On the Boundedness of Bessel-Riesz Operators on Generalized Morrey Spaces

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The talk will be about Bessel-Riesz operators, which may be viewed as a variant of Riesz potentials or fractional integral operators. We prove the boundedness of these operators on generalized Morrey spaces by using the usual dyadic decomposition, a Hedberg-type inequality, and the boundedness of Hardy-Littlewood maximal operator.

Our results reveal that the norm of Bessel-Riesz operators on such spaces is dominated by the norm of the associated kernels.
Abstract & Previous Works

Previous Works, Among Others . . .


For $\gamma \geq 0$ and $0 < \alpha < n$, define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x - y) f(y) \, dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, $p \geq 1$, where

$$K_{\alpha,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1 + |x|)^\gamma}, \quad x \in \mathbb{R}^n.$$

$I_{\alpha,\gamma}$ is called Bessel-Riesz operator, while $K_{\alpha,\gamma}$ is called Bessel-Riesz kernel.
Hardy-Littlewood-Sobolev Inequality for $I_\alpha$

For $\gamma = 0$, $I_{\alpha,0} =: I_\alpha$ is known as the Riesz potential or fractional integral operator.

For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, i.e. there exists a constant $C = C_{p,q} > 0$ such that

$$\|I_\alpha f\|_{L^q} \leq C \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}^n),$$

provided that $1 < p < \frac{\alpha}{n}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

This result is due to Hardy & Littlewood and Sobolev, and the above inequality is known as Hardy-Littlewood-Sobolev inequality.
Morrey Spaces

For $1 \leq p \leq q$, the (classical) Morrey space $M^p_q := M^p_q(\mathbb{R}^n)$ is defined as the set of all locally integrable functions $f$ on $\mathbb{R}^n$ for which

$$
\|f\|_{M^p_q} := \sup_{a \in \mathbb{R}^n, r > 0} |B(a, r)|^{1/q} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(y)|^p dy \right)^{1/p} < \infty.
$$

Here $B(a, r)$ denotes the ball in $\mathbb{R}^n$ centered at $a$ with radius $r$, and $|B(a, r)| = c \cdot r^n$ denotes its volume.

**Note:** If $p = q$, then $M^p_q = L^q$.

Moreover, $M^p_q$ is a Banach space.
Inclusion Property of Morrey Spaces

For $1 \leq p_1 \leq p_2 \leq q < \infty$, the following inclusions hold:

$$L^q = M_q^q \subseteq M_q^{p_2} \subseteq M_q^{p_1} \subseteq M_q^1.$$

Moreover, if $1 < p_1 < p_2 < q$, then the inclusions are strict.
For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is bounded from $M^p_q$ to $M^s_t$, i.e. there exists $C = C_{p,q,s,t}$ such that

$$\|I_\alpha f\|_{M^s_t} \leq C \|f\|_{M^p_q},$$

provided that $1 < p \leq q < \frac{n}{\alpha}$, $1 < s \leq t < \infty$, $\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{d}$, and $\frac{p}{q} = \frac{s}{t}$.

This result is due to D. Adams (1975) and F. Chiarenza & M. Frasca (1987).
Generalized Morrey Spaces

For $1 \leq p < \infty$ and a certain function $\phi : (0, \infty) \rightarrow (0, \infty)$, the \textit{generalized Morrey space} $M^p_\phi = M^p_\phi(\mathbb{R}^d)$ consists of $f$ for which

$$\|f\|_{M^p_\phi} := \sup_{a \in \mathbb{R}^n, r > 0} \frac{1}{\phi(r)} \left( \frac{1}{|B(a, r)|} \int_{B(a, r)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$ 

Here $\phi$ is nonincreasing and $r \mapsto \phi(r)^p r^n$ is nondecreasing on $(0, \infty)$. Consequently, $\phi$ satisfies the \textit{doubling condition}: $\exists C > 0$ s.t.

$$\frac{1}{C} \leq \frac{\phi(r)}{\phi(u)} \leq C \text{ whenever } \frac{1}{2} \leq \frac{r}{u} \leq 2.$$ 

Note that if $\phi(r) := r^{-n/q}$ for some $q > p$, then $M^p_\phi = M^p_q$ — the classical Morrey spaces.
For $0 < \alpha < n$, the fractional integral operator $I_\alpha$ is bounded from $M^p_\phi$ to $M^q_\psi$, i.e. there exists $C = C_{p,q,\phi,\psi} > 0$ such that
\[
\|I_\alpha f\|_{M^p_\phi} \leq C \|f\|_{M^q_\psi}, \quad f \in M^p_\phi(\mathbb{R}^n),
\]
provided that $1 < p < \frac{n}{\alpha}$, $\int_r^\infty \frac{\phi(u)}{u} \, du \leq C \phi(r)$, and $\phi(r) \leq C r^\beta$ for every $r > 0$, $q = \frac{\beta p}{\alpha + \beta}$, and $\psi(r) = \phi(r)^{\frac{p}{q}}$ for every $r > 0$.

This result is obtained by G. & Eridani (2009).
In the next section, we shall reprove the boundedness of $I_{\alpha,\gamma}$ for $\gamma > 0$ on generalized Morrey spaces using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator $M$ on these spaces.

**Theorem 3.1**

(Nakai) Let $M$ be given by $Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy$. Then, for $1 < p \leq \infty$, we have

$$\|Mf\|_{M_p^\phi} \leq C \|f\|_{M_p^\phi}, \quad f \in M_p^\phi(\mathbb{R}^n).$$

Our aim is to show that the norm of Bessel-Riesz operators is dominated by the norm of their kernels on (generalized) Morrey spaces.
Properties of the Kernel

Let $0 < \alpha < n$ and $\gamma > 0$ (we shall always assume this, unless otherwise stated).

Then $K_{\alpha,\gamma} \in L^t(\subseteq M^{s,t}, 1 \leq s \leq t)$ for $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, with

$$
\| K_{\alpha,\gamma} \|_{L^t} \sim \left( \sum_{k=1}^{\infty} \frac{(2^k R)^{(\alpha-n)t+n}}{(1 + 2^k R)^{\gamma t}} \right)^{\frac{1}{t}}
$$

where $R > 0$ is fixed but arbitrary.
By Young’s inequality, we have

$$\|I_{\alpha,\gamma} f\|_{L^q} \leq \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}$$

for $1 \leq p < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, and $\frac{1}{q} = \frac{1}{p} + \frac{1}{t} - 1$.

Hence $\|I_{\alpha,\gamma}\|_{L^q} \leq \|K_{\alpha,\gamma}\|_{L^t}$ for those $q$ and $t$.

On Morrey spaces, however, we do not have Young’s inequality. Thus, to prove the boundedness of $I_{\alpha,\gamma}$, we have to use a different approach.
The Boundedness of $I_{\alpha, \gamma}$ on Morrey Spaces

Using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on (classical) Morrey spaces, we obtain

**Theorem 4.1**

(Idris, G., Lindiarni, Eridani) *For $0 < \alpha < n$ and $\gamma > 0$, we have*

$$\| I_{\alpha, \gamma} f \|_{M_{q_2}^{p_2}} \leq C \| K_{\alpha, \gamma} \|_{M_t^s} \| f \|_{M_{q_1}^{p_1}}, \quad f \in M_{q_1}^{p_1}(\mathbb{R}^n),$$

*for $1 < p_1 \leq q_1 < t'$, $1 \leq s \leq t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$,*

$$\frac{1}{p'_2} = \frac{1}{p_1} - \frac{q_1}{p_1 t'}, \quad \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{t'}.$$
The Boundedness of $I_{\alpha,\gamma}$ on Generalized Morrey Spaces

Theorem 4.2

If $\phi(r) \leq Cr^\beta$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \leq \beta < -\alpha$, $1 < p_1 < t'$, and

$$\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha},$$

then we have

$$\left\| I_{\alpha,\gamma} f \right\|_{M_{\psi}^{p_2}} \leq C \left\| K_{\alpha,\gamma} \right\|_{M_s^{t'}} \left\| f \right\|_{M_{\phi}^{p_1}}, \quad f \in M_{\phi}^{p_1}(\mathbb{R}^n)$$

for $1 \leq s \leq t$, $p_2 = \frac{\beta p_1}{\alpha + \beta}$, and $\psi(r) = \phi(r)^{p_1/p_2}$. 
Write $I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$.

We estimate $I_1$ using dyadic decomposition as follow:

$$|I_1(x)| \leq \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|x-y|^{\alpha-n}|f(y)|}{(1 + |x-y|)^\gamma} \, dy$$

$$\leq C_1 \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n}}{(1 + 2^k R)^\gamma} \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)| \, dy$$

$$= C_2 \mathcal{M} f(x) \sum_{k=-\infty}^{-1} \frac{(2^k R)^{\alpha-n+n/s}}{(1 + 2^k R)^\gamma} \left(2^k R\right)^{n/s'},$$

where $1 \leq s \leq t$. 
By Hölder’s inequality, we have

\[ |I_1(x)| \leq C_2 \mathcal{M} f(x) \left( \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1 + 2^k R)^{\gamma s}} \right)^{1/s} \left( \sum_{k=-\infty}^{-1} (2^k R)^n \right)^{1/s'} . \]

We also have

\[ \sum_{k=-\infty}^{-1} \frac{(2^k R)^{(\alpha-n)s+n}}{(1 + 2^k R)^{\gamma s}} \lesssim \int_{0 <|x| < R} K_{s,\alpha,\gamma}^s(x) \, dx , \]

so that

\[ |I_1(x)| \leq C_3 \mathcal{M} f(x) \left( \int_{0 <|x| < R} K_{s,\alpha,\gamma}^s(x) \, dx \right)^{1/s} R^{n/s'} \]

\[ \leq C_3 \|K_{\alpha,\gamma}\|_{M_{s}^t} \mathcal{M} f(x) R^{n/t'} . \]
Next, we estimate $I_2$. By using Hölder’s inequality, we obtain

$$|I_2(x)| \leq C_4 \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha-n+n/p_1'}}{(1 + 2^k R)^\gamma} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} |f(y)|^{p_1} dy \right)^{1/p_1}.$$

It thus follows that

$$|I_2(x)| \leq C_5 \|f\|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \frac{(2^k R)^{\alpha} \phi(2^k R)}{(1 + 2^k R)^\gamma} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} dy \right)^{1/s} (2^k R)^{n/s}$$

$$\leq C_6 \|f\|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \phi(2^k R) (2^k R)^{n/t'} \left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s (x - y) dy \right)$$

$$\leq C_6 \|f\|_{M_{\phi}^{p_1}} \sum_{k=0}^{\infty} \phi(2^k R) (2^k R)^{n/s-n/t} (2^k R)^{n/s-n/t}.$$
The Proof IV

Because $\phi(r) \leq C r^\beta$ and
\[
\left( \int_{2^k R \leq |x-y| < 2^{k+1} R} K_{\alpha,\gamma}^s(x-y) dy \right)^{1/s} \leq C R^{n/s-n/t} \leq \| K_{\alpha,\gamma} \|_{M^s_t}
\]
for every $k = 0, 1, 2, \ldots$, we get

\[
|I_2(x)| \leq C_7 \| K_{\alpha,\gamma} \|_{M^s_t} \| f \|_{M^p_1} \sum_{k=0}^{\infty} (2^k R)^{\beta+n/t'}
\]

\[
\leq C_8 \| K_{\alpha,\gamma} \|_{M^s_t} \| f \|_{M^p_1} R^\beta R^{n/t'}.
\]

From the two estimates, we obtain

\[
|I_{\alpha,\gamma} f(x)| \leq C_9 \| K_{\alpha,\gamma} \|_{M^s_t} \left( \mathcal{M} f(x) R^{n/t'} + \| f \|_{M^p_1} R^{n/t'+\beta} \right),
\]

for every $x \in \mathbb{R}^n$. 
The Proof V

Now, for each \( x \in \mathbb{R}^n \), choose \( R > 0 \) such that \( R^{\beta} = \frac{\mathcal{M}f(x)}{\|f\|_{M^p_\phi}} \). Hence we get

\[
|I_{\alpha,\gamma}f(x)| \leq C_9 \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^p_\phi}^{-\alpha/\beta} \mathcal{M}f(x)^{1+\alpha/\beta}.
\]

Put \( p_2 := \frac{\beta p_1}{\alpha+\beta} \). For arbitrary \( a \in \mathbb{R}^n \) and \( r > 0 \), we have

\[
\left(\int_{|x-a|<r}|I_{\alpha,\gamma}f(x)|^{p_2} \, dx\right)^{1/p_2} \leq C_9 \|K_{\alpha,\gamma}\|_{M^s_t} \|f\|_{M^p_\phi}^{1-p_1/p_2} \left(\int_{|x-a|<r}|\mathcal{M}f(x)|^{p_1} \, dx\right)^{1/p_2}.
\]
Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get the following Hedberg’s type inequality:

$$\| I_{\alpha, \gamma} f \|_{M_{\psi}^{p_2}} \leq C_{10} \| K_{\alpha, \gamma} \|_{M_{\hat{t}}^{s}} \| f \|_{M_{\phi}^{p_1}}^{1-p_1/p_2} \| \mathcal{M} f \|_{M_{\phi}^{p_1}}^{p_1/p_2},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$.

With the boundedness of the maximal operator on generalized Morrey spaces (Nakai’s Theorem), we obtain the desired result:

$$\| I_{\alpha, \gamma} f \|_{M_{\psi}^{p_2}} \leq C_{p_1, \phi} \| K_{\alpha, \gamma} \|_{M_{\hat{t}}^{s}} \| f \|_{M_{\phi}^{p_1}}^{p_1/p_2}.$$

The proof is complete.
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Currently we are working on the generalized Bessel-Riesz operator \( I_{\rho, \gamma} \) for some function \( \rho : (0, \infty) \rightarrow (0, \infty) \), supported by ITB Research and Innovation Program 2016.

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