

A CONTRACTIVE MAPPING THEOREM ON THE n -NORMED SPACE OF p -SUMMABLE SEQUENCES

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Abstract. We revisit the space ℓ^p of p -summable sequences of real numbers which is equipped with an n -norm. We derive a norm from the n -norm in a certain way, and show that this norm is equivalent to the usual norm on ℓ^p . We then use this fact together with others to prove a contractive mapping theorem on the n -normed space ℓ^p .

Keywords: p -summable sequence spaces, norm equivalence, contractive mapping theorem

1. INTRODUCTION

The theory of 2-normed spaces was first introduced by Gähler [1] in the 1960's. Extensions to the theory of n -normed spaces were developed in [2, 3, 4]. Related works may be found in [6, 7, 9, 10]. For a fixed number $n \in \mathbb{N}$, an n -norm on a real vector space X (of dimension at least n) is a mapping $\|\cdot, \dots, \cdot\| : X^n \rightarrow \mathbb{R}$ which satisfies the following four conditions:

- (1.1) $\|x_1, \dots, x_n\| = 0$ if and only if x_1, \dots, x_n are linearly dependent,
- (1.2) $\|x_1, \dots, x_n\|$ is invariant under permutation,
- (1.3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$ for every $\alpha \in \mathbb{R}$,
- (1.4) $\|x_1 + x'_1, x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x'_1, x_2, \dots, x_n\|$.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n -normed space.

Geometrically, $\|x_1, \dots, x_n\|$ may be interpreted as the volume of the n -dimensional parallelepiped spanned by x_1, \dots, x_n . Note that in an n -normed space, we have

$$\|x_1 + y, x_2, \dots, x_n\| = \|x_1, x_2, \dots, x_n\|$$

for any $y = \alpha_2 x_2 + \dots + \alpha_n x_n$.

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n -normed space. A sequence (x_k) in X is said to *converge* to an $x \in X$ (in the n -norm) if

$$\lim_{k \rightarrow \infty} \|x_k - x, y_2, \dots, y_n\| = 0,$$

for every $y_2, \dots, y_n \in X$. Also, a sequence (x_k) in X is called a *Cauchy* sequence if

$$\lim_{k, l \rightarrow \infty} \|x_k - x_l, y_2, \dots, y_n\| = 0,$$

for every $y_2, \dots, y_n \in X$.

If every Cauchy sequence (x_k) in X converges to some $x \in X$, then X is said to be *complete*. A complete n -normed space is called an n -Banach Space.

Our interest here is in the existence of a fixed point of a contractive mapping on an n -Banach space. In particular, we shall consider the space ℓ^p of p -summable sequences of real

numbers, where $1 \leq p \leq \infty$, that is, the space of all sequences $x = (\xi_j)$ for which $\sum_j |\xi_j|^p < \infty$ when $1 \leq p < \infty$ (or $\sup_j |\xi_j| < \infty$ when $p = \infty$). For $1 \leq p < \infty$, we equip ℓ^p with the following n -norm (for a fixed number $n \in \mathbb{N}$)

$$\|x_1, \dots, x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \left| \det [\xi_{i j_k}]_{i,k} \right|^p \right]^{\frac{1}{p}},$$

where $x_i := (\xi_{ij}) \in \ell^p$, $i = 1, \dots, n$. For $p = \infty$, the n -norm is given by

$$\|x_1, \dots, x_n\|_\infty := \sup_{j_1} \cdots \sup_{j_n} \left| \det [\xi_{i j_k}]_{i,k} \right|.$$

The n -normed space $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ was first studied by Gunawan [5].

In [5], Gunawan formulated a contractive mapping theorem for $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ as follows:

Theorem 1.1 [5] *Let T be a self-mapping of ℓ^p which is contractive in the sense that there exists a constant $C \in (0, 1)$ such that the inequality*

$$\|Tx - Tx', y_2, \dots, y_n\|_p \leq C \|x - x', y_2, \dots, y_n\|_p$$

holds for all $x, x', y_2, \dots, y_n \in \ell^p$. Then T has a unique fixed point, that is, there exists a unique $x \in \ell^p$ such that $Tx = x$.

We notice that the hypothesis on T in the above theorem is quite strong. The aim of this note is to prove a similar theorem but with a much weaker condition. Additional results will also be presented.

2. MAIN RESULTS

Our main result is the following theorem, which is more general than the previous theorem of Gunawan.

Theorem 2.1 *Let T be a self-mapping of ℓ^p which is contractive with respect to a linearly independent set $\{a_1, \dots, a_n\}$ in ℓ^p , that is, there exists a constant $C \in (0, 1)$ such that*

$$\|Tx - Tx', a_{i_2}, \dots, a_{i_n}\|_p \leq C \|x - x', a_{i_2}, \dots, a_{i_n}\|_p$$

holds for all $x, x' \in \ell^p$ and $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. Then T has a unique fixed point, that is, there exists a unique $x \in \ell^p$ such that $Tx = x$.

To prove the above theorem, we need several lemmas and propositions. The first lemma below from [5] gives a relationship between the n -norm $\|\cdot, \dots, \cdot\|_p$ and the usual norm $\|\cdot\|_p$ on ℓ^p . From now on, we assume that $1 \leq p < \infty$, and leave the case where $p = \infty$ to the reader.

Lemma 2.2 [5] *The inequality*

$$\|x_1, \dots, x_n\|_p \leq (n!)^{1-1/p} \|x_1\|_p \cdots \|x_n\|_p$$

holds for every $x_1, \dots, x_n \in \ell^p$.

On an n -normed space, we can derive a norm from the n -norm in a certain way. In particular, on the space ℓ^p of p -summable sequences, we have the following result.

Proposition 2.3 *Let $\{a_1, \dots, a_n\}$ be a linearly independent set on ℓ^p . Then, the following function*

$$\|x\|_p^* := \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}}$$

defines a norm on ℓ^p .

Proof. One may verify that $\|x\|_p^*$ satisfies the four properties of a norm. In particular, we may check that if $\|x\|_p^* = 0$, then $\|x, a_{i_2}, \dots, a_{i_n}\|_p = 0$ for every $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. This means that x is in the linear span of $\{a_{i_2}, \dots, a_{i_n}\}$ for every $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$. This forces us to conclude that $x = 0$. Furthermore, one may check that the triangle inequality for $\|\cdot\|_p^*$ follows immediately from the triangle inequality for the n -norm and the Minkowski's inequality for sums. \square

The next lemma is required to show that the norm $\|\cdot\|_p^*$ is actually equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p .

Lemma 2.4 *For every $x, y_1, \dots, y_n \in \ell^p$, we have*

$$\begin{aligned} \|x\|_p \|y_1, \dots, y_n\|_p &\leq n \|y_1\|_p \|x, y_2, y_3, \dots, y_n\|_p + \\ &+ \|y_2\|_p \|x, y_1, y_3, \dots, y_n\|_p + \dots + \|y_n\|_p \|x, y_1, y_2, \dots, y_{n-1}\|_p. \end{aligned}$$

Proof. The case $n = 2$ has been proved in [8]. Here we shall only give the proof for $n = 3$, leaving the proof for $n > 3$ to the reader. Let $x := (\xi_j) \in \ell^p$ and $y_i := (\eta_{ij}) \in \ell^p$, $i = 1, 2, 3$. We observe that

$$\|x\|_p^p \|y_1, y_2, y_3\|_p^p = \frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \xi_k \begin{vmatrix} \eta_{1j_1} & \eta_{1j_2} & \eta_{1j_3} \\ \eta_{2j_1} & \eta_{2j_2} & \eta_{2j_3} \\ \eta_{3j_1} & \eta_{3j_2} & \eta_{3j_3} \end{vmatrix} \right|^p.$$

By Minkowski's inequality, we have

$$\begin{aligned}
& \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \xi_k \begin{vmatrix} \eta_{1j_1} & \eta_{1j_2} & \eta_{1j_3} \\ \eta_{2j_1} & \eta_{2j_2} & \eta_{2j_3} \\ \eta_{3j_1} & \eta_{3j_2} & \eta_{3j_3} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} \\
& \leq \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \eta_{1j_1} \begin{vmatrix} \eta_{2j_2} & \eta_{2k} & \eta_{2j_3} \\ \xi_{j_2} & \xi_k & \xi_{j_3} \\ \eta_{3j_2} & \eta_{3k} & \eta_{3j_3} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} + \\
& \quad \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \eta_{1j_2} \begin{vmatrix} \eta_{2j_1} & \eta_{2k} & \eta_{2j_3} \\ \eta_{3j_1} & \eta_{3k} & \eta_{3j_3} \\ \xi_{j_1} & \xi_k & \xi_{j_3} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} + \\
& \quad \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \eta_{1j_3} \begin{vmatrix} \eta_{2j_1} & \eta_{2k} & \eta_{2j_2} \\ \xi_{j_1} & \xi_k & \xi_{j_2} \\ \eta_{3j_1} & \eta_{3k} & \eta_{3j_2} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} + \\
& \quad \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \eta_{2k} \begin{vmatrix} \eta_{1j_1} & \eta_{1j_3} & \eta_{1j_2} \\ \eta_{3j_1} & \eta_{3j_3} & \eta_{3j_2} \\ \xi_{j_1} & \xi_{j_3} & \xi_{j_2} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} + \\
& \quad \left(\frac{1}{3!} \sum_k \sum_{j_1} \sum_{j_2} \sum_{j_3} \left| \eta_{3k} \begin{vmatrix} \eta_{1j_1} & \eta_{1j_3} & \eta_{1j_2} \\ \xi_{j_1} & \xi_{j_3} & \xi_{j_2} \\ \eta_{2j_1} & \eta_{2j_3} & \eta_{2j_2} \end{vmatrix} \right|^p \right)^{\frac{1}{p}} \\
& = 3 \|y_1\|_p \|x, y_2, y_3\|_p + \|y_2\|_p \|x, y_1, y_3\|_p + \|y_3\|_p \|x, y_1, y_2\|_p.
\end{aligned}$$

Hence we obtain

$$\|x\|_p \|y_1, y_2, y_3\|_p \leq 3 \|y_1\|_p \|x, y_2, y_3\|_p + \|y_2\|_p \|x, y_1, y_3\|_p + \|y_3\|_p \|x, y_1, y_2\|_p,$$

as desired. \square

Proposition 2.5 *Let $\{a_1, \dots, a_n\}$ be a linearly independent set on ℓ^p . Then the norm $\|\cdot\|_p^*$ defined in Proposition 2.3 is equivalent to the usual norm $\|\cdot\|_p$ on ℓ^p . Precisely, we have*

$$\begin{aligned}
\frac{n \|a_1, \dots, a_n\|_p}{(2n-1) [\|a_1\|_p + \dots + \|a_n\|_p]} \|x\|_p & \leq \|x\|_p^* \\
& \leq (n!)^{1-\frac{1}{p}} \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right]^{\frac{1}{p}} \|x\|_p
\end{aligned}$$

for every $x \in \ell^p$.

Proof. For any $x \in \ell^p$ and any subset $\{i_2, \dots, i_n\}$ of $\{1, 2, \dots, n\}$, we observe that

$$\|x, a_{i_2}, \dots, a_{i_n}\|_p \leq (n!)^{1-1/p} \|x\|_p \|a_{i_2}\|_p \cdots \|a_{i_n}\|_p.$$

Hence

$$\begin{aligned} \|x\|_p^* &= \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}} \\ &\leq (n!)^{1-\frac{1}{p}} \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right]^{\frac{1}{p}} \|x\|_p. \end{aligned}$$

By the inequality in Lemma 2.4, we have

$$\|x\|_p \|a_1, a_2, \dots, a_n\|_p \leq n \|a_1\|_p \|x, a_2, a_3, \dots, a_n\|_p + \|a_2\|_p \|x, a_1, a_3, \dots, a_n\|_p + \cdots + \|a_n\|_p \|x, a_1, a_2, \dots, a_{n-1}\|_p$$

$$\begin{aligned} \|x\|_p \|a_2, a_1, \dots, a_n\|_p &\leq n \|a_2\|_p \|x, a_1, a_3, \dots, a_n\|_p + \|a_1\|_p \|x, a_2, a_3, \dots, a_n\|_p + \cdots + \|a_n\|_p \|x, a_1, a_2, \dots, a_{n-1}\|_p \\ &\vdots \end{aligned}$$

$$\|x\|_p \|a_n, a_1, \dots, a_{n-1}\|_p \leq n \|a_n\|_p \|x, a_1, a_2, \dots, a_{n-1}\|_p + \|a_1\|_p \|x, a_n, a_2, \dots, a_{n-1}\|_p + \cdots + \|a_{n-1}\|_p \|x, a_n, a_1, \dots, a_{n-2}\|_p,$$

whence

$$\begin{aligned} n \|x\|_p \|a_1, a_2, \dots, a_n\|_p &\leq (2n-1) \|a_1\|_p \|x, a_2, a_3, \dots, a_n\|_p + \cdots + (2n-1) \|a_n\|_p \|x, a_1, a_2, \dots, a_{n-1}\|_p. \end{aligned}$$

Next, we observe that

$$\begin{aligned} \|x, a_2, a_3, \dots, a_n\|_p &\leq \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}} = \|x\|_p^* \\ &\vdots \\ \|x, a_1, a_2, \dots, a_{n-1}\|_p &\leq \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x, a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{\frac{1}{p}} = \|x\|_p^*. \end{aligned}$$

Hence, we obtain

$$n \|x\|_p \|a_1, a_2, \dots, a_n\|_p \leq (2n-1) [\|a_1\|_p + \cdots + \|a_n\|_p] \|x\|_p^*.$$

This completes the proof. \square

Since $(\ell^p, \|\cdot\|_p)$ is a Banach space, we have the following corollary.

Corollary 2.6 *The normed space $(\ell^p, \|\cdot\|_p^*)$ is a Banach space.*

Now we come to the proof of our main theorem, namely Theorem 2.1. All the previous results make the proof very simple.

Proof of Theorem 2.1. For any $x, x' \in \ell^p$, we observe that

$$\begin{aligned} \|Tx - Tx'\|_p^* &= \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|Tx - Tx', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\ &\leq \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} C^p \|x - x', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\ &= C \left[\sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|x - x', a_{i_2}, \dots, a_{i_n}\|_p^p \right]^{1/p} \\ &= C \|x - x'\|_p^*. \end{aligned}$$

This shows that T is a contractive mapping with respect to $\|\cdot\|_p^*$. Since $(\ell^p, \|\cdot\|_p^*)$ is a Banach space, by the contractive mapping theorem for Banach spaces we conclude that T has a unique fixed point in ℓ^p . \square

3. FURTHER RESULTS

As a consequence of Proposition 2.5, we also have the following theorem.

Theorem 3.1 *A sequence $(x_k) \in \ell^p$ converges to an $x \in \ell^p$ in $\|\cdot, \dots, \cdot\|_p$ if and only if (x_k) converges to x in $\|\cdot\|_p$. Similarly, $(x_k) \in \ell^p$ is a Cauchy sequence with respect to $\|\cdot, \dots, \cdot\|_p$ if and only if (x_k) is a Cauchy sequence with respect to $\|\cdot\|_p$.*

Proof. Let $(x_k) \in \ell^p$ converge to an $x \in \ell^p$ in $\|\cdot\|_p$, that is, $\lim_{k \rightarrow \infty} \|x_k - x\|_p = 0$. Hence, for every $y_2, \dots, y_n \in \ell^p$, we have (by Lemma 2.2)

$$\lim_{k \rightarrow \infty} \|x_k - x, y_2, \dots, y_n\|_p \leq \lim_{k \rightarrow \infty} (n!)^{1-1/p} \|x_k - x\|_p \|y_2\|_p \cdots \|y_n\|_p = 0.$$

This tells us that (x_k) converges to x in $\|\cdot, \dots, \cdot\|_p$.

Conversely, let $(x_k) \in \ell^p$ converge to $x \in \ell^p$ in $\|\cdot, \dots, \cdot\|_p$, that is,

$$\lim_{k \rightarrow \infty} \|x_k - x, y_2, \dots, y_n\|_p = 0,$$

for every $y_2, \dots, y_n \in \ell^p$. Let $\{a_1, \dots, a_n\}$ be a linearly independent set on ℓ^p and $\|\cdot\|_p^*$ be defined as in Proposition 2.3. Then, for any $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$, we have

$$\lim_{k \rightarrow \infty} \|x_k - x, a_{i_2}, \dots, a_{i_n}\|_p = 0,$$

so that $\lim_{k \rightarrow \infty} \|x_k - x\|_p^* = 0$. By Proposition 2.5, we then conclude that $\lim_{k \rightarrow \infty} \|x_k - x\|_p = 0$, which means that (x_k) converges to x in $\|\cdot\|_p$.

The proof of the second part is similar, and so we are done. \square

Corollary 3.2 *The n -normed space $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ is an n -Banach space.*

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