Bounded linear functionals on the n-normed space of p-summable sequences

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The notion of n-normed spaces was introduced by S. Gähler in the 1960's. Later on, the concept of n-inner product spaces was developed by A. Misiak at the end of the 1980's.

If in a normed space one speaks about the length of a vector, then in an n-normed space we discuss about the volume of the parallelepiped spanned by a set of n-vectors.

In this talk, we shall be interested in bounded linear functionals on an n-normed space X, using the n-norm as our main tool.

We study the duality properties and show that the space X' of bounded linear functionals on X also forms an n-normed space.

We shall present more results on bounded multilinear n-functionals on the space of p-summable sequences being equipped with an n-norm.

Who's Who, Among Others . . .

S. Gähler and collaborators, incl. C. Diminnie and A. White (1960's-1970's)
K. Iseki (1975-1976)
S.N. Lai and A.K. Singh (1978)
A. Khan and A. Siddiqui (1982)
G. Godini (1985)
A. Misiak (1989)
M.S. Khan dan M.D. Khan (1993)
D.R. Jain and R. Chugh (1995)
Y.J. Cho and collaborators, incl. S.S. Kim and N.J. Huang (1990's)
H. Mazaheri and collaborators (2007-now)
H. Gunawan and collaborators (2000-now)

Definition of n-normed spaces

Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here d can be infinite.) A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an n-norm on X. The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Notes

In an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$, we have

$\|x_1, \ldots, x_n\| \geq 0$ for every $x_1, \ldots, x_n \in X$

and

$\|x_1, x_2, \ldots, x_n\| = \|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2, \ldots, x_n\|$ for every $x_1, \ldots, x_n \in X$ and $\alpha_2, \ldots, \alpha_n \in \mathbb{R}$.

Standard example

If X is a real inner product space of dimension $d \geq n$, then the following formula defines an n-norm on X:

$$\|x_1, \dots, x_n\|^S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X.

Geometrically, $\|x_1, \dots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X.

General example (Gähler's formula)

If X is a normed space, one may define an n-norm on X by using Gähler's formula

$$\|x_1, \ldots, x_n\|^G := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i = 1, \ldots, n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X.

Definition of n-inner product spaces

A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X^{n+1} satisfying the following five properties

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$; and $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(I3) $\langle x_0, x_1 | x_2, \ldots, x_n \rangle = \langle x_1, x_0 | x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x_0, x_1 | x_2, \ldots, x_n \rangle = \alpha \langle x_0, x_1 | x_2, \ldots, x_n \rangle$ for every $\alpha \in \mathbb{R}$;

(I5)
$\langle x_0 + x_0', x_1 | x_2, \ldots, x_n \rangle = \langle x_0, x_1 | x_2, \ldots, x_n \rangle + \langle x_0', x_1 | x_2, \ldots, x_n \rangle$,

is called an n-inner product on X. The pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ is called an n-inner product space.

If $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ is an n-inner product on X, then we have **the Cauchy-Schwarz inequality**

$$\langle x_0, x_1 | x_2, \dots, x_n \rangle^2 \leq \langle x_0, x_0 | x_2, \dots, x_n \rangle \langle x_1, x_1 | x_2, \dots, x_n \rangle.$$

Moreover, the following function

$$\|x_1, x_2, \dots, x_n\| := \langle x_1, x_1 | x_2, \dots, x_n \rangle^{1/2}$$

defines an n-norm on X.

Standard example

If X is a real inner product space of dimension $d \geq n$, then the following formula defines an n-inner product on X:

$$\langle x_0, x_1 | x_2, \ldots, x_n \rangle_S := \begin{vmatrix} \langle x_0, x_1 \rangle & \langle x_0, x_2 \rangle & \cdots & \langle x_0, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X.

Geometrically, $\langle x_0, x_1 | x_2, \ldots, x_n \rangle_S$ has something to do with the angle between two n-dimensional parallelepipeds in X, one spanned by $x_0, x_2, \ldots, x_n$ and the other spanned by $x_1, x_2, \ldots, x_n$. (Here the two parallelepipeds intersect on the $(n-1)$ dimensional base spanned by $x_2, \ldots, x_n$.)

Bounded linear functionals (of 1st index)

Let $(X, \|\cdot, \dots, \cdot\|)$ be a real n-normed space and $f : X \to \mathbb{R}$ be a linear functional on X.

Fix a linearly independent set $Y := \{y_1, \dots, y_n\}$ in X.

We say that f is *bounded with respect to* Y if and only if there exists $K > 0$ such that

$$|f(x)| \leq K \sum \|x, y_{i_2}, \dots, y_{i_n}\| \tag{1}$$

for all $x \in X$, where the sum is taken over $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ with $i_2 < \dots < i_n$.

Note

One might ask why we do not just take a linearly independent set $\{y_2, \ldots, y_n\}$ in X and put $|f(x)| \leq K \|x, y_2, \ldots, y_n\|$ for all $x \in X$.

The drawback with this is that for a nonzero vector x in the linear span of $\{y_2, \ldots, y_n\}$, we have $\|x, y_2, \ldots, y_n\| = 0$ while $f(x) \neq 0$.

This problem is overcome by taking a set of n linearly independent vectors and form the sum as in (1). Indeed, one might observe that the sum is equal to 0 if and only if $x = 0$.

For simplicity, we shall say 'bounded' instead of 'bounded with respect to Y'.

Clearly the space X_1' of all linear functionals which are bounded on X forms a vector space.

Now, for $f \in X_1'$, we define

$$\|f\|_1 := \inf\{K > 0 \,:\, (1) \text{ holds}\}. \tag{2}$$

It is easy to see that

$$\|f\|_1 = \sup\{|f(x)| \,:\, \sum \|x, y_{i_2}, \ldots, y_{i_n}\| \leq 1\}$$

Moreover, the formula (2) defines a norm on X_1'.

An example

Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space, and $\|\cdot, \dots, \cdot\| = \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle^{1/2}$ be the induced n-norm on X.

With respect to the set $Y := \{y_1, \dots, y_n\}$, define $f : X \to \mathbb{R}$ by

$$f(x) := \sum \langle x, y_{i_1} | y_{i_2}, \dots, y_{i_n} \rangle, \tag{3}$$

where the sum is taken over $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ with $i_2 < \dots < i_n$ and $i_1 \in \{1, \dots, n\} \setminus \{i_2, \dots, i_n\}$.

Fact 1 *The linear functional f defined by (3) is bounded with* $\|f\|_1 = \|y_1, \dots, y_n\|$.

Proof. For every $x \in X$, we have

$$\begin{aligned} |f(x)| &\le \sum |\langle x, y_{i_1} | y_{i_2}, \dots, y_{i_n} \rangle| \\ &\le \sum \|x, y_{i_2}, \dots, y_{i_n}\| \, \|y_{i_1}, y_{i_2}, \dots, y_{i_n}\| \\ &= \|y_1, \dots, y_n\| \sum \|x, y_{i_2}, \dots, y_{i_n}\|. \end{aligned}$$

Thus f is bounded with $\|f\|_1 \le \|y_1, \dots, y_n\|$.

For $x := \|y_1, \dots, y_n\|^{-1} y_1$, one has $|f(x)| = \|y_1, \dots, y_n\|$; and so we conclude that $\|f\|_1 = \|y_1, \dots, y_n\|$. □

Bounded linear functionals of p-th index

Fix a linearly independent set $Y := \{y_1, \dots, y_n\}$ in X and $1 \le p \le \infty$. We say that f is *bounded of p-th index* (with respect to Y) if and only if there exists $K > 0$ such that

$$|f(x)| \le K \left(\sum \|x, y_{i_2}, \dots, y_{i_n}\|^p\right)^{1/p} \tag{4}$$

where the sum is taken over $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ with $i_2 < \cdots < i_n$. [If $p = \infty$, then the sum is the maximum of all possible values of $\|x, y_{i_2}, \dots, y_{i_n}\|$.]

As in the case where $p = 1$, the space X'_p of all linear functionals which are bounded of p-index on X forms a vector space.

Now, for $f \in X'_p$, we define

$$\|f\|_p := \inf\{K > 0 \,:\, (4) \text{ holds}\}. \tag{5}$$

One then has

$$\|f\|_p = \sup\{|f(x)| \,:\, \sum \|x, y_{i_2}, \ldots, y_{i_n}\|^p \le 1\}.$$

Moreover, the formula (5) defines a norm on X'_p.

The following theorem tells us that X_1' and X_p' are identical as a set.

Theorem 2 *Let f be a linear functional on X. If f is bounded (of 1st index), then f is bounded of p-th index; and vice versa. In other words, $X_p' = X_1'$. Furthermore, the norms are equivalent:*

$$\|f\|_1 \leq \|f\|_p \leq n^{1/p'} \|f\|_1,$$

for every $f \in X_1'$.

`Remark`. We may now denote by X' the set of all bounded linear functionals on X and call it the *dual space* of X (with respect to Y).

Another example

Let $Y := \{y_1, \dots, y_n\}$ be a linearly independent set in X and $y \neq y_i$ for $i = 1, \dots, n$. Define $f_y : X \to \mathbb{R}$ by

$$f_y(x) := \sum \langle x, y | y_{i_2}, \dots, y_{i_n} \rangle, \tag{6}$$

where the sum is taken over $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ with $i_2 < \cdots < i_n$.

Then f_y is linear. Moreover, we have:

Fact 3 *The linear functional f_y defined by (6) is bounded of 2nd index with $\|f_y\|_2 = \left(\sum \|y, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}$.*

Theorem 4 [8] *Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space and $\|\cdot, \ldots, \cdot\| = \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle^{1/2}$ be the induced n-norm on X. With respect to the linearly independent set $Y = \{y_1, \ldots, y_n\}$, the mapping $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ given by*

$$\langle x, y \rangle := \sum \langle x, y | y_{i_2}, \ldots, y_{i_n} \rangle \tag{7}$$

defines an inner product on X, and its induced norm $\| \cdot \|_2 : X \to \mathbb{R}$ is given by

$$\|x\|_2 := \left(\sum \|x, y_{i_2}, \ldots, y_{i_n}\|^2\right)^{1/2}. \tag{8}$$

An analog of Riesz-Fréchet Theorem

Corollary 5 *If* $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ *is complete with respect to the norm* $\| \cdot \|_2$ *in (8), then for every linear functional* f *which is bounded of 2nd index on* X *there exists a unique* $y \in X$ *such that*

$$f(x) = \langle x, y \rangle, \quad x \in X,$$

where $\langle \cdot, \cdot \rangle$ *is the inner product in (7). Moreover, we have* $\|y\|_2 = \|f\|_2$.

Theorem 6 *Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space, X' be the dual space of X (with respect to Y), and $\|\cdot\|_2$ be the derived norm on X given by*

$$\|x\|_2 := \left(\sum \|x, y_{i_2}, \dots, y_{i_n}\|^2\right)^{1/2}.$$

Then, the function $\|\cdot, \dots, \cdot\|' : (X')^n \to \mathbf{R}$ given by

$$\|f_1, \dots, f_n\|' := \sup_{x_i \in X,\ \|x_i\|_2 \le 1} \begin{vmatrix} f_1(x_1) & \dots & f_n(x_1) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \dots & f_n(x_n) \end{vmatrix}$$

defines an n-norm on X'.

The space of p-summable sequences

Using Gähler's formula, ℓ^p may be equipped with the following n-norm:

$$\|x_1, \dots, x_n\|_p^G := \sup_{y_i \in \ell^{p'},\ \|y_i\|_{p'} \le 1} \begin{vmatrix} \sum_k x_{1k} y_{1k} & \cdots & \sum_k x_{1k} y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk} y_{1k} & \cdots & \sum_k x_{nk} y_{nk} \end{vmatrix}, \tag{9}$$

where p' denotes the dual exponent of p.

Another formula of n-norm can also be defined on ℓ^p, namely

$$\|x_1, \dots, x_n\|_p^H := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \left| \det \begin{pmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{pmatrix} \right|^p \right]^{\frac{1}{p}}, \quad (10)$$

where $x_i = \{x_{ij}\}$, $i = 1, \dots, n$.

As shown in [13], the two n-norms are equivalent. On ℓ^2, both n-norms coincide with the standard n-norm $\|\cdot, \cdots, \cdot\|^S$ [6].

Observe that the determinant in the right hand side of (9) can be rewritten as

$$\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n}\end{vmatrix}\begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n}\end{vmatrix}.$$

By Hölder's inequality, we find that it is dominated by

$$\|x_1,\dots,x_n\|_p^H\|y_1,\dots,y_n\|_{p'}^H.$$

This inspires us to define another n-norm on ℓ^p, namely

$$\|x_1,\dots,x_n\|_p^I := \sup_{y_i\in\ell^{p'},\,\|y_1,\dots,y_n\|_{p'}^H\le 1} \begin{vmatrix} \sum_k x_{1k}y_{1k} & \cdots & \sum_k x_{1k}y_{nk} \\ \vdots & \ddots & \vdots \\ \sum_k x_{nk}y_{1k} & \cdots & \sum_k x_{nk}y_{nk} \end{vmatrix}. \tag{11}$$

Fact 7 *The three n-norms $\|\cdot,\dots,\cdot\|_p^I$, $\|\cdot,\dots,\cdot\|_p^H$, and $\|\cdot,\dots,\cdot\|_p^G$ are equivalent.*

Multilinear n-functionals

By a *multilinear n-functional* on a real vector space X we mean a mapping $F : X^n \to \mathbb{R}$ which is linear in each variable.

A multilinear n-functional F is *bounded* on an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ if and only if there exists $K > 0$ such that

$$|F(x_1, \dots, x_n)| \leq K \, \|x_1, \dots, x_n\| \tag{12}$$

for every $x_1, \dots, x_n \in X$.

For a bounded multilinear n-functional F, we define

$$\|F\| := \inf\{K > 0 \, : \, (12) \text{ holds}\},$$

or equivalently

$$\|F\| := \sup\{|F(x_1, \dots, x_n)| \, : \, \|x_1, \dots, x_n\| \leq 1\}.$$

To give an example, let $y_1, \dots, y_n \in \ell^{p'}$ where p' is the dual exponent of p. Define

$$F(x_1, \dots, x_n) := \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix}$$

for $x_1, \dots, x_n \in \ell^p$. Since

$$|F(x_1, \dots, x_n)| \le \|x_1, \dots, x_n\|_p^H \|y_1, \dots, y_n\|_{p'}^H,$$

we see that F is bounded on $(\ell^p, \|\cdot, \dots, \cdot\|_p^H)$ with

$$\|F\| \le \|y_1, \dots, y_n\|_{p'}^H.$$

For $p = 2$, we have the following fact [6].

Fact 8 *Consider the* n*-normed space* $(\ell^2, \|\cdot, \ldots, \cdot\|_2^H)$ *and let* F *be the* n*-functional*

$$F(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \begin{vmatrix} y_{1j_1} & \cdots & y_{1j_n} \\ \vdots & \ddots & \vdots \\ y_{nj_1} & \cdots & y_{nj_n} \end{vmatrix},$$

where $y_1, \ldots, y_n$ *are fixed. Then we have*

$$\|F\| = \|y_1, \ldots, y_n\|_2^H.$$

Remark. The set X^* of all bounded multilinear n-functionals on ℓ^p may be indentified as a space of p'-summable sequences with multiple indices [12].

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