On the Boundedness of Fractionals Integral Operators

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Outline

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For $0 < \alpha < d$, we define the fractional integral (also known as the Riesz potential) $I_\alpha f$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\,dy, \quad x \in \mathbb{R}^d,$$

for any suitable function f on $\mathbb{R}^d$. Clearly $I_\alpha f$ is well-defined for any locally bounded, compactly supported function f on $\mathbb{R}^d$. It is well-known that I_α is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, that is,

$$\|I_\alpha f : L^q\| \le C\,\|f : L^p\|,$$

if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, with $1 < p < \frac{d}{\alpha}$. This result was proved by Hardy and Littlewood [5, 6] and Sobolev [10] around the 1930's. Further development on the subject can be found in [11, 12].

Next, let $\mathbb{R}^+ := (0, \infty)$. For $1 \le p < \infty$ and a suitable function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, we define the generalized Morrey space $L^{p,\phi} = L^{p,\phi}(\mathbb{R}^d)$ to be the set of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^d)$ for which

$$\|f : L^{p,\phi}\| := \sup_B \frac{1}{\phi(B)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Here the supremum are taken over all open balls $B = B(a, r)$ in $\mathbb{R}^d$ and $\phi(B) = \phi(r)$, where $r \in \mathbb{R}^+$. For certain functions ϕ, the spaces $L^{p,\phi}$ reduce to some classical spaces. For instance, if $\phi(r) = r^{(\lambda-d)/p}$, where $0 \le \lambda \le d$, then $L^{p,\phi}$ is the classical Morrey space $L^{p,\lambda}$. For a brief history of the Morrey space and related spaces, see [8]. For more recent results, see [9, 13] and the references therein.

In this talk, we shall revisit Nakai's theorems on the fractional integrals on the generalized Morrey spaces [7].

In particular, we find that the sufficient condition imposed by Nakai for the boundedness of the operator turns out to be necessary.

In other words, we obtain a characterization for which the fractional integral operators are bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

Let us begin with some assumptions and relevant facts that follow. As customary, the letters $C,\ C_i,\ C_p$ and $C_{p,q}$ denote positive constants, which may depend on the parameters such as α, p, q and the dimension d of the ambient space, but not on the function f or the variable x. These constants may vary from line to line.

In the definition of $L^{p,\phi}$, the function ϕ is assumed to satisfy the following conditions:

$$\begin{aligned} \phi \text{ is } \textit{almost decreasing} \quad &: \quad t \le r \Rightarrow \phi(r) \le C_1\,\phi(t); \\ r^d\phi(r)^p \text{ is } \textit{almost increasing} \quad &: \quad t \le r \Rightarrow t^d\phi(t)^p \le C_2\,r^d\phi(r)^p, \end{aligned}$$

with $C_1,\ C_2 > 0$ being independent of r and t. These two conditions implies that

$$\phi \text{ satisfies } \textit{the doubling condition} : \quad 1 \le \frac{t}{r} \le 2 \Rightarrow \frac{1}{C_3} \le \frac{\phi(t)}{\phi(r)} \le C_3,$$

for some $C_3 > 0$ (which is also independent of r and t). Throughout this talk, we shall always assume that ϕ satisfies these conditions.

In [7], Nakai showed that I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ if ϕ satisfies an additional condition, namely

$$\int_r^\infty t^{\alpha-1}\phi(t)\,dt \le C_4\, r^\alpha \phi(r), \tag{1}$$

and

$$r^\alpha \phi(r) \le C_5 \psi(r), \tag{2}$$

for every $r \in \mathbb{R}^+$.

By taking $\phi(r) = r^{(\lambda-d)/p}$ with $0 \le \lambda < d - \alpha p$ and $\psi(r) = r^\alpha \phi(r)$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, Nakai's result contains Spanne's, which states that I_α is bounded form $L^{p,\lambda}$ to $L^{q,\mu}$ for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, $0 \le \lambda < d - \alpha p$ and $\mu = \frac{q}{p}\lambda$ [8]. See also [3] for related results.

In the following, we shall show that the condition (2) is necessary for the fractional integral operator I_α to be bounded from $L^{p,\phi}$ to $L^{q,\psi}$. To do so, we need some lemmas. The first lemma shows particularly that the space $L^{p,\phi}$ is not trivial.

Lemma 2.1. *If $B_0 := B(a_0, r_0)$, then $\chi_{B_0} \in L^{p,\phi}$ where χ_{B_0} is the characteristic function of the ball B_0. Moreover, there exists $C > 0$ such that*

$$\frac{1}{\phi(r_0)} \leq \|\chi_{B_0} : L^{p,\phi}\| \leq \frac{C}{\phi(r_0)}.$$

Proof. Let $B := B(a,r)$ denote an arbitrary ball in $\mathbb{R}^d$. It is easy to see that

$$\|\chi_{B_0} : L^{p,\phi}\| = \sup_B \frac{1}{\phi(r)} \left(\frac{|B \cap B_0|}{|B|}\right)^{1/p} \geq \frac{1}{\phi(r_0)}.$$

Now, if $r \leq r_0$, then $\phi(r_0) \leq C\,\phi(r)$ and

$$\frac{1}{\phi(r)} \left(\frac{|B \cap B_0|}{|B|}\right)^{1/p} \leq \frac{1}{\phi(r)} \leq \frac{C}{\phi(r_0)}.$$

On the other hand, if $r_0 \leq r$, we have $r_0^d \phi(r_0)^p \leq C\, r^d \phi(r)^p$ and

$$\frac{1}{\phi(r)} \left(\frac{|B \cap B_0|}{|B|}\right)^{1/p} = \frac{C|B \cap B_0|^{1/p}}{r^{d/p}\phi(r)} \leq \frac{C|B_0|^{1/p}}{r^{d/p}\phi(r)} \leq \frac{C r_0^{1/p}}{r_0^{d/p}\phi(r_0)} \leq \frac{C}{\phi(r_0)}$$

This completes the proof. □

Lemma 2.2. *If* $B_0 := B(a_0, r_0)$, *then* $r_0^\alpha \leq C\, I_\alpha \chi_{B_0}(x)$ for every $x \in B_0$.

Proof. If $x, y \in B_0 := B(a_0, r_0)$, then

$$|x - y| \leq |x - a_0| + |a_0 - y| < 2r_0.$$

If we integrate both sides of the inequality $r_0^{\alpha-d} \leq C\, |x - y|^{\alpha-d}$ over B_0, then we get the desired estimate. □

The following theorem gives a characterization of the functions ϕ and ψ for which I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$.

Theorem 2.3. *Suppose that* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$*, where* $1 < p < \frac{d}{\alpha}$*. Suppose further that* $r^\alpha\,\phi(r)$ *satisfies the integral condition (1). Then,* I_α *bounded from* $L^{p,\phi}$ *to* $L^{q,\psi}$ *if and only if* $r^\alpha\phi(r) \le C\,\psi(r)$ *for every* $r \in \mathbb{R}^+$*.*

Proof. The sufficient part is proved in [7]. We shall now prove the necessary part. Assume that I_α is bounded from $L^{p,\phi}$ to $L^{q,\psi}$, and let $B_0 := B(a_0, r_0)$. If $x \in B_0$, then $r_0^\alpha \le C\, I_\alpha \chi_{B_0}(x)$. Integrating over B_0, we get

$$r_0^\alpha \le C \left(\frac{1}{|B_0|} \int_{B_0} |I_\alpha \chi_{B_0}(x)|^q \, dx \right)^{1/q} \le C\, \psi(r_0) \| I_\alpha \chi_{B_0} : L^q_\psi \| \\ \le C\, \psi(r_0) \| \chi_{B_0} : L^p_\phi \| \le C\, \psi(r_0)\, \phi(r_0)^{-1}.$$

Note that the first inequality follows from Lemma 2.2, while the last one follows from Lemma 2.1. Since this is true for every $r_0 \in \mathbb{R}^+$, we are done. □

In [4], there is the following theorem that serves as an extension of Adams and Chiarenza–Frasca's result on the fractional integral operator I_α [1, 2].

Theorem 3.1 (Gunawan-Eridani). *Suppose that* $1 < p < \frac{d}{\alpha}$ *and* ϕ^p *satisfies the integral condition, namely*

$$\int_r^\infty \frac{\phi^p(t)}{t}\,dt \le C_6 \phi^p(r), \tag{3}$$

for every $r \in \mathbb{R}^+$. *If* $\phi(r) \le C\, r^\beta$ *for* $-\frac{d}{p} \le \beta < -\alpha$, *then, for* $q = \frac{\beta\, p}{\alpha+\beta}$, *there exists* $C_{p,\beta} > 0$ *such that*

$$\|I_\alpha f : L^{q,\phi^{p/q}}\| \le C_{p,\beta}\, \|f : L^{p,\phi}\|.$$

As in the previous part, we also have the characterization of ϕ for which I_α is bounded from $L^{p,\phi}$ to $L^{q,\phi^{p/q}}$.

Theorem 3.2 *Suppose that* $1 < p < \frac{d}{\alpha}$ *and* ϕ^p *satisfies the integral condition (3). If* $-\frac{d}{p} \leq \beta < -\alpha$ *and* $q = \frac{\beta\, p}{\alpha+\beta}$*, then* I_α *bounded from* L^p_ϕ *to* $L^q_{\phi^{p/q}}$ *if and only if* $\phi(r) \leq C\, r^\beta$ for every $r \in \mathbb{R}^+$.

Proof. The proof of the sufficient part can be found in [4]. As for the necessary part, we have the following observation: if $B_0 := B(a_0, r_0)$, then

$$r_0^{\alpha} \le C \left(\frac{1}{|B_0|} \int_{B_0} |I_\alpha \chi_{B_0}(x)|^q \, dx \right)^{1/q} \le C\, \phi(r_0)^{p/q} \| I_\alpha \chi_{B_0} : L^{q,\phi^{p/q}} \| \\ \le C\, \phi(r_0)^{p/q} \| \chi_{B_0} : L^{p,\phi} \| \le C\, \phi(r_0)^{p/q}\, \phi(r_0)^{-1},$$

which may be rewritten as $\phi(r_0) \le C r_0^{\beta}$. Since this inequality is valid for every $r_0 \in \mathbb{R}^+$, the theorem is proved. □

[1] D. R. Adams, "A note on Riesz potentials", *Duke Math. J.* **42** (1975), 765–778.

[2] F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7** (1987), 273–279.

[3] Eridani, H. Gunawan and E. Nakai, "On generalized fractional integral operators", *Sci. Math. Jpn.* **60** (2004), 539–550.

[4] H. Gunawan and Eridani, "Fractional integrals and generalized Olsen inequalities", *Kyungpook Math. J.* **49** (2009), 31–39.

[5] G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals. I", *Math. Zeit.* **27** (1927), 565–606.

[6] G. H. Hardy and J. E. Littlewood, "Some properties of fractional integrals. II", *Math. Zeit.* **34** (1932), 403–439.

[7] E. Nakai, "Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces", *Math. Nachr.* **166** (1994), 95–103.

[8] J. Peetre, "On the theory of $\mathcal{L}_{p,\lambda}$ spaces", *J. Funct. Anal.* **4** (1969), 71–87.

[9] Y. Sawano, "Generalized Morrey spaces for non-doubling measures", *Non-linear Differential Equations and Applications* **15** (2008), 413–425.

[10] S. L. Sobolev, "On a theorem in functional analysis" (Russian), *Mat. Sob.* **46** (1938), 471–497 [English translation in *Amer. Math. Soc. Transl. ser. 2* **34** (1963), 39–68].

[11] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, Princeton, New Jersey, 1970.

[12] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

[13] S. Sugano and H. Tanaka, "Boundedness of fractional integral operators on generalized Morrey spaces", *Sci. Math. Jpn. Online* **8** (2003), 233–242.