Application of RKHS Theory to a Minimization Problem with Prescribed Nodes

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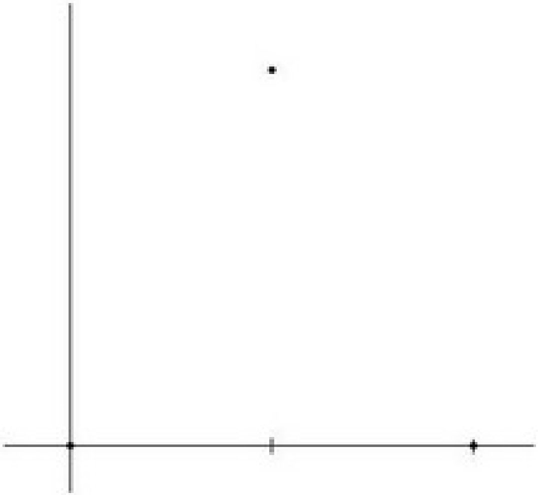
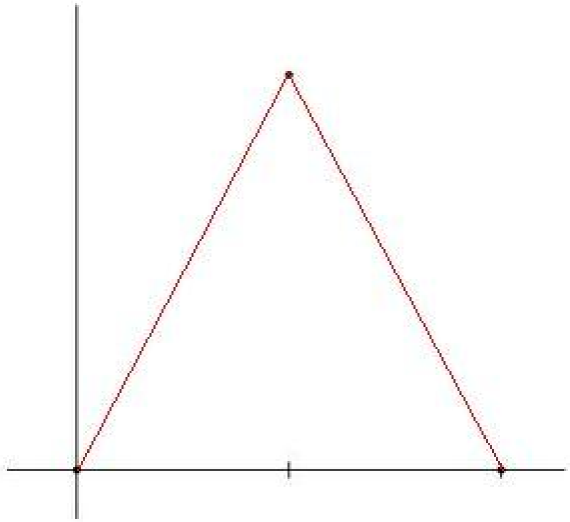
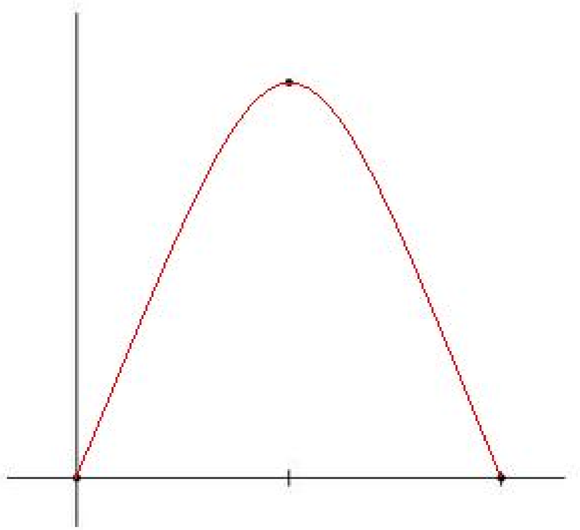
1-D Minimization Problem







Given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots, (x_N, c_N)$ with

$$0 = x_0 < x_1 < \cdots < x_N = 1 \text{ and } c_0 = c_N = 0,$$

we are interested in finding a sufficiently smooth function u on $[0, 1]$ that passes through the given points and minimizes an energy integral.

For example, we want u that minimizes the energy integral

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx,$$

where $u^{(\alpha)}$ denotes the fractional derivative of u of order α.

For $\alpha = 1$, the energy

$$E_1(u) := \int_0^1 |u'(x)|^2 dx.$$

represents the tension (or the potential energy of axial load) of u.

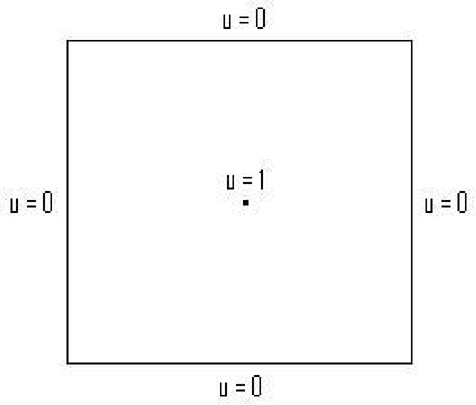
For $\alpha = 2$, the energy

$$E_2(u) := \int_0^1 |u''(x)|^2 dx.$$

represents the curvature (or the strain energy of bending) of u.

2-D Minimization Problem

Find a cts fn $u : [0,1]^2 \to \mathbb{R}$ which minimizes the energy $E_\alpha(u)$ and satisfies the boundary and the interior conditions:



An Energy Minimizing Surface

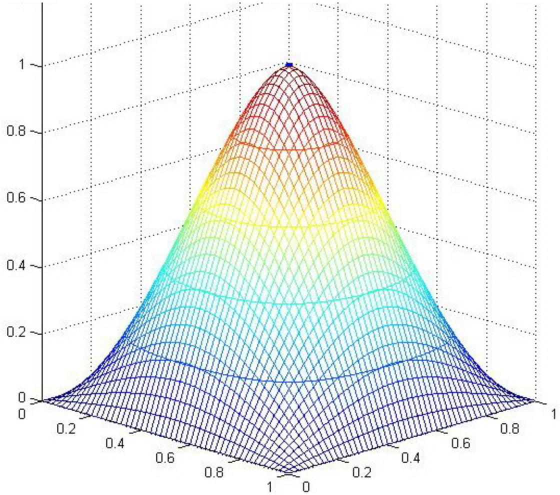
The Energy Integral $E_\alpha(\cdot)$

For $\alpha = 2$, the energy $E_\alpha(u) = E_2(u)$ is given by

$$E_2(u) := \int_0^1 \int_0^1 |-\Delta u(x,y)|^2 dx\, dy.$$

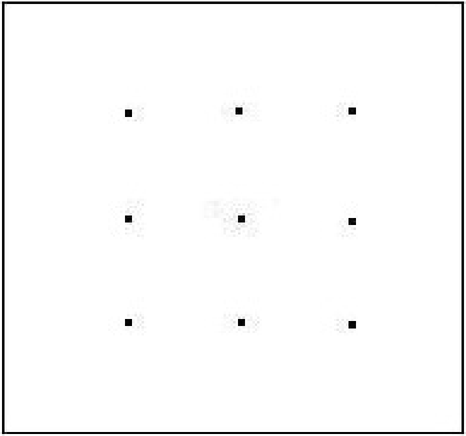
Here $-\Delta$ denotes the positive definite Laplacian in $\mathbb{R}^2$, and $E_2(u)$ represents the (total) *curvature* of u on $[0,1]^2$.

The surface passing through (0.5,0.5,1) with minimum curvature



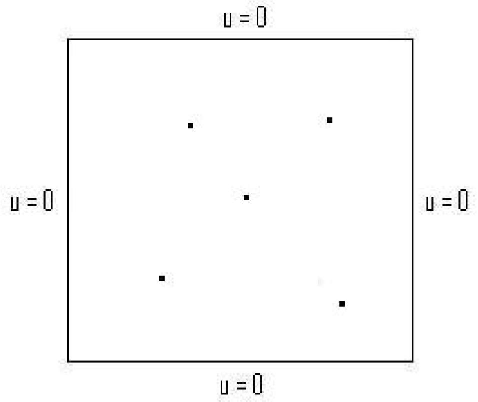
The Interior Conditions

In general, the interior conditions may be prescribed at $M \times N$ points (x_i, y_j), say $u(x_i, y_j) = c_{ij}$, $i = 1, \ldots, M$, $j = 1, \ldots, N$. E.g., for $M = N = 3$:



The Interior Conditions

More generally, the interior conditions can be prescribed at K points inside the unit square. For example:



Since we are looking for a function $u(x,y)$ which vanishes at the boundary, we write $u(x,y)$ as a double Fourier sine series, that is,

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin m\pi x \sin n\pi y,$$

where $a_{m,n}$'s are the Fourier coefficients of u. Through Parseval's identity, the energy integral to be minimized is

$$E_\alpha(u) := \int_0^1 \int_0^1 |(-\Delta)^{\frac{\alpha}{2}} u(x,y)|^2 dx\, dy = \frac{\pi^{2\alpha}}{4} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (m^2+n^2)^\alpha a_{m,n}^2.$$

The 1- and 2-dimensional case have been studied by Gunawan *et al.* [3, 4]. In this talk, we shall present the use the theory of reproducing kernel Hilbert spaces to study the problem (in a more general setting).

The d-Dimensional Problem

Let $0 \leq \alpha < \infty$. We define a Hilbert space H_α to be the set of functions f on $[0,1]^d$ of the form

$$f(x_1, \ldots, x_n) := \sum_{m_1, \ldots, m_d \in \mathbb{N}} a_{m_1 \cdots m_d} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d)$$

for which

$$\|f\|_{H_\alpha} := \frac{\pi^{2\alpha}}{2^d} \sum_{m_1, \ldots, m_d \in \mathbb{N}} (m_1^2 + \cdots + m_d^2)^\alpha |a_{m_1 \cdots m_d}|^2 < \infty.$$

The above norm is induced from the inner product

$$\langle f, g \rangle_{H_\alpha} = \frac{\pi^{2\alpha}}{2^d} \sum_{m_1, \ldots, m_d \in \mathbb{N}} (m_1^2 + \cdots + m_d^2)^\alpha a_{m_1 \cdots m_d} b_{m_1 \cdots m_d},$$

where $a_{m_1 \cdots m_d}$ and $b_{m_1 \cdots m_d}$ are the coefficients of f and g, respectively.

We are then interested in studying the following problem:

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to the prescribed nodes:

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \dots, N,$$

where $\mathbf{p}_k := (p_{k1}, \dots, p_{kd}) \in (0,1)^d$ and $c_k \in \mathbb{R}$ are given. [Note here that the points $\mathbf{p}_k$'s are 'inside' the unit cube $[0,1]^d$.]

Our first result is the following:

Theorem 1. Let $\alpha > \frac{d}{2}$. The solution to the minimization problem

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(p_1, \ldots, p_d) = 1,$$

is given by $F(x_1, \ldots, x_d) :=$

$$A \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \sin(m_1 \pi x_1) \cdots \sin(m_d \pi x_d).$$

where $A^{-1} := \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{(\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d))^2}{(m_1^2 + \cdots + m_d^2)^\alpha}$.

RKHS Theory

Let E be a compact subspace of $\mathbb{R}^d$ (containing at least N points) and $K : E \times E \to \mathbb{F}$ be a positive definite kernel, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by H_K the corresponding reproducing kernel Hilbert space, which is defined as the completion of the pre-Hilbert space $H_K^0 := \mathrm{span}_{\mathbb{F}}\{K(\cdot, \mathbf{p}) \,:\, \mathbf{p} \in E\}$ — equipped uniquely with the inner product $\langle \cdot, \cdot \rangle_{H_K^0}$ so that

$$\langle K(\cdot, \mathbf{p}), K(\cdot, \mathbf{q}) \rangle_{H_K^0} = K(\mathbf{q}, \mathbf{p}),$$

for every $\mathbf{p}, \mathbf{q} \in E$.

A well-known fact for the theory of reproducing kernel Hilbert spaces is that

$$f(\mathbf{p}) = \langle f, K(\cdot, \mathbf{p}) \rangle_{H_K}$$

for every $f \in H_K$ and $\mathbf{p} \in E$. Accordingly, we have the following proposition.

Proposition 2. For every p, we have

$$\{f \in H_K \, : \, f \perp K(\cdot, \mathbf{p})\} = \{f \in H_K \, : \, f(\mathbf{p}) = 0\}.$$

As a direct consequence, we have the following result.

Proposition 3. Let $\mathbf{p} \in E$, so that $K(\mathbf{p}, \mathbf{p}) > 0$. Then the minimization problem:

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}) = 1,$$

has a unique solution given by $F := \dfrac{K(\cdot, \mathbf{p})}{K(\mathbf{p}, \mathbf{p})}$.

Proof of Proposition 3

Proof. Clearly $F(\mathbf{p}) = 1$. Now, if a function $g \in H_K$ satisfies $g(\mathbf{p}) = 1$, then

$$g - F \in \{f \in H_K \,:\, f(\mathbf{p}) = 0\}.$$

By Proposition 2, we have $g - F \perp K(\cdot, \mathbf{p})$, and accordingly $g - F \perp F$. It then follows that

$$\begin{aligned} \|g\|_{H_K}^2 &= \|g - F\|_{H_K}^2 + 2\mathrm{Re}\langle g - F, F\rangle_{H_K} + \|F\|_{H_K}^2 \\ &= \|g - F\|_{H_K}^2 + \|F\|_{H_K}^2 \\ &\geq \|F\|_{H_K}^2, \end{aligned}$$

and the equality is attained if and only if $g = F$. □

Theorem 1 can now be seen as a corollary of Proposition 3.

Indeed, let $E := [0,1]^d$ and put

$$K_\alpha(\mathbf{x},\mathbf{p}) := \frac{2^d}{\pi^{2\alpha}} \sum_{m_1,\ldots,m_d\in\mathbb{N}} \frac{\prod_{i=1}^d \sin(m_i\pi p_i)}{(m_1^2+\cdots+m_d^2)^\alpha} \prod_{i=1}^d \sin(m_i\pi x_i).$$

Observe that $K(\mathbf{p},\mathbf{p}) > 0$ for every $\mathbf{p} \in E$.

Moreover, if $f(\mathbf{x}) = \prod_{i=1}^{d} \sin(M_i \pi x_i)$, then

$$
\begin{aligned}
\langle f, K_\alpha(\cdot, \mathbf{p}) \rangle
&= \left\langle \prod_{i=1}^{d} \sin(M_i \pi \cdot_i), K(\cdot, \mathbf{p}) \right\rangle_{H_\alpha} \\
&= \frac{2^d}{\pi^{2\alpha}} \frac{\prod_{i=1}^{d} \sin(M_i \pi p_i)}{(M_1^2 + \cdots + M_d^2)^\alpha} \left\langle \prod_{i=1}^{d} \sin(M_i \pi \cdot_i), \prod_{i=1}^{d} \sin(M_i \pi \cdot_i) \right\rangle_{H_\alpha} \\
&= \prod_{i=1}^{d} \sin(M_i \pi p_i) \\
&= f(\mathbf{p}).
\end{aligned}
$$

It follows that $f(\mathbf{p}) = \langle f, K_\alpha(\cdot, \mathbf{p})\rangle$ for every $f \in H_\alpha$ and $p \in E$. This shows that H_α is the reproducing kernel Hilbert space with kernel K_α. Hence, the solution to the minimization problem

$$\text{Minimize } \|f\|_{H_\alpha}$$

subject to

$$f(\mathbf{p}) = 1,$$

is $F(\mathbf{x}) := \frac{K(\mathbf{x},\mathbf{p})}{K(\mathbf{p},\mathbf{p})}$, namely

$$F(\mathbf{x}) = \sum_{m_1,\ldots,m_d \in \mathbb{N}} A \frac{\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d)}{(m_1^2 + \cdots + m_d^2)^\alpha} \prod_{i=1}^{d} \sin(m_i \pi x_i),$$

where $A^{-1} := \sum_{m_1,\ldots,m_d \in \mathbb{N}} \frac{(\sin(m_1 \pi p_1) \cdots \sin(m_d \pi p_d))^2}{(m_1^2 + \cdots + m_d^2)^\alpha}$.

A more general result than Proposition 3 is formulated as follows.

Proposition 4. Suppose that we are given a finite set $\{c_k : k = 1, \ldots, N\} \subset \mathbb{F}$ and $P := \{\mathbf{p}_k : k = 1, \ldots, N\} \subset E$ such that the matrix $[K(\mathbf{p}_j, \mathbf{p}_k)]_{j,k=1}^N$ is positive definite. Then the minimization problem

$$\text{Minimize } \|f\|_{H_K}$$

subject to

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N,$$

has a unique solution that lies in $\text{span}\{f_k : k = 1, \ldots, N\}$, where $f_k := K(\cdot, \mathbf{p}_k)$.

As a corollary, we have the following theorem on our original interest.

Theorem 5. Given $\mathbf{p}_k \in (0,1)^d$ and $c_k \in \mathbb{R}$ for $k = 1, \ldots, N$, the minimization problem:

$$\text{Minimize } \|f\|_{H_\alpha}$$

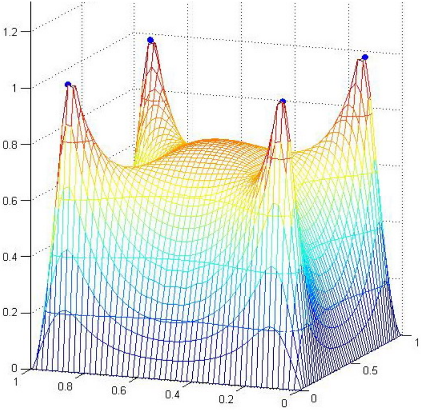
subject to the prescribed nodes:

$$f(\mathbf{p}_k) = c_k, \quad k = 1, \ldots, N,$$

has a unique solution $F := b_1 f_1 + \cdots + b_N f_N$, where $f_k := K_\alpha(\cdot, \mathbf{p}_k)$, $k = 1, \ldots, N$.

Since the solution to the minimization problem with several prescribed nodes is a linear combination of the minimizers with one prescribed node, in order to investigate its Hölder continuity, it suffices for us to investigate the Hölder continuity of the minimizer with one prescribed node, whose formula is given in Theorem 1.

A 2-D surface passing through four prescribed points with minimum $E_{1.5}(u)$



Theorem 6. Let $\alpha > \frac{d}{2}$, $\beta = \alpha - \frac{d}{2}$, and F be the solution to the minimization problem as in Proposition 1.

1. If $\beta \in \mathbb{N}$, then $F \in C^{\beta-1}(\mathbb{R}^d)$ with bounded partial derivatives up to order $\beta - 1$ and

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{(\beta-1)}(x) + F^{(\beta-1)}(y) - 2F^{(\beta-1)}((x+y)/2)|}{|x-y|} < \infty.$$

2. If $\beta \in (0,\infty) \setminus \mathbb{N}$, then $F \in C^{[\beta]}(\mathbb{R}^d)$ and

$$\sup_{x,y\in\mathbb{R}^d} \frac{|F^{([\beta])}(x) + F^{([\beta])}(y) - 2F^{([\beta])}((x+y)/2)|}{|x-y|^{\beta-[\beta]}} < \infty.$$

Related Works I

[1] A.R. Alghofari, *Problems in Analysis Related to Satellites*, Ph.D. Thesis, The University of New South Wales, Sydney, 2005.

[2] L. Ambarwati and H. Gunawan, "On the product of Chebyshev systems", preprint.

[3] H. Gunawan, F. Pranolo and E. Rusyaman, "An interpolation method that minimizes an energy integral of fractional order", *Proceedings of Asian Symposium on Computer Mathematics 2007* (published by Springer-Verlag in 2008).

[4] H. Gunawan, E. Rusyaman and L. Ambarwati, "Surfaces with prescribed nodes and minimum energy integral of fractional order", submitted.

Related Works II

[5] H. Pottmann and M. Hofer, *A variational approach to spline curves on surfaces*, *Comput. Aided Geom. Design* **22** (2005), 693–709.

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[8] G.G. Lorentz, *Approximation of Functions*, AMS Chelsea Publishing, Providence, 1966.

Related Works III

[9] M. Sugimoto and N. Tomita, "The dilation property of modulation spaces and their inclusion relation with Besov spaces", *J. Funct. Anal.* **248** (2007), 79–106.

[10] M. H. Taibleson, "On the theory of Lipschitz spaces of distributions on Euclidean n-space. I. Principal properties", *J. Math. Mech.* **13** (1964), 407–479.

[11] J. Wallner, "Existence of set-interpolating and energy-minimizing curves", *Comput. Aided Geom. Design* **21** (2004), 883–892.

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