THE HARDY-LITTLEWOOD MAXIMAL OPERATOR ON DISCRETE MORREY SPACES

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ABSTRACT. We discuss the Hardy-Littlewood maximal operator on discrete Morrey spaces of arbitrary dimension. In particular, we obtain its boundedness on the discrete Morrey spaces using a discrete version of the Fefferman-Stein inequality. As a corollary, we also obtain the boundedness of some Riesz potentials on discrete Morrey spaces.

1. Introduction

While the Hardy-Littlewood maximal operator is well-known, discrete Morrey spaces were only studied recently in [5] (see also [1] for related works). In this paper, we investigate the boundedness of the (discrete) Hardy-Littlewood maximal operator on discrete Morrey spaces of arbitrary dimension. Some important properties of this operator (and many others) on the $\ell^p(\mathbb{Z}^d)$ spaces were discussed in [9]. See also [6, 10, 11, 12, 13] for related works on discrete analogues in harmonic analysis.

The boundedness of the (continuous) Hardy-Littlewood maximal operator on the (continuous) Morrey spaces was first studied in [2], whose results were later extended in [7, 8] to some generalizations of Morrey spaces. The driving force behind the results in [2, 7, 8] is the so-called Fefferman-Stein inequality [4, Lemma 1], a result specifically regarding integrable functions defined on $\mathbb{R}^d$. We illustrate in Theorem 2.1 how this inequality, despite its reliance on various tools only available in the continuous setting, can be transformed into a natural discrete analogue. As a consequence of Theorem 2.1, we obtain the boundedness of the discrete Hardy-Littlewood maximal operator on the discrete Morrey spaces in Theorem 3.2.

We begin with some notation and definitions. First we set $\omega := \mathbb{N} \cup \{0\}$ and use this notation throughout the paper. For $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and $N \in \omega$ define

$$S_{m,N} := \{k \in \mathbb{Z}^d : \|k - m\|_\infty \le N\},$$

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where as usual $\|(x_1,\ldots,x_d)\|_\infty := \max\{|x_i| : 1 \le i \le d\}$ for $(x_1,\ldots,x_d) \in \mathbb{R}^d$. Again following standard conventions, we denote the cardinality of a set S by $|S|$. Then we have $|S_{m,N}| = (2N+1)^d$ for all $m \in \mathbb{Z}^d$ and each $N \in \omega$. Given $1 \le p \le q < \infty$ we define the discrete Morrey space $\ell^p_q(\mathbb{Z}^d)$ to be the space of all functions $x\colon \mathbb{Z}^d \to \mathbb{R}$ for which

$$\|x\|_{\ell^p_q(\mathbb{Z}^d)} := \sup_{m\in\mathbb{Z}^d, N\in\omega} |S_{m,N}|^{\frac{1}{q}-\frac{1}{p}} \left(\sum_{k\in S_{m,N}} |x(k)|^p \right)^{1/p} < \infty.$$

By following the proof of [5, Proposition 2.2], one can readily prove that $\|\cdot\|_{\ell^p_q(\mathbb{Z}^d)}$ defines a norm on $\ell^p_q(\mathbb{Z}^d)$ and that $\ell^p_q(\mathbb{Z}^d)$ is a Banach space with respect to this norm. Indeed, [5, Proposition 2.2] proves the given result for $d = 1$, and its proof is easily adaptable to higher dimensions.

We wish to study the (discrete) Hardy-Littlewood maximal operator on these discrete Morrey spaces of arbitrary dimension. To begin, define the discrete Hardy-Littlewood maximal operator (or for emphasis, the "odd" discrete Hardy-Littlewood maximal operator) M by

$$Mx(m) := \sup_{N\in\omega} \frac{1}{|S_{m,N}|} \sum_{k\in S_{m,N}} |x(k)| \quad \big(x \in \mathbb{R}^{\mathbb{Z}^d},\ m \in \mathbb{Z}^d\big).$$

The operator M is a discrete analogue of the "centered continuous" Hardy-Littlewood maximal operator, which is defined by

$$\bar{M}f(y) := \sup_{r>0} \frac{1}{(2r)^d} \int_{Q_{y,r}} |f(z)|dz \quad (f \in L^1_{loc}(\mathbb{R}^d),\ y \in \mathbb{R}^d),$$

where $Q_{y,r} := \{t \in \mathbb{R}^d : \|t-y\|_\infty \le r\}$. While the "odd" discrete Hardy-Littlewood maximal operator will be our primary interest, the following rendition of this function will prove useful in obtaining a discrete analogue of the Fefferman-Stein inequality.

The "even" discrete Hardy-Littlewood maximal operator $\hat{M}$ is defined for $x \in \mathbb{R}^{\mathbb{Z}^d}$ and $m = (m_1,\ldots,m_d) \in \mathbb{Z}^d$ by

$$\hat{M}x(m) := \sup_{N\in\mathbb{N}} \frac{1}{|R_{m,N}|} \sum_{k\in R_{m,N}} |x(k)|,$$

where

$$R_{m,N} := S_{m,N} \setminus \{(k_1,\ldots,k_d) \in \mathbb{Z}^d : k_i = m_i + N \text{ for some } 1 \le i \le d\} \quad (m \in \mathbb{Z}^d,\ N \in \mathbb{N}),$$

so that $|R_{m,N}| = (2N)^d$. Additionally, we define the "uncentered" discrete Hardy-Littlewood maximal operator $\tilde{M}$ by

$$\tilde{M}x(m) := \sup_{S\ni m} \frac{1}{|S|} \sum_{k\in S} |x(k)| \quad (x \in \mathbb{R}^{\mathbb{Z}^d},\ m \in \mathbb{Z}^d),$$

where the supremum above is taken over all sets of the form $S = S_{k,N}$, for some $k \in \mathbb{Z}^d$ and $N \in \omega$, that contain m.

We say that two operators $T_1, T_2 \colon \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}^{\mathbb{Z}^d}$ are *equivalent* if there exist $C_1, C_2 > 0$ such that $C_1 T_1 x(k) \le T_2 x(k) \le C_2 T_1 x(k)$ hold for all $x \in \mathbb{R}^{\mathbb{Z}^d}$ and every $k \in \mathbb{Z}^d$. Regarding the operators $M, \hat{M}$, and $\tilde{M}$, we have the following lemma which will be useful in our discussion in the next sections. We leave its proof to the reader.

Lemma 1.1. *The operators $M, \hat{M}$, and $\tilde{M}$ are pairwise equivalent.*

2. The Discrete Fefferman-Stein Inequality

In this section, we provide a discrete version of the Fefferman-Stein inequality ([4, Lemma 1]). Theorem 2.1 below will be used to obtain the boundedness of the discrete Hardy-Littlewood maximal operator on discrete Morrey spaces in Section 3.

We call a function $x \in \mathbb{R}^{\mathbb{Z}^d}$ *positive* if $x(k) \ge 0$ for each $k \in \mathbb{Z}^d$. Given $A \subseteq \mathbb{R}^d$, we denote the characteristic function of A by χ_A.

Theorem 2.1. *Let $1 < p < \infty$. There exists $K > 0$ such that for all $x \in \mathbb{R}^{\mathbb{Z}^d}$ and each positive $\phi \in \mathbb{R}^{\mathbb{Z}^d}$,*

(1) $\sum_{k\in\mathbb{Z}^d} \big(Mx(k)\big)^p \phi(k) \le K \sum_{k\in\mathbb{Z}^d} |x(k)|^p M\phi(k)$,
(2) $\sum_{k\in\mathbb{Z}^d} \big(\hat{M}x(k)\big)^p \phi(k) \le K \sum_{k\in\mathbb{Z}^d} |x(k)|^p \hat{M}\phi(k)$, *and*
(3) $\sum_{k\in\mathbb{Z}^d} \big(\tilde{M}x(k)\big)^p \phi(k) \le K \sum_{k\in\mathbb{Z}^d} |x(k)|^p \tilde{M}\phi(k)$.

Proof. We prove statement (2), from which statements (1) and (3) will follow from Lemma 1.1. For each $k = (k_1, ..., k_d) \in \mathbb{Z}^d$ define the d-dimensional cube of volume one by

$$\mathcal{C}_k := \{(y_1, ..., y_d) \in \mathbb{R}^d : k_i \le y_i < k_i + 1 \text{ for all } 1 \le i \le d\}.$$

Next note that for every $a \in \mathbb{R}^{\mathbb{Z}^d}$ the function $\bar{a}$ defined by

$$\bar{a}(t) := \sum_{k\in\mathbb{Z}^d} a(k)\chi_{\mathcal{C}_k}(t) \quad (t \in \mathbb{R}^d)$$

is a member of $L^1_{loc}(\mathbb{R}^d)$ and

$$\sum_{k\in R_{m,N}} a(k) = \int_{Q_{m,N}} \bar{a}(t)dt \quad (m \in \mathbb{Z}^d,\ N \in \omega),$$

where again $Q_{m,N} = \{t \in \mathbb{R}^d : \|t - m\|_\infty \le N\}$. Therefore, we have

$$\sum_{k\in\mathbb{Z}^d} a(k) = \int_{\mathbb{R}^d} \bar{a}(t)dt.$$

Next let $x, \phi \in \mathbb{R}^{\mathbb{Z}^d}$, and suppose ϕ is positive. Then $\bar{\phi}(t) \geq 0$ for every $t \in \mathbb{R}^d$, and in light of the remarks above,

$$\sum_{k\in\mathbb{Z}^d} \big(\hat{M}x(k)\big)^p \phi(k) = \int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} \left(\sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \sum_{i\in R_{k,N}} |x(i)| \right)^p \phi(k)\chi_{\mathcal{C}_k}(t)dt.$$

Furthermore,

$$\begin{aligned}\int_{\mathbb{R}^d} \sum_{k\in\mathbb{Z}^d} &\left(\sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \sum_{i\in R_{k,N}} |x(i)| \right)^p \phi(k)\chi_{\mathcal{C}_k}(t)dt \\ &= \sum_{k\in\mathbb{Z}^d} \int_{\mathcal{C}_k} \left(\sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \int_{Q_{k,N}} |\bar{x}(s)|ds \right)^p \phi(k)dt.\end{aligned}$$

Note that for each $k \in \mathbb{Z}^d$ and all $t \in \mathcal{C}_k$ we have

$$\int_{Q_{k,N}} |\bar{x}(s)|ds \leq \int_{Q_{t,N+1}} |\bar{x}(s)|ds.$$

Hence

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} &\int_{\mathcal{C}_k} \left(\sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \int_{Q_{k,N}} |\bar{x}(s)|ds \right)^p \phi(k)dt \\ &\leq 2^d \sum_{k\in\mathbb{Z}^d} \int_{\mathcal{C}_k} \left(\sup_{N\in\mathbb{N}} \frac{1}{(2(N+1))^d} \int_{Q_{t,N+1}} |\bar{x}(s)|ds \right)^p \phi(k)dt \\ &\leq 2^d \int_{\mathbb{R}^d} \big(\bar{M}\bar{x}(t)\big)^p \bar{\phi}(t)dt.\end{aligned}$$

By the Fefferman-Stein inequality [4, Lemma 1], there exists $K > 0$ such that

$$\int_{\mathbb{R}^d} \big(\bar{M}\bar{x}(t)\big)^p \bar{\phi}(t)dt \leq K \int_{\mathbb{R}^d} |\bar{x}(t)|^p \, \bar{M}\bar{\phi}(t)dt.$$

Thus we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \big(\hat{M}x(k)\big)^p \phi(k) &\leq 2^d K \int_{\mathbb{R}^d} |\bar{x}(t)|^p \, \bar{M}\bar{\phi}(t)dt \\ &= 2^d K \sum_{k\in\mathbb{Z}^d} \int_{\mathcal{C}_k} |x(k)|^p \left(\sup_{r>0} \frac{1}{(2r)^d} \int_{Q_{t,r}} \bar{\phi}(s)ds \right) dt.\end{aligned}$$

For each $k \in \mathbb{Z}^d$, let k^* denote the midpoint of $\mathcal{C}_k$. Notice that for each $k \in \mathbb{Z}^d$ we have

$$\int_{\mathcal{C}_k} |x(k)|^p \left(\sup_{r>0} \frac{1}{(2r)^d} \int_{Q_{t,r}} \bar{\phi}(s)ds \right) dt \leq \int_{\mathcal{C}_k} |x(k)|^p \left(\sup_{r>0} \frac{1}{(2r)^d} \int_{Q_{k^*,r}} \bar{\phi}(s)ds \right) dt.$$

Next suppose that $0 < r < \frac{1}{2}$. Then for all $k \in \mathbb{Z}^d$,

$$\frac{1}{(2r)^d}\int_{Q_{k^*,r}} \bar{\phi}(s)ds = \frac{1}{(2r)^d}\phi(k)r^d = \frac{1}{2^d}\phi(k) \leq \frac{1}{(2\cdot\frac{1}{2})^d}\int_{Q_{k^*,\frac{1}{2}}} \bar{\phi}(s)ds.$$

Hence

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}\int_{\mathcal{C}_k}|x(k)|^p\left(\sup_{r>0}\frac{1}{(2r)^d}\int_{Q_{t,r}}\bar{\phi}(s)ds\right)dt\\
&\quad\leq \sum_{k\in\mathbb{Z}^d}|x(k)|^p\left(\sup_{r\geq\frac{1}{2}}\frac{1}{(2r)^d}\int_{Q_{k^*,r}}\bar{\phi}(s)ds\right)\\
&\quad\leq \sum_{k\in\mathbb{Z}^d}|x(k)|^p\left(\sup_{r\geq\frac{1}{2}}\frac{1}{(2r)^d}\int_{Q_{k,\lfloor r+2\rfloor}}\bar{\phi}(s)ds\right)\\
&\quad\leq 5^d\sum_{k\in\mathbb{Z}^d}|x(k)|^p\left(\sup_{r\geq\frac{1}{2}}\frac{1}{(2\lfloor r+2\rfloor)^d}\int_{Q_{k,\lfloor r+2\rfloor}}\bar{\phi}(s)ds\right)\\
&\quad\leq 5^d\sum_{k\in\mathbb{Z}^d}|x(k)|^p\left(\sup_{N\in\mathbb{N}}\frac{1}{(2N)^d}\sum_{i\in R_{k,N}}\phi(i)\right)\\
&\quad= 5^d\sum_{k\in\mathbb{Z}^d}|x(k)|^p\hat{M}\phi(k).
\end{aligned}$$

Therefore,

$$\sum_{k\in\mathbb{Z}^d}\left(\hat{M}x(k)\right)^p\phi(k) \leq (10)^d K\sum_{k\in\mathbb{Z}^d}|x(k)|^p\hat{M}\phi(k),$$

as desired. □

3. The Boundedness of the Discrete Maximal Operator

We use the methods of F. Chiarenza and M. Frasca in [2] in this section, and we additionally require the following lemma.

Lemma 3.1. *Let* $1 \leq p \leq q < \infty$. *For any* $x \in \ell^p_q(\mathbb{Z}^d)$ *we have* $Mx \in \ell^\infty(\mathbb{Z}^d)$ *and* $\|Mx\|_{\ell^\infty(\mathbb{Z}^d)} \leq \|x\|_{\ell^p_q(\mathbb{Z}^d)}$.

Proof. Let $x \in \ell^p_q(\mathbb{Z}^d)$, and put $m^* \in \mathbb{Z}^d$. Then

$$Mx(m^*) = \sup_{N\in\omega}\frac{1}{(2N+1)^d}\sum_{k\in S_{m^*,N}}|x(k)| \leq \sup_{m\in\mathbb{Z}^d,N\in\omega}\frac{1}{(2N+1)^d}\sum_{k\in S_{m,N}}|x(k)|.$$

In [5, Lemma 2.3], it is shown for $d = 1$ that for any $m \in \mathbb{Z}^d$ and $N \in \omega$,

$$\frac{1}{(2N+1)^d} \sum_{k \in S_{m,N}} |x(k)| \leq \left(\frac{1}{(2N+1)^d} \sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}}.$$

One may readily check that the proof of [5, Lemma 2.3] also holds for general $d \in \mathbb{N}$. Hence

$$\begin{aligned}
\sup_{m \in \mathbb{Z}^d, N \in \omega} \frac{1}{(2N+1)^d} \sum_{k \in S_{m,N}} |x(k)| &\leq \sup_{m \in \mathbb{Z}^d, N \in \omega} (2N+1)^{-\frac{d}{p}} \left(\sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}} \\
&\leq \sup_{m \in \mathbb{Z}^d, N \in \omega} (2N+1)^{\frac{d}{q} - \frac{d}{p}} \left(\sum_{k \in S_{m,N}} |x(k)|^p \right)^{\frac{1}{p}} \\
&= \|x\|_{\ell^p_q},
\end{aligned}$$

which proves the lemma. □

We next present the main result of this paper.

Theorem 3.2. *Let $1 < p \leq q < \infty$. For all $x \in \ell^p_q(\mathbb{Z}^d)$ we have $Mx \in \ell^p_q(\mathbb{Z}^d)$, and there exists $C > 0$ such that $\|Mx\|_{\ell^p_q(\mathbb{Z}^d)} \leq C\|x\|_{\ell^p_q(\mathbb{Z}^d)}$ holds for all $x \in \ell^p_q(\mathbb{Z}^d)$.*

Proof. Let $m \in \mathbb{Z}^d$, and put $N \in \mathbb{N}$ (the case $N = 0$ will be handled later). By Theorem 2.1 (1) there exists $K > 0$ such that

$$\sum_{k \in \mathbb{Z}^d} (Mx(k))^p \chi_{S_{m,N}}(k) \leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p M\chi_{S_{m,N}}(k),$$

and thus

$$\begin{aligned}
\sum_{k \in S_{m,N}} (Mx(k))^p &\leq K \sum_{k \in \mathbb{Z}^d} |x(k)|^p M\chi_{S_{m,N}}(k) \\
&= K \sum_{k \in S_{m,2N}} |x(k)|^p M\chi_{S_{m,N}}(k) + K \sum_{j=1}^{\infty} \sum_{k \in S_{m,2^{j+1}N} \setminus S_{m,2^j N}} |x(k)|^p M\chi_{S_{m,N}}(k).
\end{aligned}$$

Next note that for every $k \in \mathbb{Z}^d$ we have

$$M\chi_{S_{m,N}}(k) = \sup_{t \in \omega} \frac{1}{(2t+1)^d} \sum_{i \in S_{k,t}} \chi_{S_{m,N}}(i) = \sup_{t \in \omega} \frac{1}{(2t+1)^d} |S_{k,t} \cap S_{m,N}|.$$

Let $j \in \mathbb{N}$, and assume $k \in S_{m,2^{j+1}N} \setminus S_{m,2^j N}$. Then $\|k - m\|_\infty - N > 0$. Now observe that

(1) $S_{k,t} \cap S_{m,N} \neq \varnothing$ if and only if $\|k - m\|_\infty \leq t + N$, that is $t \geq \|k - m\|_\infty - N$, and
(2) $S_{k,t} \cap S_{m,N} = S_{m,N}$ when $\|k - m\|_\infty \leq t - N$, that is when $t \geq \|k - m\|_\infty + N$.

It follows from (1) and (2) above that

$$\begin{aligned}\sup_{t\in\omega}\frac{1}{(2t+1)^d}|S_{k,t}\cap S_{m,N}| &= \sup_{\|k-m\|_\infty-N\leq t\leq\|k-m\|_\infty+N}\frac{1}{(2t+1)^d}|S_{k,t}\cap S_{m,N}|\\ &\leq \frac{(2N+1)^d}{\big(2(\|k-m\|_\infty-N)+1\big)^d}\\ &\leq \left(\frac{3}{2}\right)^d\frac{N^d}{(\|k-m\|_\infty-N)^d}.\end{aligned}$$

Also observe that for every $k\in\mathbb{Z}^d$ we have $M\chi_{S_{m,N}}(k)\leq M\mathbf{1}(k)=1$, where $\mathbf{1}$ denotes the constant function on $\mathbb{Z}^d$ taking value one. Hence

$$\begin{aligned}&K\sum_{k\in S_{m,2N}}|x(k)|^pM\chi_{S_{m,N}}(k)+K\sum_{j=1}^\infty\sum_{k\in S_{m,2^{j+1}N}\setminus S_{m,2^jN}}|x(k)|^pM\chi_{S_{m,N}}(k)\\ &\quad\leq K\sum_{k\in S_{m,2N}}|x(k)|^p+\left(\frac{3}{2}\right)^dK\sum_{j=1}^\infty\sum_{k\in S_{m,2^{j+1}N}\setminus S_{m,2^jN}}|x(k)|^p\frac{N^d}{(\|k-m\|_\infty-N)^d}.\end{aligned}$$

Now if $k\in S_{m,2^{j+1}N}\setminus S_{m,2^jN}$ then $\|k-m\|_\infty-N>2^jN-N\geq 2^{j-1}N$. Thus

$$\begin{aligned}&K\sum_{k\in S_{m,2N}}|x(k)|^p+\left(\frac{3}{2}\right)^dK\sum_{j=1}^\infty\sum_{k\in S_{m,2^{j+1}N}\setminus S_{m,2^jN}}|x(k)|^p\frac{N^d}{(\|k-m\|_\infty-N)^d}\\ &\quad\leq K\sum_{k\in S_{m,2N}}|x(k)|^p+\left(\frac{3}{2}\right)^dK\sum_{j=1}^\infty\sum_{k\in S_{m,2^{j+1}N}}|x(k)|^p\frac{N^d}{(2^{j-1}N)^d}\\ &\quad= K\sum_{k\in S_{m,2N}}|x(k)|^p+\left(\frac{3}{2}\right)^dK\sum_{j=1}^\infty\frac{1}{(2^d)^{j-1}}\sum_{k\in S_{m,2^{j+1}N}}|x(k)|^p.\end{aligned}$$

Next observe that for every $t\in\mathbb{Z}^d$ and all $n\in\omega$ we have

$$\sum_{k\in S_{t,n}}|x(k)|^p\leq\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2n+1)^{d-\frac{dp}{q}}.$$

Hence

$$
\begin{aligned}
K \sum_{k\in S_{m,2N}} |x(k)|^p &+ \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{1}{(2^d)^{j-1}} \sum_{k\in S_{m,2^{j+1}N}} |x(k)|^p \\
&\leq K\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(4N+1)^{d-\frac{dp}{q}} + \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{1}{(2^d)^{j-1}} \|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2^{j+2}N+1)^{d-\frac{dp}{q}} \\
&\leq 2^{d-\frac{dp}{q}} K\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2N+1)^{d-\frac{dp}{q}} + \left(\frac{3}{2}\right)^d K \sum_{j=1}^{\infty} \frac{(2^{j+1})^{d-\frac{dp}{q}}}{(2^d)^{j-1}} \|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2N+1)^{d-\frac{dp}{q}} \\
&= 2^{d-\frac{dp}{q}} K\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2N+1)^{d-\frac{dp}{q}} + 3^d(2^{d-\frac{dp}{q}})\left(\frac{1}{1-2^{-\frac{dp}{q}}}-1\right) K\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2N+1)^{d-\frac{dp}{q}} \\
&\leq C\|x\|^p_{\ell^p_q(\mathbb{Z}^d)}(2N+1)^{d-\frac{dp}{q}},
\end{aligned}
$$

where $\frac{C}{2} = \left(2^{d-\frac{dp}{q}}K\right) \vee \left(3^d(2^{d-\frac{dp}{q}})\left(\frac{1}{1-2^{-\frac{dp}{q}}}-1\right)K\right)$. Thus for every $m \in \mathbb{Z}^d$ and all $N \in \mathbb{N}$,

$$
(2N+1)^{\frac{d}{q}-\frac{d}{p}} \left(\sum_{k\in S_{m,N}} \big(Mx(k)\big)^p\right)^{\frac{1}{p}} \leq C^{1/p}\|x\|_{\ell^p_q(\mathbb{Z}^d)}.
$$

That this inequality holds for $N = 0$ (with $C = 1$) follows from Lemma 3.1. Therefore,

$$
\|Mx\|_{\ell^p_q(\mathbb{Z}^d)} \leq (C^{1/p} \vee 1)\|x\|_{\ell^p_q(\mathbb{Z}^d)},
$$

which completes the proof. □

As an application of Theorem 3.2, we obtain the boundedness of some Riesz potentials on discrete Morrey spaces.

Theorem 3.3. *Let $0 < \alpha < d$ and $1 < p < q < \frac{d}{\alpha}$. Define*

$$
I_\alpha x(k) = \sum_{i\in\mathbb{Z}^d\setminus\{k\}} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} \quad (x \in \ell^p_q(\mathbb{Z}^d),\ k \in \mathbb{Z}^d). \tag{1}
$$

Set $s = \frac{dp}{d-\alpha q}$ and $t = \frac{qs}{p}$. Then $I_\alpha x \in \ell^s_t(\mathbb{Z}^d)$ for every $x \in \ell^p_q(\mathbb{Z}^d)$, and there exists a $C > 0$ such that

$$
\|I_\alpha x\|_{\ell^s_t(\mathbb{Z}^d)} \leq C\|x\|_{\ell^p_q(\mathbb{Z}^d)} \quad (x \in \ell^p_q(\mathbb{Z}^d)).
$$

Proof. Let $x \in \ell_q^p(\mathbb{Z}^d)$, and let $m \in \mathbb{Z}^d$. Then

$$\begin{aligned} Mx(m) &= \sup_{N\in\omega} \frac{1}{(2N+1)^d} \sum_{k\in S_{m,N}} |x(k)| \\ &\leq |x(m)| \vee \sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \sum_{k\in S_{m,N}} |x(k)| \\ &\leq 2^d \left(\frac{1}{2^d} \sum_{k\in S_{m,1}} |x(m)| \vee \sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \sum_{k\in S_{m,N}} |x(k)| \right) \\ &\leq 2^d \sup_{r\geq 1} \frac{1}{(2r)^d} \sum_{k\in\mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sup_{r\geq 1} \frac{1}{(2r)^d} \sum_{k\in\mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)| &\leq \sup_{r\geq 1} \frac{1}{(2\lfloor r\rfloor)^d} \sum_{k\in\mathbb{Z}^d, \|m-k\|_\infty \leq \lfloor r\rfloor} |x(k)| \\ &= \sup_{N\in\mathbb{N}} \frac{1}{(2N)^d} \sum_{k\in S_{m,N}} |x(k)| \\ &\leq \left(\frac{3}{2}\right)^d Mx(m). \end{aligned}$$

Thus

$$\left(\frac{2}{3}\right)^d \sup_{r\geq 1} \frac{1}{(2r)^d} \sum_{k\in\mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)| \leq Mx(m) \leq 2^d \sup_{r\geq 1} \frac{1}{(2r)^d} \sum_{k\in\mathbb{Z}^d, \|m-k\|_\infty \leq r} |x(k)|. \quad (2)$$

Next let $r \geq 1$, and put $k \in \mathbb{Z}^d$. Then

$$I_\alpha x(k) = \sum_{i\in\mathbb{Z}^d\setminus\{k\}} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} = \sum_{0<\|k-i\|_\infty\leq r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} + \sum_{\|k-i\|_\infty > r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}}.$$

Define

$$I_1 := \sum_{0<\|k-i\|_\infty\leq r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}} \quad \text{and} \quad I_2 := \sum_{\|k-i\|_\infty > r} \frac{x(i)}{\|k-i\|_\infty^{d-\alpha}}.$$

Then

$$|I_1| \leq \sum_{j=0}^{\infty} \sum_{r2^{-j-1}<\|k-i\|_\infty\leq r2^{-j}} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}}.$$

If $\|k-i\|_\infty > r2^{-j-1}$ then $\|k-i\|_\infty^{\alpha-d} < r^{\alpha-d}2^{-j\alpha+jd-\alpha+d}$. Thus

$$\sum_{j=0}^{\infty} \sum_{r2^{-j-1}<\|k-i\|_\infty \leq r2^{-j}} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}} < \sum_{j=0}^{\infty} \sum_{r2^{-j-1}<\|k-i\|_\infty \leq r2^{-j}} |x(i)| r^{\alpha-d}2^{-j\alpha+jd-\alpha+d}$$
$$\leq r^\alpha 2^{d-\alpha} \sum_{j=0}^{\infty} 2^{-j\alpha} \frac{1}{(r2^{-j})^d} \sum_{0<\|k-i\|_\infty \leq r2^{-j}} |x(i)|.$$

Let $J = \max\{j \in \omega : r2^{-j} \geq 1\}$. Since $\sum_{0<\|k-i\|_\infty \leq r2^{-j}} |x(i)|$ is an empty sum for all $j > J$, we have

$$r^\alpha 2^{d-\alpha} \sum_{j=0}^{\infty} 2^{-j\alpha} \frac{1}{(r2^{-j})^d} \sum_{0<\|k-i\|_\infty \leq r2^{-j}} |x(i)| = r^\alpha 2^{d-\alpha} \sum_{j=0}^{J} 2^{-j\alpha} \frac{1}{(r2^{-j})^d} \sum_{0<\|k-i\|_\infty \leq r2^{-j}} |x(i)|.$$

Using (2), there exists a constant C_0 (for brevity, we do not record the precise value of this constant) for which

$$r^\alpha 2^{d-\alpha} \sum_{j=0}^{J} 2^{-j\alpha} \frac{1}{(r2^{-j})^d} \sum_{0<\|k-i\|_\infty \leq r2^{-j}} |x(i)| \leq C_0 r^\alpha Mx(k).$$

Next note that

$$|I_2| \leq \sum_{j=0}^{\infty} \sum_{2^j r<\|k-i\|_\infty \leq 2^{j+1}r} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}}.$$

If $\|k-i\|_\infty > 2^j r$ then $\|k-i\|_\infty^{\alpha-d} < (2^j r)^{\alpha-d}$. Hence we obtain

$$\sum_{j=0}^{\infty} \sum_{2^j r<\|k-i\|_\infty \leq 2^{j+1}r} \frac{|x(i)|}{\|k-i\|_\infty^{d-\alpha}} < \sum_{j=0}^{\infty} (2^j r)^{\alpha-d} \sum_{\|k-i\|_\infty \leq 2^{j+1}r} |x(i)|$$
$$\leq \sum_{j=0}^{\infty} (2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}} \left(\sum_{\|k-i\|_\infty \leq 2^{j+1}r} 1 \right)^{\frac{1}{p'}} (2^j r)^{\frac{d}{q}-\frac{d}{p}} \left(\sum_{\|k-i\|_\infty \leq 2^{j+1}r} |x(i)|^p \right)^{\frac{1}{p}},$$

where we use Hölder's inequality in the inequality above with reference to the Hölder conjugate p' of p. Moreover, we have

$$\begin{aligned}
&\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}}\left(\sum_{\|k-i\|_\infty\le 2^{j+1}r}1\right)^{\frac{1}{p'}}(2^j r)^{\frac{d}{q}-\frac{d}{p}}\left(\sum_{\|k-i\|_\infty\le 2^{j+1}r}|x(i)|^p\right)^{\frac{1}{p}}\\
&\quad\le\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}}\Big(2^{j+2}r+1\Big)^{\frac{d}{p'}}(2^j\lfloor r\rfloor)^{\frac{d}{q}-\frac{d}{p}}\left(\sum_{\|k-i\|_\infty\le 2^{j+1}\lfloor r\rfloor}|x(i)|^p\right)^{\frac{1}{p}}\\
&\quad\le 8^{\frac{d}{p}-\frac{d}{q}}\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}}\Big(2^{j+2}r+1\Big)^{\frac{d}{p'}}(2^{j+2}\lfloor r\rfloor+1)^{\frac{d}{q}-\frac{d}{p}}\left(\sum_{i\in S_{k,2^{j+1}\lfloor r\rfloor}}|x(i)|^p\right)^{\frac{1}{p}}\\
&\quad\le 8^{\frac{d}{p}-\frac{d}{q}}\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}}\Big(2^{j+2}r+1\Big)^{\frac{d}{p'}}\|x\|_{\ell^p_q(\mathbb{Z}^d)}.
\end{aligned}$$

It is readily checked that there exist constants $C_1, C_2 > 0$ such that

$$\begin{aligned}
8^{\frac{d}{p}-\frac{d}{q}}\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}}\Big(2^{j+2}r+1\Big)^{\frac{d}{p'}}\|x\|_{\ell^p_q(\mathbb{Z}^d)}&\le C_1\sum_{j=0}^{\infty}(2^j r)^{\alpha-d+\frac{d}{p}-\frac{d}{q}+\frac{d}{p'}}\|x\|_{\ell^p_q(\mathbb{Z}^d)}\\
&=C_2\|x\|_{\ell^p_q(\mathbb{Z}^d)}r^{\alpha-\frac{d}{q}}.
\end{aligned}$$

Thus for $C_3 = C_0 \vee C_2$,

$$|I_\alpha x(k)|\le C_3\Big(r^\alpha Mx(k)+r^{\alpha-\frac{d}{q}}\|x\|_{\ell^p_q(\mathbb{Z}^d)}\Big). \tag{3}$$

Suppose for the moment that $k \in \mathbb{Z}^d$ satisfies $Mx(k) \neq 0$. By Lemma 3.1 we can take $r := \left(\frac{\|x\|_{\ell^p_q(\mathbb{Z}^d)}}{Mx(k)}\right)^{q/d} \ge 1$ in (3) above and obtain

$$\begin{aligned}
|I_\alpha x(k)|&\le C_3\left[\left(\left(\frac{\|x\|_{\ell^p_q(\mathbb{Z}^d)}}{Mx(k)}\right)^{q/d}\right)^{\alpha}Mx(k)+\left(\left(\frac{\|x\|_{\ell^p_q(\mathbb{Z}^d)}}{Mx(k)}\right)^{q/d}\right)^{\alpha-\frac{d}{q}}\|x\|_{\ell^p_q(\mathbb{Z}^d)}\right]\\
&=2C_3\big(Mx(k)\big)^{1-\frac{\alpha q}{d}}\|x\|^{\frac{\alpha q}{d}}_{\ell^p_q(\mathbb{Z}^d)}.
\end{aligned}$$

On the other hand, if $Mx(k) = 0$ then $x = 0$, and so $I_\alpha x(k) = 0$. Thus the inequality

$$|I_\alpha x(k)|\le 2C_3\big(Mx(k)\big)^{1-\frac{\alpha q}{d}}\|x\|^{\frac{\alpha q}{d}}_{\ell^p_q(\mathbb{Z}^d)}$$

holds in this case as well. Hence

$$\begin{aligned}
\|I_\alpha x\|_{\ell^s_t(\mathbb{Z}^d)} &= \sup_{m\in\mathbb{Z},N\in\omega}\left(\frac{1}{(2N+1)^{d-\frac{ds}{t}}}\sum_{k\in S_{m,N}}|I_\alpha x(k)|^s\right)^{\frac{1}{s}}\\
&\le \sup_{m\in\mathbb{Z},N\in\omega}\left(\frac{1}{(2N+1)^{d-\frac{ds}{t}}}\sum_{k\in S_{m,N}}\left|2C_3\big(Mx(k)\big)^{1-\frac{\alpha q}{d}}\|x\|^{\alpha q/d}_{\ell^p_q(\mathbb{Z}^d)}\right|^s\right)^{\frac{1}{s}}\\
&= 2C_3\|x\|^{\alpha q/d}_{\ell^p_q(\mathbb{Z}^d)}\sup_{m\in\mathbb{Z},N\in\omega}\left(\frac{1}{(2N+1)^{d-\frac{dp}{q}}}\sum_{k\in S_{m,N}}\big(Mx(k)\big)^p\right)^{\frac{p}{ps}}\\
&= 2C_3\|x\|^{\alpha q/d}_{\ell^p_q(\mathbb{Z}^d)}\|Mx\|^{\frac{p}{s}}_{\ell^p_q(\mathbb{Z}^d)}.
\end{aligned}$$

By Theorem 3.2, there exists $C>0$ such that

$$2C_3\|x\|^{\alpha q/d}_{\ell^p_q(\mathbb{Z}^d)}\|Mx\|^{\frac{p}{s}}_{\ell^p_q(\mathbb{Z}^d)} \le C\|x\|^{\alpha q/d}_{\ell^p_q(\mathbb{Z}^d)}\|x\|^{p/s}_{\ell^p_q(\mathbb{Z}^d)} = C\|x\|_{\ell^p_q(\mathbb{Z}^d)},$$

as desired. □

Remark 3.4. The operator defined in (1) may be considered as the discrete fractional integral operator (for the continuous version, see for instance [3]). The proof that we presented above uses an analogue of Hedberg's inequality, which we obtain right after we have inequality (3). The boundedness of this operator on the $\ell^p(\mathbb{Z}^d)$ spaces can be found in [12].

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