Discrete Morrey Spaces

H. Gunawan*

*http://personal.fmipa.itb.ac.id/hgunawan/

Analysis and Geometry Group
Bandung Institute of Technology
Bandung, INDONESIA

SNAMA 2016 – Makassar, 14-15 October 2016

Outline

Outline

Outline

Outline

Outline

PART I

Introduction

Many operators that are initially studied on Lebesgue spaces $L^p(\mathbb{R}^d)$ have discrete analogues on $\ell^p(\mathbb{Z}^d)$.

Some of these operators have also been studied on 'continuous' Morrey spaces $M^p_q(\mathbb{R}^d)$.

In this paper, we are interested in studying discrete analogues of Morrey spaces and their generalizations.

In particular, we discuss the inclusion property of these spaces and prove some necessary and sufficient conditions for this property.

The Space ℓ_q^p

Let $m \in \mathbb{Z}$, $N \in \omega := \mathbb{N} \cup \{0\}$, and $S_{m,N} := \{m - N, \ldots, m, \ldots, m + N\}$. Then $|S_{m,N}| = 2N + 1$, the cardinality of $S_{m,N}$.

Let $1 \leq p \leq q < \infty$. We denote by $\ell_q^p = \ell_q^p(\mathbb{Z})$ the set of real (or complex) sequences $x = (x_k)_{k\in\mathbb{Z}}$ such that

$$\|x\|_{\ell_q^p} := \sup_{m\in\mathbb{Z},N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} < \infty.$$

Clearly ℓ_q^p is a vector space, which we call the *discrete Morrey space*.

When $p = q$, we have $\ell_p^p = \ell^p$, the space of p-summable sequences with integer indices.

A Larger Space than ℓ^p

For a sequence x to be in ℓ^p_q, x has to have some decay, but not as fast as those in ℓ^p. In general, for $p < q$, ℓ^p_q is a larger space than ℓ^p.

Proposition. For $1 \le p \le q < \infty$, we have $\ell^p \subseteq \ell^p_q$.

Proof. We have for all $m \in \mathbb{Z}$ and $N \in \omega$, $0 < |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \le 1$, and thus

$$|S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} \le \left(\sum_{k \in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}}.$$

Taking the supremum over $m \in \mathbb{Z}$ and $N \in \omega$, we get $\|x\|_{\ell^p_q} \le \|x\|_{\ell^p}$.

The Inclusion is Proper

Let $1 \le p < q < \infty$. Consider the sequence $x = (x_k)_{k\in\mathbb{Z}}$ given by $x_k = |k|^{-1/q}$ when $k \neq 0$ and $x_0 = 1$. Since $p/q < 1$, the series

$$\sum_{k\in\mathbb{Z}} |x_k|^p = 1 + 2\sum_{k=1}^{\infty} \frac{1}{k^{\frac{p}{q}}}$$

is divergent, thus $x \notin \ell^p(\mathbb{Z})$.

Next, for all $m \in \mathbb{Z}$ and $N \in \mathbb{N}$, we have

$$\sum_{k\in S_{m,N}} |x_k|^p = \sum_{k\in S_{0,N}} |x_k|^p \le 1 + 2\sum_{k=1}^{N} \frac{1}{k^{\frac{p}{q}}}.$$

Estimating the above sum by $\int_1^N x^{-\frac{p}{q}}\, dx$ and multiplying by $|S_{m,N}|^{\frac{p}{q}-1}$, one can show that $x \in \ell^p_q$.

It is a Banach Space

Proposition. For $1 \leq p \leq q < \infty$, the mapping $\| \cdot \|_{\ell^p_q}$ defines a norm on ℓ^p_q. Moreover, $(\ell^p_q, \| \cdot \|_{\ell^p_q})$ is a Banach space.

An Inequality

The following lemma will be useful in studying the relation between two discrete Morrey spaces.

Lemma. For all $1 \le p_1 \le p_2 < \infty$, $m \in \mathbb{Z}$, and $N \in \omega$, we have

$$\left(\frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \le \left(\frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x_k|^{p_2} \right)^{\frac{1}{p_2}},$$

where $x_k \in \mathbb{R}$ for all $k \in S_{m,N}$.

Monotonicity

Proposition. For all $1 \leq p_1 \leq p_2 \leq q < \infty$, we have $\ell_q^{p_2} \subseteq \ell_q^{p_1}$ with

$$\|x\|_{\ell_q^{p_1}} \leq \|x\|_{\ell_q^{p_2}}$$

for every $x \in \ell_q^{p_2}$.

Weak Morrey Spaces

For $1 \le p \le q < \infty$, we define the *weak discrete Morrey space* $w\ell^p_q$ to be the set of all real (or complex) sequences $x = (x_k)_{k\in\mathbb{Z}}$ for which $\|x\|_{w\ell^p_q} < \infty$, where $\|\cdot\|_{w\ell^p_q}$ given by

$$\|x\|_{w\ell^p_q} = \sup_{m\in\mathbb{Z}, N\in\omega, \gamma>0} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \gamma \Big|\{k \in S_{m,N} : \ |x_k| > \gamma\}\Big|^{\frac{1}{p}}.$$

Note that when $p = q$, we have $w\ell^p := w\ell^p_p$, which is the weak ℓ^p space.

The Weak Contains the Strong

The following example shows that the weak type spaces have more elements than the (strong) discrete Morrey spaces.

The sequence $x = (x_k)_{k\in\mathbb{Z}}$ given by $x_k = |k|^{-1/p}$ when $k \neq 0$ and $x_0 = 1$ is not in ℓ^p. Nevertheless, for any $m \in \mathbb{Z},\ N \in \omega, \gamma \in (0,1]$, we have

$$\gamma|\{k \in S_{m,N} : |x_k| > \gamma\}|^{\frac{1}{p}} \leq \gamma\big[1 + 2|\{k \in \{1,\ldots,N\} \ : \ k^{-\frac{1}{p}} > \gamma\}|\big]^{\frac{1}{p}}$$
$$< \gamma \cdot \big[1 + \frac{2}{\gamma}\big] \leq 3.$$

Thus $(x_k)_{k\in\mathbb{Z}}$ is in $w\ell^p_p$.

A Larger Space than ℓ_q^p

Theorem. For $1 \leq p \leq q < \infty$, $\ell_q^p \subseteq w\ell_q^p$ and $\|x\|_{w\ell_q^p} \leq \|x\|_{\ell_q^p}$ for every $x \in \ell_q^p$.

Proof. Let $x \in \ell_q^p$, $\gamma > 0$, $m \in \mathbb{Z}$, and $N \in \omega$. We have,

$$\begin{aligned} & |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\gamma\Big|\{k \in S_{m,N} : \ |x_k| > \gamma\}\Big|^{\frac{1}{p}} \\ = \ & |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_{m,N},|x_k|>\gamma} \gamma^p\right)^{\frac{1}{p}} \\ \leq \ & |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}}\left(\sum_{k\in S_{m,N}} |x_k|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Taking the supremum over $m \in \mathbb{Z}$, $N \in \omega$, and $\gamma > 0$, we obtain $\|x\|_{w\ell_q^p} \leq \|x\|_{\ell_q^p}$. Therefore, if $x \in \ell_q^p$, then $x \in w\ell_q^p$.

It is a Quasi-Normed Space

Proposition. For $1 \leq p \leq q < \infty$, $\| \cdot \|_{w\ell_q^p}$ is a quasi-norm, so that $(w\ell_q^p, \| \cdot \|_{w\ell_q^p})$ is a quasi-normed space.

Remark. At the present we do not know whether $w\ell_q^p$ is complete or not with respect to the quasi-norm $\| \cdot \|_{w\ell_q^p}$.

Inclusion between Two Weak Discrete Morrey Spaces

Proposition Let $1 \leq p_1 \leq p_2 \leq q < \infty$. Then, $w\ell_q^{p_2} \subseteq w\ell_q^{p_1}$ and $\|x\|_{w\ell_q^{p_1}} \leq \|x\|_{w\ell_q^{p_2}}$ for every $x \in w\ell_q^{p_2}$.

PART II

The Class $\mathcal{G}_p$

By $\lesssim$ and $\gtrsim$ we mean that the inequalities are satisfied up to a constant $C > 0$, that is, $x \lesssim y$ means $x \leq Cy$ for some $C > 0$.

We denote by $\mathcal{G}_p$ the set of all function $\phi : (0, \infty) \to (0, \infty)$ such that ϕ is almost decreasing ($r \leq s$ implies that $\phi(r) \gtrsim \phi(s)$), and the mapping $t \mapsto t^{\frac{1}{p}}\phi(t)$ is almost increasing (that is, $r \leq s$ implies that $r^{\frac{1}{p}}\phi(r) \lesssim s^{\frac{1}{p}}\phi(s)$).

Note that $\phi \in \mathcal{G}_p$ implies that ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that

$$\frac{1}{C} \leq \frac{\phi(r)}{\phi(s)} \leq C$$

for every $\frac{1}{2} \leq \frac{r}{s} \leq 2$.

Generalized Discrete Morrey Spaces

For $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$, the *generalized discrete Morrey space* ℓ^p_ϕ is defined as the set of all real (or complex) sequences $x = (x_k)_{k=1}^\infty$ such that

$$\|x\|_{\ell^p_\phi} = \sup_{m\in\mathbb{Z}, N\in\omega} \frac{1}{\phi(2N+1)} \left(\frac{1}{|S_{m,N}|} \sum_{k\in S_{m,N}} |x_k|^p \right)^{\frac{1}{p}} < \infty.$$

Note that the discrete Morrey space ℓ^p_q $(1 \le p \le q < \infty)$ may be obtained from ℓ^p_ϕ by choosing the function $\phi(r) = r^{-\frac{1}{q}}$.

A Lemma for the Characteristic Sequence

Lemma. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. For $m_0 \in \mathbb{Z}$ and $N_0 \in \omega$, let ξ^{m_0,N_0} be the characteristic sequence given by

$$\xi_k^{m_0,N_0} = \begin{cases} 1, & \text{if } k \in S_{m_0,N_0}; \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Then,

$$\frac{1}{\phi(2N_0+1)} \leq \|\xi^{m_0,N_0}\|_{\ell^p_\phi} \lesssim \frac{1}{\phi(2N_0+1)}.$$

Strong Equivalence

Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$ and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:

(i) $\phi_2(2N+1) \lesssim \phi_1(2N+1)$, for all $N \in \omega$.

(ii) $\ell^{p_2}_{\phi_2} \subseteq \ell^{p_1}_{\phi_1}$ with $\|x\|_{\ell^{p_1}_{\phi_1}} \lesssim \|x\|_{\ell^{p_2}_{\phi_2}}$ for every $x \in \ell^{p_2}_{\phi_2}$.

The Proof

Suppose (i) holds. Let $x \in \ell^{p_2}_{\phi_2}$. For any $m \in \mathbb{Z}$ and $N \in \omega$, we have

$$\begin{aligned} & \frac{1}{\phi_1(2N+1)} \left(\frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \\ \lesssim \; & \frac{1}{\phi_2(2N+1)} \left(\frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x_k|^{p_1} \right)^{\frac{1}{p_1}} \\ \lesssim \; & \frac{1}{\phi_2(2N+1)} \left(\frac{1}{|S_{m,N}|} \sum_{k \in S_{m,N}} |x_k|^{p_2} \right)^{\frac{1}{p_2}} . \end{aligned}$$

Note that we use the monotonicity in the last inequality. Taking the supremum over $m \in \mathbb{Z}$ and $N \in \omega$, we obtain $\|x\|_{\ell^{p_1}_{\phi_1}} \lesssim \|x\|_{\ell^{p_2}_{\phi_2}}$.

The Proof (Continued)

Now suppose (ii) holds. Let $m_0 \in \mathbb{Z}$, $N_0 \in \omega$, and ξ^{m_0,N_0} be the characteristic sequence defined by (1) as in previous lemma. Then, $\|\xi^{m_0,N_0}\|_{\ell^{p_1}_{\phi_1}} \lesssim \|\xi^{m_0,N_0}\|_{\ell^{p_2}_{\phi_2}}$ by our assumption. The lemma for the characteristic function gives us

$$\frac{1}{\phi_1(2N_0+1)} \le \|\xi^{m_0,N_0}\|_{\ell^{p_1}_{\phi_1}} \quad \text{and} \quad \|\xi^{m_0,N_0}\|_{\ell^{p_2}_{\phi_2}} \lesssim \frac{1}{\phi_2(2N_0+1)}.$$

We conclude that

$$\frac{1}{\phi_1(2N_0+1)} \lesssim \frac{1}{\phi_2(2N_0+1)} \quad \text{or} \quad \phi_2(2N_0+1) \lesssim \phi_1(2N_0+1);$$

and this completes the proof since we choose arbitrary $N_0 \in \omega$.

Generalized Weak Discrete Morrey Spaces

For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, the *generalized weak discrete Morrey space* $w\ell^p_\phi$ is the set of all real (or complex) sequences $x = (x_k)_{k \in \mathbb{Z}}$ for which $\|x\|_{w\ell^p_\phi} < \infty$, and $\| \cdot \|_{w\ell^p_\phi}$ is defined by

$$\|x\|_{w\ell^p_\phi} = \sup_{m \in \mathbb{Z}, N \in \omega, \gamma > 0} \frac{\gamma}{\phi(2N+1)} \left(\frac{\left|\{k \in S_{m,N} : \ |x_k| > \gamma\}\right|}{|S_{m,N}|} \right)^{\frac{1}{p}}.$$

Proposition. For $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, $\ell^p_\phi \subseteq w\ell^p_\phi$ with

$$\|x\|_{w\ell^p_\phi} \leq \|x\|_{\ell^p_\phi}$$

for every $x \in \ell^p_\phi$.

Another Lemma for the Characteristic Sequence

Lemma. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. If $m_0 \in \mathbb{Z}$ and $N_0 \in \omega$, and ξ^{m_0,N_0} is the characteristic sequence defined by (1). Then,

$$\frac{1}{2\phi(2N_0+1)} \leq \|\xi^{m_0,N_0}\|_{w\ell^p_\phi} \lesssim \frac{1}{\phi(2N_0+1)}.$$

Weak Equivalence

Theorem. Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$ and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:

(i) $\phi_2(2N+1) \lesssim \phi_1(2N+1)$, for all $N \in \omega$.

(ii) $w\ell^{p_2}_{\phi_2} \subseteq w\ell^{p_1}_{\phi_1}$ with $\|x\|_{w\ell^{p_1}_{\phi_1}} \lesssim \|x\|_{w\ell^{p_2}_{\phi_2}}$ for every $x \in w\ell^{p_2}_{\phi_2}$.

Acknowledgement

The work is joint with Eder Kikianty, and is supported by ITB Research and Innovation Program 2016.

REFERENCES

D.R. Adams, "A note on Riesz potentials", *Duke Math. J.* **42** (1975), 765–778.

F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7** (1987), 273–279.

H. Gunawan, "A note on the generalized fractional integral operator", *J. Indones. Math. Soc.* **9** (2003), no. 1, 39–43.

O. Kovrizhkin, "On the norms of discrete analogues of convolution operators", *Proc. Amer. Math. Soc.* **140** (2012), 1349–1352.

A. Magyar, E.M. Stein, and S. Wainger, "Discrete analogues in harmonic analysis: spherical averages", *Annals Math.* **155** (2002), 189–208.

E. Nakai, "Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces", *Math. Nachr.* **166** (1994), 95–103.

D.M. Oberlin, "Two discrete fractional integrals", *Math. Res. Lett.* **8** (2001), 1–6.

A. Osançlıol, "Inclusions between weighted Orlicz spaces", *J. Inequal. Appl.* **2014**:390 (2014), 8 pp.

Y. Sawano and H. Tanaka, "Morrey space for non-doubling measures", *Acta Math. Sinica* **21** (2005), no. 6, 1535–1544.

E.M. Stein and S. Wainger, "Discrete analogues of singular Radon transform", *Bull. Amer. Math. Soc.* **23** (1990), 537–544.

E.M. Stein and S. Wainger, "Discrete analogues in harmonic analysis I: ℓ^2 estimates for singular Radon transforms", *Amer. J. Math.* **21** (1999), 1291–1336.

E.M. Stein and S. Wainger, "Discrete analogues in harmonic analysis II: fractional integration", *J. d'Analyse Math.* **80** (2000), 335–355.

E.M. Stein and S. Wainger, "Two discrete fractional integral operators revisited", *J. d'Analyse Math.* **87** (2002), 451–479.