WEAK (p,q) INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED NON-HOMOGENEOUS MORREY SPACES

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Abstract

In this paper, we prove weak (p,q) inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces for $1 \leq p < q < \infty$. The proof involves an inequality for the modified Hardy-Littlewood maximal operator and the Chebyshev inequality. Our results generalize those obtained by Garcia-Cuerva and Gatto [1] and also extend those by Sihwaningrum *et al.* [5].
Key words: Fractional integral operators, generalized non-homogeneous Morrey spaces, doubling condition, Chebyshev inequality.

1 Introduction

Let μ be a positive Borel measure on $\mathbb{R}^d$. We say that the space $(\mathbb{R}^d, \mu)$ is *non-homogeneous* if μ satisfies the *growth condition* of order n with $0 \leq n \leq d$, that is, there exists a constant $C > 0$ such that

$$\mu(B(a,r)) \leq C\, r^n$$

for any ball $B(a,r)$ centered at $a \in \mathbb{R}^d$ with radius $r > 0$. When $0 < n \leq d$, we define the fractional integral operator I_α, for $0 < \alpha < n$, by

$$I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y), \quad x \in \mathbb{R}^d,$$

for any suitable function f on $\mathbb{R}^d$. Note that when $n = d$ and $\mu = m$ being the Lebesgue measure on $\mathbb{R}^d$, I_α is the classical fractional integral operator introduced by Hardy and Littlewood [3] and Sobolev [6].

One of the important results about the classical fractional integral operator I_α is the Hardy-Littlewood-Sobolev inequality, which amounts to the boundedness of I_α from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $1 < p < \frac{d}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$.

Meanwhile, for $p = 1$, we have a weak type inequality for I_α: if $\frac{1}{q} = 1 - \frac{\alpha}{d}$, then there exists a constant $C > 0$ such that

$$m\left(\left\{x \in \mathbb{R}^d : |I_\alpha f(x)| > \gamma\right\}\right) \leq C\left(\frac{\|f\|_{L^1(\mathbb{R}^d)}}{\gamma}\right)^q$$

for every $\gamma > 0$ (see [7]).

Next, for $1 \leq p < \infty$, we define the non-homogeneous Lebesgue space $L^p(\mu) = L^p(\mathbb{R}^d, \mu)$ to be the set of all measurable functions f such that

$$\|f\|_{L^p(\mu)} = \left(\int_{\mathbb{R}^d} |f(y)|^p d\mu(y)\right)^{1/p} < \infty.$$

In [1], Garcia-Cuerva and Gatto proved a weak type inequality for I_α on these spaces, as in the following theorem.

Theorem 1.1 *[1] If $1 \leq p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a constant $C > 0$ such that*

$$\mu\left(\left\{x \in \mathbb{R}^d : |I_\alpha f(x)| > \gamma\right\}\right) \leq C\left(\frac{\|f\|_{L^p(\mu)}}{\gamma}\right)^q \tag{1.1}$$

for every $\gamma > 0$.

Note that, by using Theorem 1.1 and Marcinkiewicz interpolation theorem, one may obtain the boundedness of I_α from $L^p(\mu)$ to $L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Now, for $1 \leq p < \infty$ and a function $\phi : (0, \infty) \to (0, \infty)$, let us consider the generalized non-homogeneous Morrey space $L^{p,\phi}(\mu) = L^{p,\phi}\left(\mathbb{R}^d, \mu\right)$, which consists of all functions $f \in L^p_{loc}(\mu)$ such that

$$\|f\|_{L^{p,\phi}(\mu)} = \sup_{B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{B(a,r)} |f(y)|^p \, d\mu(y)\right)^{1/p} < \infty.$$

Note that, if $\phi(r) = r^{-n/p}$, then we get $L^{p,\phi}(\mu) = L^p(\mu)$.

In this paper, we shall always assume that ϕ satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that $\frac{1}{C} \leq \frac{\phi(r)}{\phi(s)} \leq C$ whenever $\frac{1}{2} \leq \frac{r}{s} \leq 2$. One may observe that for any function ϕ that satisfies the doubling condition, we have

$$\frac{1}{C}\phi(2^{j+1}r) \leq \int_{2^j r}^{2^{j+1}r} \frac{\phi(t)}{t} dt \leq C\phi(2^{j+1}r)$$

for every $j \in \mathbb{Z}$ and $r > 0$ (see [2]).

A generalization of Theorem 1.1 on generalized non-homogeneous Morrey spaces is given in the following theorem.

Theorem 1.2 *[5] Suppose that* $\int_r^\infty \frac{\phi(t)}{t}dt \leq C\phi(r)$ *for every* $r > 0$ *and for some* $\lambda \in [0, n-\alpha)$ *we have*

$$\int_r^\infty t^{\alpha-1}\phi(t)dt \leq Cr^{\lambda+\alpha-n}, \ r > 0.$$

If $\frac{1}{q} = 1 - \frac{\alpha}{n-\lambda}$*, then there exists a constant* $C > 0$ *such that for any function* $f \in L^{1,\phi}(\mu)$ *and any ball* $B(a,r) \subseteq \mathbb{R}^d$ *we have*

$$\mu\left\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\right\} \leq Cr^n\phi(r)\left(\frac{\|f\|_{L^{1,\phi}(\mu)}}{\gamma}\right)^q \tag{1.2}$$

for every $\gamma > 0$.

Note that, if $\phi(r) = r^{-n}$ and $\lambda = 0$, then this result reduces to the previous inequality (1.1) for $p = 1$.

In this paper, we will prove weak type inequalities for I_α on generalized non-homogeneous Morrey spaces which extend (1.2). We shall use some inequality involving the modified Hardy-Littlewood maximal operator M^n, which is given by

$$M^n f(x) = \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, d\mu(y).$$

In addition, we shall also invoke the Chebyshev inequality, which is presented in the following theorem.

Theorem 1.3 *[4] Let* μ *be a positive Borel measure on* $\mathbb{R}^d$ *and* E *be a measurable subset of* $\mathbb{R}^d$*. If* f *is an integrable function on* E*, then for every* $\gamma > 0$ *we have*

$$\mu\left(\{x \in E : |f(x)| > \gamma\}\right) \leq \frac{1}{\gamma}\int_E |f(x)|\, d\mu(x).$$

Throughout the paper, C denotes a positive constant which is independent of the function f and the variable x, and may have different values from line to line.

2 Main Results

In the proof of Theorem 1.1, Garcia-Cuerva and Gatto [1] use the following inequality — which will also be useful for us here.

Lemma 2.1 *[1] For any ball $B(x,R) \subseteq \mathbb{R}^d$, we have*

$$\int_{B(x,R)} \frac{1}{|x-y|^{n-\alpha}} d\mu(y) \leq C\, R^{\alpha}. \tag{2.1}$$

In addition, to prove weak type inequalities for I_α, we also need the following lemmas.

Lemma 2.2 *Let $1 \leq p < \infty$. If ϕ satisfies*

$$\int_r^\infty \frac{\phi(t)^p}{t}\, dt \leq C\phi(r)^p$$

for every $r > 0$, then for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a,r) \subseteq \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |f(y)|^p M^n \chi_{B(a,r)}(y) d\mu(y) \leq C\, r^n \phi(r)^p ||f||^p_{L^{p,\phi}(\mu)}. \tag{2.2}$$

Proof. The proof is adapted from [5]. Since μ satisfies the growth condition, we have $M^n\chi_{B(a,r)}(y) \leq C$ for $y \in B(a,2r)$ and $M^n\chi_{B(a,r)}(y) \leq C2^{-jn}$ for $y \in B(a,2^{j+1}r)\backslash B(a,2^j r)$ where $j \in \mathbb{N}$. By using the definition of $\|f\|_{L^{p,\phi}(\mu)}$ and the doubling condition of ϕ^p, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} |f(y)|^p M^n \chi_{B(a,r)}(y) d\mu(y) \\
&\leq \int_{B(a,2r)} |f(y)|^p M^n \chi_{B(a,r)}(y) d\mu(y) \\
&\quad + \sum_{j=1}^{\infty} \int_{B(a,2^{j+1}r)\backslash B(a,2^j r)} |f(y)|^p M^n \chi_{B(a,r)}(y) d\mu(y) \\
&\leq C \left(\int_{B(a,2r)} |f(y)|^p d\mu(y) + \sum_{j=1}^{\infty} \int_{B(a,2^{j+1}r)\backslash B(a,2^j r)} \frac{|f(y)|^p}{2^{jn}} d\mu(y) \right) \\
&\leq C \left((2r)^n \phi(2r)^p \|f\|^p_{L^{p,\phi}(\mu)} + \sum_{j=1}^{\infty} \left(2^{j+1}r\right)^n \phi(2^{j+1}r)^p \frac{\|f\|^p_{L^{p,\phi}(\mu)}}{2^{jn}} \right) \\
&\leq C\, r^n \|f\|^p_{L^{p,\phi}(\mu)} \sum_{j=0}^{\infty} \phi(2^{j+1}r)^p \\
&\leq C\, r^n \|f\|^p_{L^{p,\phi}(\mu)} \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1}r} \frac{\phi(t)^p}{t} dt \\
&\leq C\, r^n \|f\|^p_{L^{p,\phi}(\mu)} \int_r^\infty \frac{\phi(t)^p}{t} dt \\
&\leq C\, r^n \phi(r)^p \|f\|^p_{L^{p,\phi}(\mu)},
\end{aligned}$$

which is the desired inequality. ■

Lemma 2.3 *Let $B(y,R)$ be a ball centered at $y \in \mathbb{R}^d$ with radius $R > 0$, then for any ball $B(a,r) \subseteq \mathbb{R}^d$ we have*

$$\int_{B(y,R)} \frac{\chi_{B(a,r)}(x)}{|x-y|^{n-\alpha}} d\mu(x) \le C\, R^\alpha M^n \chi_{B(a,r)}(y). \tag{2.3}$$

Proof. By using the definition of the maximal operator M^n, we get

$$\begin{aligned}
\int_{B(y,R)} \frac{\chi_{B(a,r)}(x)}{|x-y|^{n-\alpha}} d\mu(x) &= \sum_{j=-\infty}^{-1} \int_{B(y,2^{j+1}R)\setminus B(y,2^jR)} \frac{\chi_{B(a,r)}(x)}{|x-y|^{n-\alpha}}\, d\mu(x) \\
&\le \sum_{j=-\infty}^{-1} \frac{1}{(2^jR)^{n-\alpha}} \int_{B(y,2^{j+1}R)} \chi_{B(a,r)}(x)\, d\mu(x) \\
&\le 2^n R^\alpha M^n \chi_{B(a,r)}(y) \sum_{j=-\infty}^{-1} 2^{j\alpha} \\
&\le C\, R^\alpha M^n \chi_{B(a,r)}(y),
\end{aligned}$$

as desired. ■

With Theorem 1.3 and Lemmas 2.1–2.3, we are now ready to prove weak type inequalities for I_α on generalized non-homogeneous Morrey spaces.

Theorem 2.4 *Let $1 \le p < q < \infty$. Suppose that $\inf_{r>0} \phi(r) = 0$ and ϕ satisfies*

$$\int_r^\infty \frac{\phi(t)^p}{t}\, dt \le C\, \phi(r)^p \quad \text{and} \quad r^\alpha \phi(r) + \int_r^\infty t^{\alpha-1}\phi(t)dt \le C\, \phi(r)^{p/q}$$

for every $r > 0$, then for any function $f \in L^{p,\phi}(\mu)$ and any ball $B(a,r) \subseteq \mathbb{R}^d$ we have

$$\mu\left(\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}\right) \le C\, r^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma}\right)^q,$$

for every $\gamma > 0$.

Proof. For every $x \in B(a,r)$, write $I_\alpha f(x) = I_1(x) + I_2(x)$ where

$$I_1(x) = \int_{B(x,R)} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y) \text{ and } I_2(x) = \int_{\mathbb{R}^d\setminus B(x,R)} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y).$$

By using Hölder's inequality and the definition of $\|f\|_{L^{p,\phi}(\mu)}$, we get

$$
\begin{aligned}
|I_2(x)| &\leq \int_{\mathbb{R}^d\setminus B(x,R)} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \\
&= \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}R)\setminus B(x,2^jR)} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \\
&\leq \sum_{j=0}^{\infty} \frac{1}{(2^jR)^{n-\alpha}} \int_{B(x,2^{j+1}R)} |f(y)| d\mu(y) \\
&= \frac{2^n}{2^\alpha} \sum_{j=0}^{\infty} \frac{(2^{j+1}R)^\alpha}{(2^{j+1}R)^n} \int_{B(x,2^{j+1}R)} |f(y)| d\mu(y) \\
&\leq C \sum_{j=0}^{\infty} \frac{(2^{j+1}R)^\alpha}{(2^{j+1}R)^n} \left(\int_{B(x,2^{j+1}R)} |f(y)|^p d\mu(y)\right)^{1/p} \left(\mu(B(x,2^{j+1}R))\right)^{1-\frac{1}{p}} \\
&\leq C\, \|f\|_{L^{p,\phi}(\mu)} \sum_{j=0}^{\infty} (2^{j+1}R)^\alpha \phi(2^{j+1}R).
\end{aligned}
$$

Since $t \mapsto t^\alpha \phi(t)$ satisfies the doubling condition, we have

$$
\begin{aligned}
|I_2(x)| &\leq C\|f\|_{L^{p,\phi}(\mu)} \sum_{j=0}^{\infty} \int_{2^jR}^{2^{j+1}R} t^{\alpha-1}\phi(t)\, dt \\
&= C\, \|f\|_{L^{p,\phi}(\mu)} \int_R^\infty t^{\alpha-1}\phi(t) dt \\
&\leq C\, \|f\|_{L^{p,\phi}(\mu)} \phi(R)^{p/q}.
\end{aligned}
$$

Let $\tilde{\gamma} = \left(\frac{\gamma}{2C\|f\|_{L^{p,\phi}(\mu)}}\right)^{q/p}$. By our assumptions on ϕ, we can find $R > 0$ such that $\phi(R) \leq \tilde{\gamma}$. For this R, we obtain

$$
|I_2(x)| \leq C\, \|f\|_{L^{p,\phi}(\mu)} \tilde{\gamma}^{p/q} = \frac{\gamma}{2}.
$$

Define $E_\gamma = \{x \in B(a,r) : |I_\alpha f(x)| > \gamma\}$. Since $|I_\alpha f(x)| \leq |I_1(x)| + |I_2(x)|$, we have

$$
\mu\left(E_\gamma\right) \leq \mu\left(\left\{x \in B(a,r) : |I_1(x)| > \frac{\gamma}{2}\right\}\right).
$$

By using Hölder's inequality and the inequality (2.1), we get

$$
\begin{aligned}
|I_1(x)| &\leq \left(\int_{B(x,R)} \frac{|f(y)|^p}{|x-y|^{n-\alpha}} d\mu(y)\right)^{\frac{1}{p}} \left(\int_{B(x,R)} \frac{1}{|x-y|^{n-\alpha}} d\mu(y)\right)^{1-\frac{1}{p}} \\
&\leq C\, R^{\alpha\left(1-\frac{1}{p}\right)} \left(\int_{B(x,R)} \frac{|f(y)|^p}{|x-y|^{n-\alpha}} d\mu(y)\right)^{\frac{1}{p}}.
\end{aligned}
$$

By using the last inequality, the Chebyshev inequality, the inequalities (2.2) and (2.3), together with the condition $r^{\alpha}\phi(r) \le C\phi(r)^{p/q}$, we get

$$\begin{aligned}
\mu\left(E_{\gamma}\right) &\le \mu\left(\left\{x \in B(a,r) : \int_{B(x,R)} \frac{|f(y)|^p}{|x-y|^{n-\alpha}} d\mu(y) > \left(\frac{\gamma}{2C\, R^{\alpha\left(1-\frac{1}{p}\right)}}\right)^p\right\}\right) \\
&\le C\frac{2^p R^{\alpha p\left(1-\frac{1}{p}\right)}}{\gamma^p} \int_{B(a,r)} \int_{B(x,R)} \frac{|f(y)|^p}{|x-y|^{n-\alpha}} d\mu(y) d\mu(x) \\
&\le C\frac{R^{\alpha(p-1)}}{\gamma^p} \int_{\mathbb{R}^d} \int_{B(x,R)} \frac{|f(y)|^p}{|x-y|^{n-\alpha}} \chi_{B(a,r)}(x) d\mu(y) d\mu(x) \\
&= C\frac{R^{\alpha(p-1)}}{\gamma^p} \int_{\mathbb{R}^d} |f(y)|^p \int_{B(y,R)} \frac{\chi_{B(a,r)}(x)}{|x-y|^{n-\alpha}} d\mu(x) d\mu(y) \\
&\le C\frac{R^{\alpha p}}{\gamma^p} \int_{\mathbb{R}^d} |f(y)|^p M^n \chi_{B(a,r)}(y) d\mu(y) \\
&\le C\frac{\phi(R)^{\frac{p^2}{q}-p}}{\gamma^p} r^n \phi(r)^p \|f\|^p_{L^{p,\phi}(\mu)} \\
&\le \frac{C}{\gamma^p} \tilde{\gamma}^{\frac{p^2}{q}-p} r^n \phi(r)^p \|f\|^p_{L^{p,\phi}(\mu)} \\
&\le C\, r^n \phi(r)^p \left(\frac{\|f\|_{L^{p,\phi}(\mu)}}{\gamma}\right)^q .
\end{aligned}$$

This completes the proof. ■

Remark 2.5
(a) Note that $\phi(r) = r^{-\frac{n}{p}}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ satisfy the hypotheses in Theorem 2.4. Here $L^{p,\phi}(\mu) = L^p(\mu)$, and so we obtain

$$\mu\left(\left\{x \in B\left(a,r\right) : |I_{\alpha}f(x)| > \gamma\right\}\right) \le C\left(\frac{\|f\|_{L^p(\mu)}}{\gamma}\right)^q ,$$

for every $\gamma > 0$. This inequality holds for any ball $B(a,r) \subseteq \mathbb{R}^d$, and so we have

$$\mu\left(\left\{x \in \mathbb{R}^d : |I_{\alpha}f(x)| > \gamma\right\}\right) \le C\left(\frac{\|f\|_{L^p(\mu)}}{\gamma}\right)^q ,$$

for every $\gamma > 0$, which is the inequality in Theorem 1.1.
(b) By substituting $\frac{1}{q} = 1 - \frac{\alpha}{n-\lambda}$ for some $\lambda \in [0, n-\alpha)$ to $r^{\alpha}\phi(r) \le C\phi(r)^{1/q}$, we have

$$\phi(r) \le Cr^{\lambda-n},$$

for every $r > 0$. Hence, $\int_r^{\infty} t^{\alpha-1}\phi(t)dt \le C\, r^{\lambda+\alpha-n}$ for every $r > 0$, which is one of the hypotheses in Theorem 1.2.

(c) In [5], the weak type inequality for I_α on generalized non-homogeneous Morrey space is obtained as a consequence of the weak type inequality for M^n on generalized non-homogeneous Morrey spaces and a Hedberg type inequality for I_α. In this paper, we use Chebyshev inequality and a mild condition on ϕ, namely $\inf_{r>0} \phi(r) = 0$, in addition to the doubling condition.

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References

[1] J. Garcia-Cuerva and A.E. Gatto (2004), Boundedness properties of fractional integrals associated to non-doubling measures, *Studia Math.* **162**(3), 245–261.

[2] H. Gunawan (2003), A note on the generalized fractional integral operators, *J. Indonesian Math. Soc.* **9**(1), 39–43.

[3] G.H. Hardy and J.E. Littlewood (1927), Some properties of fractional integrals. I, *Math. Zeit.* **27**, 565–606.

[4] H.L. Royden and P.M. Fitzpatrick (2010), *Real Analysis* (Fourth Edition), Pearson, London.

[5] I. Sihwaningrum, S. Maryani, H. Gunawan (2012), Weak type inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces, *Anal. Theory Appl.* **28**(1), 65–72.

[6] S.L. Sobolev (1938), On a theorem in functional analysis (Russian), *Mat. Sob.* **46**, 471–497 [English translation in *Amer. Math. Soc. Transl. ser. 2* **34**(1963), 39–68].

[7] E.M. Stein (1970), *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, New Jersey.