phi(r) \leq Cr^\beta \) and \( \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha, \gamma}^s(x-y) dy \right)^{1/s} \leq ||K_{\alpha, \gamma}\|_{M^s_t} \) for every \( k = 0, 1, 2, \ldots \), we get

\[
|I_2(x)| \leq C_7 ||K_{\alpha, \gamma}\|_{M^s_t} \|f\|_{M^{p_1}_\phi} \sum_{k=0}^{\infty} (2^k R)^{\beta+n/t'}
\leq C_8 ||K_{\alpha, \gamma}\|_{M^s_t} \|f\|_{M^{p_1}_\phi} R^\beta R^{n/t'}.
\]

From the two estimates, we obtain

\[
|I_{\alpha, \gamma} f(x)| \leq C_9 ||K_{\alpha, \gamma}\|_{M^s_t} \left( Mf(x) R^{n/t'} + ||f||_{M^{p_1}_\phi} R^{n/t' + \beta} \right),
\]

for every \( x \in \mathbb{R}^n \).

Now, for each \( x \in \mathbb{R}^n \), choose \( R > 0 \) such that \( R^\beta = \frac{Mf(x)}{||f||_{M^{p_1}_\phi}} \).
Hence we get a Hedberg-type inequality

\[ |I_{\alpha,\gamma}f(x)| \leq C_9 \| K_{\alpha,\gamma} \|_{M^s_t} \| f \|_{M^p_\phi}^{-\alpha/\beta} \mathcal{M} f(x)^{1+\alpha/\beta}, \]

which holds for every \( x \in \mathbb{R}^n \).

Put \( p_2 := \frac{\beta p_1}{\alpha+\beta} \). For arbitrary \( a \in \mathbb{R}^n \) and \( r > 0 \), we have

\[
\left( \int_{|x-a|<r} |I_{\alpha,\gamma}f(x)|^{p_2} \, dx \right)^{1/p_2} \\
\leq C_9 \| K_{\alpha,\gamma} \|_{M^s_t} \| f \|_{M^p_\phi}^{1-p_1/p_2} \left( \int_{|x-a|<r} |\mathcal{M} f(x)|^{p_1} \, dx \right)^{1/p_2}.
\]
The Proof VI

Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get the following inequality:

$$\| I_{\alpha, \gamma} f \|_{M^{p_2}_\psi} \leq C_{10} \| K_{\alpha, \gamma} \|_{t^s} \| f \|_{M^{p_1}_\phi}^{1-p_1/p_2} \| M f \|_{M^{p_1}_\phi}^{p_1/p_2},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$.

With the boundedness of the maximal operator on generalized Morrey spaces (Nakai’s Theorem), we obtain the desired result:

$$\| I_{\alpha, \gamma} f \|_{M^{p_2}_\psi} \leq C_{p_1, \phi} \| K_{\alpha, \gamma} \|_{t^s} \| f \|_{M^{p_1}_\phi}. $$

The proof is complete.
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Our recent results include lower estimates for the norm of Bessel-Riesz operators on (generalized) Morrey estimates.

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