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Application of reproducing kernel Hilbert spaces to a minimization problem with prescribed nodes

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Abstract. The aim of this paper is to apply the theory of reproducing kernel Hilbert spaces to a minimization problem with prescribed nodes. We reprove and at the same time generalize some results previously obtained by Gunawan *et al.* [2,3]. In addition, we also discuss the H lder continuity of the solution to the problem.

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1. Introduction

In the present paper we are interested in finding the solution to the following minimization problem on $\mathbb{R}^d$.

Let $0 \le \alpha < \infty$. We define a Hilbert space H_α to be the set of functions f on $[0,1]^d$ of the form

$$f(x_1, \dots, x_n) := \sum_{m_1,\dots,m_d \in \mathbb{N}} a_{m_1 \cdots m_d} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d)$$

for which

$$\|f\|_{H_\alpha} := \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d \in \mathbb{N}} (m_1^2 + \cdots + m_d^2)^\alpha |a_{m_1 \cdots m_d}|^2 < \infty.$$

The above norm is induced from the inner product

$$\langle f, g \rangle_{H_\alpha} = \frac{\pi^{2\alpha}}{2^d} \sum_{m_1,\dots,m_d \in \mathbb{N}} (m_1^2 + \cdots + m_d^2)^\alpha a_{m_1 \cdots m_d} b_{m_1 \cdots m_d},$$

where $a_{m_1 \cdots m_d}$ and $b_{m_1 \cdots m_d}$ are the coefficients of f and g, respectively.

We are then interested in studying the following problem:

$$\text{Minimize } \|f\|_{H_\alpha}$$

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subject to the prescribed nodes:

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N,$$

where $\mathbf{p}_k := (p_{k1}, \ldots, p_{kd}) \in (0,1)^d$ and $c_k \in \mathbb{R}$ are given. [Note here that the points $\mathbf{p}_k$'s are 'inside' the unit cube $[0,1]^d$.]

The 1- and 2-dimensional case have been studied by Gunawan *et al.* [2,3]. They show, among others, that the value $\alpha > \frac{d}{2}$ is a necessary and sufficient condition for the solution to the above problem to be continuous. In this note, we shall use the theory of reproducing kernel Hilbert spaces to study the problem (in a more general setting). Our first result is the following theorem.

Theorem 1.1. *Let $\alpha > \frac{d}{2}$. The solution to the minimization problem*

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(p_1, \ldots, p_d) = 1,$$

is given by $F(x_1, \ldots, x_d) :=$

$$A \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d).$$

where $A^{-1} := \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{(\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d))^2}{(m_1^2 + \cdots + m_d^2)^\alpha}$.

We shall give the proof of this theorem in the next section. A more general result will also be presented. In the last section, we shall also consider the H lder continuity of the solution, by using the inclusionship between Besov spaces and modulation spaces.

2. Main Results

Let E be a compact subspace of $\mathbb{R}^d$ (containing at least N points) and $K : E \times E \to \mathbb{F}$ be a positive definite kernel, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by H_K the corresponding reproducing kernel Hilbert space, which is defined as the completion of the pre-Hilbert space $H_K^0 := \text{span}_{\mathbb{F}}\{K(\cdot, \mathbf{p}) : \mathbf{p} \in E\}$ — equipped uniquely with the inner product $\langle \cdot, \cdot \rangle_{H_K^0}$ so that

$$\langle K(\cdot, \mathbf{p}), K(\cdot, \mathbf{q}) \rangle_{H_K^0} = K(\mathbf{q}, \mathbf{p}),$$

for every $\mathbf{p}, \mathbf{q} \in E$. A well-known fact for the theory of reproducing kernel Hilbert spaces is that

$$f(\mathbf{p}) = \langle f, K(\cdot, \mathbf{p}) \rangle_{H_K}$$

for every $f \in H_K$ and $\mathbf{p} \in E$. Accordingly, we have the following proposition.

Proposition 2.1. *For every p, we have*

$$\{f \in H_K \,:\, f \perp K(\cdot,\mathbf{p})\} = \{f \in H_K \,:\, f(\mathbf{p}) = 0\}.$$

As a direct consequence, we obtain the following result. Although this is known (see e.g. [x]), we give the proof for convenience.

Proposition 2.2. *Let* $\mathbf{p} \in E$*, so that* $K(\mathbf{p},\mathbf{p}) > 0$*. Then the minimization problem:*

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}) = 1,$$

has a unique solution given by $F := \dfrac{K(\cdot,\mathbf{p})}{K(\mathbf{p},\mathbf{p})}$.

Proof. Clearly $F(\mathbf{p}) = 1$. Now, if a function $g \in H_K$ satisfies $g(\mathbf{p}) = 1$, then

$$g - F \in \{f \in H_K \,:\, f(\mathbf{p}) = 0\}.$$

By Proposition 2.1, we have $g - F \perp K(\cdot,\mathbf{p})$, and accordingly $g - F \perp F$. It then follows that

$$\begin{aligned}\|g\|_{H_K}^2 &= \|g - F\|_{H_K}^2 + 2\mathrm{Re}\langle g - F, F\rangle_{H_K} + \|F\|_{H_K}^2 \\ &= \|g - F\|_{H_K}^2 + \|F\|_{H_K}^2 \\ &\geq \|F\|_{H_K}^2,\end{aligned}$$

and the equality is attained if and only if $g = F$. □

Theorem 1.1 can now be seen as a corollary of Proposition 2.2. Indeed, for $\mathbf{p} \in E := [0,1]^d$, write

$$K_\alpha(\mathbf{x},\mathbf{p}) := \frac{2^d}{\pi^{2\alpha}} \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\prod_{i=1}^d \sin(m_i \pi p_i)}{(m_1^2 + \cdots + m_d^2)^\alpha} \prod_{i=1}^d \sin(m_i \pi x_i),$$

where $\mathbf{x} = (x_1,\ldots,x_d) \in E$. Observe that $K(\mathbf{p},\mathbf{p}) > 0$. Next, if $f(\mathbf{x}) = \prod_{i=1}^d \sin(M_i \pi x_i)$, then

$$\begin{aligned}&\langle f, K_\alpha(\cdot,\mathbf{p})\rangle \\ &\quad= \left\langle \prod_{i=1}^d \sin(M_i \pi \cdot_i), K(\cdot,\mathbf{p}) \right\rangle_{H_\alpha} \\ &\quad= \frac{2^d}{\pi^{2\alpha}} \frac{\prod_{i=1}^d \sin(M_i \pi p_i)}{(M_1^2 + \cdots + M_d^2)^\alpha} \left\langle \prod_{i=1}^d \sin(M_i \pi \cdot_i), \prod_{i=1}^d \sin(M_i \pi \cdot_i) \right\rangle_{H_\alpha} \\ &\quad= \prod_{i=1}^d \sin(M_i \pi p_i) \\ &\quad= f(\mathbf{p}).\end{aligned}$$

It follows that $f(\mathbf{p}) = \langle f, K_\alpha(\cdot, \mathbf{p})\rangle$ for every $f \in H_\alpha$ and $p \in E$. This shows that H_α is the reproducing kernel Hilbert space with kernel K_α. Hence, the solution to the minimization problem

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(\mathbf{p}) = 1,$$

is $F(\mathbf{x}) := \frac{K(\mathbf{x},\mathbf{p})}{K(\mathbf{p},\mathbf{p})}$, namely

$$F(\mathbf{x}) = A \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\sin(m_1\pi p_1)\cdots\sin(m_d\pi p_d)}{(m_1^2+\cdots+m_d^2)^\alpha} \sin(m_1\pi x_1)\cdots\sin(m_d\pi x_d),$$

where $A^{-1} := \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{(\sin(m_1\pi p_1)\cdots\sin(m_d\pi p_d))^2}{(m_1^2+\cdots+m_d^2)^\alpha}$.

A more general result than Proposition 2.2 is presented as the following proposition. Again, this result is known; and we give the proof for convenience.

Proposition 2.3. *Suppose that we are given a finite set* $\{c_k : k = 1, \ldots, N\} \subset \mathbb{F}$ *and* $P := \{\mathbf{p}_k : k = 1, \ldots, N\} \subset E$ *such that the matrix* $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ *is positive definite. Then the minimization problem*

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N,$$

has a unique solution that lies in $\operatorname{span}\{f_k : k = 1, \ldots, N\}$*, where* $f_k := K(\cdot, \mathbf{p}_k)$*.*

Proof. The system of linear equations

$$b_1 f_1(\mathbf{p}_j) + \cdots + b_N f_N(\mathbf{p}_j) = c_j, \quad j = 1, \ldots, N$$

is equivalent to

$$b_1 K(\mathbf{p}_j, \mathbf{p}_1) + \cdots + b_N K(\mathbf{p}_j, \mathbf{p}_N) = c_j, \quad j = 1, \ldots, N.$$

Since $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ is positive definite, the system has a unique solution, say $F := b_1 f_1 + \cdots + b_N f_N$.

To prove that the norm is minimized, let $g \in H_K$ satisfy $g(\mathbf{p}_k) = c_k$ for $k = 1, \ldots, N$. Then, for each k, we have $(g - F)(\mathbf{p}_k) = 0$, so that $\langle g - F, f_k\rangle = 0$ or $g - F \perp f_k$. It thus follows that $g - F \perp F$, whence

$$\|g\|_{H_K}^2 = \|g - F\|_{H_K}^2 + \|F\|_{H_K}^2 \geq \|F\|_{H_K}^2,$$

and the equality is attained if and only if $g = F$. □

As a corollary, we have the following theorem on our original interest.

Theorem 2.1. *Given $\mathbf{p}_k \in (0,1)^d$ and $c_k \in \mathbb{R}$ for $k = 1, \ldots, N$, the minimization problem:*

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to the prescribed nodes:

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N,$$

has a unique solution $F := b_1 f_1 + \cdots + b_N f_N$, where $f_k := K_\alpha(\cdot, \mathbf{p}_k)$, $k = 1, \ldots, N$.

Proof. We only have to make sure that the matrix $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ is positive definite for both the existence and the uniqueness of the solution. Since the matrix is equal to the Gram matrix $[\langle K(\cdot, \mathbf{p}_k), K(\cdot, \mathbf{p}_j)\rangle_{H_\alpha}]_{j,k=1}^N$, it is sufficient to show that its determinant is nonzero. But this is so, because $\{\sin \pi x, \ldots, \sin N\pi x\}$ forms a Chebyshev system (see [4]), and the product of such Chebyshev systems can always be used to interpolate data on any rectangular grid 'inside' the cube $[0,1]^d$.

To illustrate, let us take a look at the 1-dimensional case. (For higher dimensional cases, we refer the reader to [1].) Our task reduces to verifying the linearly independence of the functions $K(\cdot, p_1), \ldots, K(\cdot, p_N)$. Recall that for $k = 1, \ldots, N$ we have

$$K(x, p_k) = \sum_{m=1}^{\infty} \frac{\sin m\pi p_k}{m^{2\alpha}} \sin m\pi x, \quad x \in [0,1].$$

Now observe that the parsial sums

$$K_N(x, p_k) = \sum_{m=1}^{N} \frac{\sin m\pi p_k}{m^{2\alpha}} \sin m\pi x, \quad x \in [0,1],$$

are linearly independent since

$$\begin{vmatrix} \sin \pi p_1 & \frac{1}{2^{2\alpha}} \sin 2\pi p_1 & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_1 \\ \sin \pi p_2 & \frac{1}{2^{2\alpha}} \sin 2\pi p_2 & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_2 \\ \vdots & \vdots & \ddots & \vdots \\ \sin \pi p_N & \frac{1}{2^{2\alpha}} \sin 2\pi p_N & \cdots & \frac{1}{N^{2\alpha}} \sin N\pi p_N \end{vmatrix} \neq 0.$$

Hence the functions $K(\cdot, p_1), \ldots, K(\cdot, p_N)$ must be linearly independent. □

3. H lder continuity

We have seen that the solution to the minimization problem with several prescribed nodes is a linear combination of the minimizers with one prescribed node. Hence, to study its H lder continuity, it suffices for us to investigate the H lder continuity of the minimizer with one prescribed node, whose formula is given in Theorem 1.1.

Theorem 3.1. *Let $\alpha > \frac{d}{2}$ and F be the solution to the minimization problem as in Theorem 1.1. Put $\beta := \alpha - \frac{d}{2}$.*

1. *If $\beta \in \mathbb{N}$, then $F \in C^{\beta-1}(\mathbb{R}^d)$ with bounded partial derivatives up to order $\beta - 1$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{(\beta-1)}(x) + F^{(\beta-1)}(y) - 2F^{(\beta-1)}((x+y)/2)|}{|x-y|} < \infty.$$

2. *If $\beta \in (0,\infty) \setminus \mathbb{N}$, then $F \in C^{[\beta]}(\mathbb{R}^d)$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{([\beta])}(x) + F^{([\beta])}(y) - 2F^{([\beta])}((x+y)/2)|}{|x-y|^{\beta-[\beta]}} < \infty.$$

Theorem 3.1 depends upon the following lemmas. We denote by $C^\infty_{\text{comp}}(\mathbb{R}^d)$ the set of all compactly supported smooth functions and by $B(r)$ the open ball given by $\{x \in \mathbb{R}^d : |x| < r\}$ for $r > 0$.

Lemma 3.1 ([6, Theorem 4])**.** *Let $\beta > 0$ and let $\psi \in C^\infty_{\text{comp}}(\mathbb{R}^d)$ such that*

$$\chi_{B(1)} \le \psi \le \chi_{B(2)}.$$

Define $\varphi_j(\xi) = \psi(2^{-j}\xi) - \psi(2^{-j+1}\xi)$ for $\xi \in \mathbb{R}^d$ and $j \in \mathbb{N}$.

1. *Let $G \in L^\infty(\mathbb{R}^d)$ and $\beta \in \mathbb{N}$. Then $G \in C^{\beta-1}(\mathbb{R}^d)$ with bounded partial derivatives up to order $\beta - 1$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|G^{(\beta-1)}(x) + G^{(\beta-1)}(y) - 2G^{(\beta-1)}((x+y)/2)|}{|x-y|} < \infty$$

if and only if

$$\|G\|_{B^\beta_{\infty,\infty}} = \|\mathscr{F}^{-1}[\psi \cdot \mathscr{F}G]\|_\infty + \sum_{j=1}^{\infty} 2^{j\beta} \|\mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F}G]\|_\infty < \infty.$$

2. *If $\beta \in (0,\infty) \setminus \mathbb{N}$, then $G \in C^{[\beta]}(\mathbb{R}^d)$ with bounded partial derivatives up to order $[\beta]$ and*

$$\sup_{x,y\in\mathbb{R}^d} \frac{|G^{([\beta])}(x) + G^{([\beta])}(y) - 2G^{([\beta])}((x+y)/2)|}{|x-y|^{\beta-[\beta]}} < \infty$$

if and only if

$$\|G\|_{B^\beta_{\infty,\infty}} = \|\mathscr{F}^{-1}[\psi \cdot \mathscr{F}G]\|_\infty + \sum_{j=1}^{\infty} 2^{j\beta} \|\mathscr{F}^{-1}[\varphi_j \cdot \mathscr{F}G]\|_\infty < \infty.$$

Note. The norm $\|\cdot\|_{B^{\beta}_{\infty,\infty}}$ in the above lemma is called the *Besov norm*.

Lemma 3.2 ([5]). *Keep the same notation as in Lemma* 3.1 *and, assuming that* $\alpha > \frac{d}{2}$, *put* $\beta := \alpha - \frac{d}{2}$. *Define*

$$\tau_m = \psi(\cdot - m) \quad (m \in \mathbb{Z}^d)$$

and, for $G \in L^{\infty}(\mathbb{R}^d)$ *and* $s \in \mathbb{R}$,

$$\|G\|_{M^s_{\infty,2}} = \left(\sum_{m \in \mathbb{Z}^d} (1+|m|)^{2s} \|\mathscr{F}^{-1}[\tau_m \cdot \mathscr{F}G]\|_{\infty} \right)^{1/2}.$$

Then there exists a constant $C > 0$ *such that*

$$\|G\|_{B^{\beta}_{\infty,\infty}} \le C\|G\|_{M^{\alpha}_{\infty,2}}$$

for all $G \in L^{\infty}(\mathbb{R}^d)$.

With the above lemma, we observe that

$$\|F\|_{B^{\alpha-d/2}_{\infty,\infty}} \le C\|F\|_{M^{\alpha}_{\infty,2}} \le C\|F\|_{H_\alpha} < \infty,$$

where we obtain the proof of Theorem 3.1.

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