On the Boundedness of Fractional Integral Operators

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For $0 < \alpha < d$, we define the fractional integral (also known as the Riesz potential) $I_\alpha f$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\alpha}} \, dy, \quad x \in \mathbb{R}^d,$$

for any suitable function $f$ on $\mathbb{R}^d$. Clearly $I_\alpha f$ is well-defined for any locally bounded, compactly supported function $f$ on $\mathbb{R}^d$. It is well-known that $I_\alpha$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, that is,

$$\|I_\alpha f : L^q\| \leq C \|f : L^p\|,$$

if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, with $1 < p < \frac{d}{\alpha}$. This result was proved by Hardy and Littlewood [5, 6] and Sobolev [10] around the 1930’s. Further development on the subject can be found in [11, 12].
Next, let $\mathbb{R}^+ := (0, \infty)$. For $1 \leq p < \infty$ and a suitable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, we define the generalized Morrey space $L^{p,\phi} = L^{p,\phi}(\mathbb{R}^d)$ to be the set of all functions $f \in L^p_{loc}(\mathbb{R}^d)$ for which

$$
\|f : L^{p,\phi}\| := \sup_B \frac{1}{\phi(B)} \left( \frac{1}{|B|} \int_B |f(y)|^p \, dy \right)^{1/p} < \infty.
$$

Here the supremum are taken over all open balls $B = B(a,r)$ in $\mathbb{R}^d$ and $\phi(B) = \phi(r)$, where $r \in \mathbb{R}^+$. For certain functions $\phi$, the spaces $L^{p,\phi}$ reduce to some classical spaces. For instance, if $\phi(r) = r^{(\lambda-d)/p}$, where $0 \leq \lambda \leq d$, then $L^{p,\phi}$ is the classical Morrey space $L^{p,\lambda}$. For a brief history of the Morrey space and related spaces, see [8]. For more recent results, see [9, 13] and the references therein.
In this talk, we shall revisit Nakai’s theorems on the fractional integrals on the generalized Morrey spaces [7].

In particular, we find that the sufficient condition imposed by Nakai for the boundedness of the operator turns out to be necessary.

In other words, we obtain a characterization for which the fractional integral operators are bounded from $L_{p,\phi}$ to $L_{q,\psi}$. 
Let us begin with some assumptions and relevant facts that follow. As customary, the letters $C$, $C_i$, $C_p$ and $C_{p,q}$ denote positive constants, which may depend on the parameters such as $\alpha$, $p$, $q$ and the dimension $d$ of the ambient space, but not on the function $f$ or the variable $x$. These constants may vary from line to line.
In the definition of $L^{p,\phi}$, the function $\phi$ is assumed to satisfy the following conditions:

- $\phi$ is almost decreasing: $t \leq r \Rightarrow \phi(r) \leq C_1 \phi(t)$;
- $r^d \phi(r)^p$ is almost increasing: $t \leq r \Rightarrow r^d \phi(t)^p \leq C_2 r^d \phi(r)^p$,

with $C_1, C_2 > 0$ being independent of $r$ and $t$. These two conditions implies that

$\phi$ satisfies the doubling condition: $1 \leq \frac{t}{r} \leq 2 \Rightarrow \frac{1}{C_3} \leq \frac{\phi(t)}{\phi(r)} \leq C_3$,

for some $C_3 > 0$ (which is also independent of $r$ and $t$). Throughout this talk, we shall always assume that $\phi$ satisfies these conditions.
In [7], Nakai showed that $I_\alpha$ is bounded from $L^{p,\phi}$ to $L^{q,\psi}$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ if $\phi$ satisfies an additional condition, namely

$$\int_r^\infty t^{\alpha-1} \phi(t) \, dt \leq C_4 r^{\alpha} \phi(r),$$

(1)

and

$$r^{\alpha} \phi(r) \leq C_5 \psi(r),$$

(2)

for every $r \in \mathbb{R}^+$. By taking $\phi(r) = r^{(\lambda-d)/p}$ with $0 \leq \lambda < d - \alpha p$ and $\psi(r) = r^{\alpha} \phi(r)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, Nakai’s result contains Spanne’s, which states that $I_\alpha$ is bounded form $L^{p,\lambda}$ to $L^{q,\mu}$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, $0 \leq \lambda < d - \alpha p$ and $\mu = \frac{q}{p} \lambda$ [8]. See also [3] for related results.
In the following, we shall show that the condition (2) is necessary for the fractional integral operator $I_\alpha$ to be bounded from $L^{p,\phi}$ to $L^{q,\psi}$. To do so, we need some lemmas. The first lemma shows particularly that the space $L^{p,\phi}$ is not trivial.

**Lemma 2.1.** If $B_0 := B(a_0, r_0)$, then $\chi_{B_0} \in L^{p,\phi}$ where $\chi_{B_0}$ is the characteristic function of the ball $B_0$. Moreover, there exists $C > 0$ such that

$$\frac{1}{\phi(r_0)} \leq \|\chi_{B_0} : L^{p,\phi}\| \leq \frac{C}{\phi(r_0)}.$$
Proof. Let $B := B(a, r)$ denote an arbitrary ball in $\mathbb{R}^d$. It is easy to see that

$$\|\chi_{B_0} : L^p, \phi\| = \sup_B \frac{1}{\phi(r)} \left( \frac{|B \cap B_0|}{|B|} \right)^{1/p} \geq \frac{1}{\phi(r_0)}.$$ 

Now, if $r \leq r_0$, then $\phi(r_0) \leq C \phi(r)$ and

$$\frac{1}{\phi(r)} \left( \frac{|B \cap B_0|}{|B|} \right)^{1/p} \leq \frac{1}{\phi(r)} \leq \frac{C}{\phi(r_0)}.$$

On the other hand, if $r_0 \leq r$, we have $r_0^d \phi(r_0)^p \leq C r^d \phi(r)^p$ and

$$\frac{1}{\phi(r)} \left( \frac{|B \cap B_0|}{|B|} \right)^{1/p} = \frac{C|B \cap B_0|^{1/p}}{r^d/p \phi(r)} \leq \frac{C|B_0|^{1/p}}{r^d/p \phi(r)} \leq \frac{C r_0^{1/p}}{r^d/p \phi(r_0)} \leq \frac{C}{\phi(r_0)}.$$

This completes the proof.
Lemma 2.2. If $B_0 := B(a_0, r_0)$, then $r_0^\alpha \leq C I_\alpha \chi_{B_0}(x)$ for every $x \in B_0$.

Proof. If $x, y \in B_0 := B(a_0, r_0)$, then

$$|x - y| \leq |x - a_0| + |a_0 - y| < 2r_0.$$ 

If we integrate both sides of the inequality $r_0^{\alpha - d} \leq C |x - y|^\alpha$ over $B_0$, then we get the desired estimate.
The following theorem gives a characterization of the functions $\phi$ and $\psi$ for which $I_\alpha$ is bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

**Theorem 2.3.** Suppose that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, where $1 < p < \frac{d}{\alpha}$. Suppose further that $r^\alpha \phi(r)$ satisfies the integral condition (1). Then, $I_\alpha$ bounded from $L^{p,\phi}$ to $L^{q,\psi}$ if and only if $r^\alpha \phi(r) \leq C \psi(r)$ for every $r \in \mathbb{R}^+$. 
Main Results

Proof. The sufficient part is proved in [7]. We shall now prove the necessary part. Assume that \( I_\alpha \) is bounded from \( L^{p,\phi} \) to \( L^{q,\psi} \), and let \( B_0 := B(a_0, r_0) \). If \( x \in B_0 \), then \( r_0^\alpha \leq C I_\alpha \chi_{B_0}(x) \). Integrating over \( B_0 \), we get

\[
r_0^\alpha \leq C \left( \frac{1}{|B_0|} \int_{B_0} |I_\alpha \chi_{B_0}(x)|^q \, dx \right)^{1/q} \leq C \psi(r_0) \| I_\alpha \chi_{B_0} : L^q_\psi \| \leq C \psi(r_0) \| \chi_{B_0} : L^p_\phi \| \leq C \psi(r_0) \phi(r_0)^{-1}.
\]

Note that the first inequality follows from Lemma 2.2, while the last one follows from Lemma 2.1. Since this is true for every \( r_0 \in \mathbb{R}^+ \), we are done. \( \square \)
In [4], there is the following theorem that serves as an extension of Adams and Chiarenza–Frasca’s result on the fractional integral operator $I_\alpha$ [1, 2].

**Theorem 3.1** (Gunawan-Eridani). *Suppose that $1 < p < \frac{d}{\alpha}$ and $\phi^p$ satisfies the integral condition, namely

$$
\int_r^\infty \frac{\phi^p(t)}{t} \, dt \leq C_6 \phi^p(r),
$$

(3)

for every $r \in \mathbb{R}^+$. If $\phi(r) \leq C \, r^\beta$ for $-\frac{d}{p} \leq \beta < -\alpha$, then, for $q = \frac{\beta p}{\alpha+\beta}$, there exists $C_{p,\beta} > 0$ such that

$$
\| I_\alpha f : L^{q,\phi^{p/q}} \| \leq C_{p,\beta} \| f : L^{p,\phi} \|.
$$
As in the previous part, we also have the characterization of $\phi$ for which $I_\alpha$ is bounded from $L^{p,\phi}$ to $L^{q,\phi^{p/q}}$.

**Theorem 3.2** Suppose that $1 < p < \frac{d}{\alpha}$ and $\phi^p$ satisfies the integral condition (3). If $-\frac{d}{p} \leq \beta < -\alpha$ and $q = \frac{\beta p}{\alpha + \beta}$, then $I_\alpha$ bounded from $L^{p,\phi}$ to $L^{q,\phi^{p/q}}$ if and only if $\phi(r) \leq C r^{\beta}$ for every $r \in \mathbb{R}^+$. 
Proof. The proof of the sufficient part can be found in [4]. As for the necessary part, we have the following observation: if \( B_0 := B(a_0, r_0) \), then

\[
\begin{align*}
    r_0^\alpha &\leq C \left( \frac{1}{|B_0|} \int_{B_0} |I_\alpha \chi_{B_0}(x)|^q \, dx \right)^{1/q} 
    \leq C \phi(r_0)^{p/q} \| I_\alpha \chi_{B_0} : L^{q,\phi^{p/q}} \| \\
    &\leq C \phi(r_0)^{p/q} \| \chi_{B_0} : L^{p,\phi} \| 
    \leq C \phi(r_0)^{p/q} \phi(r_0)^{-1},
\end{align*}
\]

which may be rewritten as \( \phi(r_0) \leq Cr_0^\beta \). Since this inequality is valid for every \( r_0 \in \mathbb{R}^+ \), the theorem is proved. \( \square \)


