The n-dual space of the space of p-summable sequences

H. Gunawan* and Y.E.P. Pangalela

*http://personal.fmipa.itb.ac.id/hgunawan/

Analysis and Geometry Group
Bandung Institute of Technology
Bandung, INDONESIA

ICFMSA, Kolkata, 7-9 December 2012

Outline

Outline

Outline

Outline

Outline

Outline

Outline

Outline

The notion of n-normed spaces was introduced by S. Gähler in the 1960's [2, 3]. If in a normed space one speaks about the length of a vector, then in an n-normed space we discuss about the volume of the parallelepiped spanned by a set of n-vectors.

In this talk, we shall be interested in n-dual spaces of an n-normed space, consisting of bounded multilinear n-functionals on the space.

The concept of bounded multilinear n-functionals on an n-normed space was initially intoduced by White [10], and studied further by Batkunde *et al.* [1] and Gozali *et al.* [4].

We shall refine the definition of bounded multilinear n-functionals, introduce the concept of n-dual spaces of an n-normed space, and then determine the n-dual spaces of ℓ^p spaces.

The results presented here are summarized from [9].

Who's Who, Among Others . . .

S. Gähler and collaborators, incl. C. Diminnie and A. White (1960's-1970's)
K. Iseki (1975-1976)
S.N. Lai and A.K. Singh (1978)
A. Khan and A. Siddiqui (1982)
G. Godini (1985)
A. Misiak (1989)
M.S. Khan dan M.D. Khan (1993)
D.R. Jain and R. Chugh (1995)
Y.J. Cho and collaborators, incl. S.S. Kim and N.J. Huang (1990's)
H. Mazaheri and collaborators (2007-now)
H. Gunawan and collaborators (2000-now)

Definition of n-normed spaces

Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here d can be infinite.) A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an n-**norm** on X. The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-**normed space**.

Note

In an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$, we have

$\|x_1, \ldots, x_n\| \geq 0$ for every $x_1, \ldots, x_n \in X$

and

$\|x_1, x_2, \ldots, x_n\| = \|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2, \ldots, x_n\|$ for every $x_1, \ldots, x_n \in X$ and $\alpha_2, \ldots, \alpha_n \in \mathbb{R}$.

Standard example

If $(X, \langle \cdot, \cdot \rangle)$ is a real inner product space of dimension $d \geq n$, then the following formula defines an n-norm on X:

$$\|x_1, \ldots, x_n\|^S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2}.$$

Geometrically, $\|x_1, \ldots, x_n\|^S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X.

General example: Gähler's formula

If X is a normed space, one may define an n-norm on X by using Gähler's formula

$$\|x_1, \ldots, x_n\|^G := \sup_{\substack{f_i \in X',\ \|f_i\| \leq 1 \\ i = 1, \ldots, n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X.

Gähler's n-norm on ℓ^p

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we can equip the space ℓ^p of p-summable sequences with Gähler's n-norm $\|\cdot, \dots, \cdot\|_p^G$, given by

$$\|x_1, \dots, x_n\|_p^G := \sup_{y_j \in \ell^q,\, \|y_j\|_q \leq 1} \left| \det \left[\sum_{k=1}^{\infty} x_{ik} y_{jk} \right]_{i,j} \right|, \quad x_1, \dots, x_n \in \ell^p.$$

Here ℓ^q is the dual space of ℓ^p, and $\| \cdot \|_q$ is the usual norm on ℓ^q.

Another n-norm on ℓ^p

On ℓ^p, we can also define

$$\|x_1, \dots, x_n\|_p^H := \left(\frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left| \det \left[x_{ik_j} \right]_{i,j} \right|^p \right)^{\frac{1}{p}}, \; x_1, \dots, x_n \in \ell^p.$$

This n-norm was introduced in [5].

The equivalence between the two n-norms on ℓ^p

As shown in [11], the two n-norms on ℓ^p are equivalent, that is,

$$(n!)^{\frac{1}{p}-1} \|x_1, \ldots, x_n\|_p^H \leq \|x_1, \ldots, x_n\|_p^G \leq (n!)^{\frac{1}{p}} \|x_1, \ldots, x_n\|_p^H \quad (1)$$

for all $x_1, \ldots, x_n \in \ell^p$.

Multilinear n-functionals

Any real-valued function f on X^n, where X is a real vector space of dimension $d \geq n$, is called an n-**functional** on X.

Furthermore, an n-functional f satisfying the following two properties:

(1) $f(x_1 + y_1, \ldots, x_n + y_n) = \displaystyle\sum_{h_i \in \{x_i, y_i\},\, 1 \leq i \leq n} f(h_1, \ldots, h_n)$

(2) $f(\alpha_1 x_1, \ldots, \alpha_n x_n) = \alpha_1 \cdots \alpha_n f(x_1, \ldots, x_n)$

is called a **multilinear** n-**functional** on X.

Bounded multilinear n-functionals

Next, suppose that f is an n-functional on a normed space $(X, \|\cdot\|)$ [respectively an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$].

If there exists a constant $K > 0$ such that

$$|f(x_1, \ldots, x_n)| \leq K\|x_1\| \cdots \|x_n\|$$

$$[\text{resp. } |f(x_1, \ldots, x_n)| \leq K\|x_1, \ldots, x_n\|\,]$$

for all $x_1, \ldots, x_n \in X$, then f is said to be **bounded** on $(X, \|\cdot\|)$ [resp. **bounded** on $(X, \|\cdot, \ldots, \cdot\|)$].

Note

It is easy to check that every bounded multilinear n-functional f on an n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ satisfies

$$f(x_1, \ldots, x_n) = 0$$

whenever $x_1, \ldots, x_n$ are linearly dependent.

Further, it is antisymmetric, that is,

$$f(x_1, \ldots, x_n) = \operatorname{sgn}(\sigma) f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

for any $x_1, \ldots, x_n \in X$ and any permutation σ of $(1, \ldots, n)$.

Here $\operatorname{sgn}(\sigma) = 1$ if σ is an even permutation and $\operatorname{sgn}(\sigma) = -1$ if σ is an odd permutation.

These properties do not hold for bounded multilinear n-functionals on a normed space $(X, \| \cdot \|)$.

n-dual spaces

Inspired by the concept of the dual space of a normed space, the space of bounded multilinear n-functionals on $(X, \|\cdot\|)$ [resp. on $(X, \|\cdot, \dots, \cdot\|)$] is called the n-**dual space** of $(X, \|\cdot\|)$ [resp. the n-**dual space** of $(X, \|\cdot, \dots, \cdot\|)$].

The n-dual space is equipped with the following norm

$$\|f\|_{n,1} := \sup_{\|x_1\|, \dots, \|x_n\| \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1\| \dots \|x_n\|}$$

$$\left[\text{resp. } \|f\|_{n,n} := \sup_{\|x_1, \dots, x_n\| \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1, \dots, x_n\|} \right].$$

In the remaining slides, we shall focus on $X = \ell^p$, where $1 \leq p < \infty$.

For convenient, we shall first discuss the 2-dual spaces of ℓ^p (as a normed space as well as a 2-normed space), and then generalize the result for all $n \geq 2$.

From now on, we shall always assume that q is the dual exponent of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$, unless otherwise stated.

The space $Y^q_{\mathbb{N}\times\mathbb{N}}$

To achieve our goals, we need to introduce the following normed space. We say that a double index sequence $\theta := (\theta_{kj})$ (of real numbers) belongs to the space $Y^q_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_p=1} \left[\sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} x_k \theta_{kj}\right|^q\right]^{\frac{1}{q}} < \infty.$$

Here $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ defines a norm on $Y^q_{\mathbb{N}\times\mathbb{N}}$. For $q=\infty$, a double index sequence $\theta := (\theta_{kj})$ is in $Y^\infty_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_1=1} \sup_{j\in\mathbb{N}} \left|\sum_{k=1}^{\infty} x_k \theta_{kj}\right| < \infty.$$

The 2-dual space of $(\ell^p, \|\cdot\|_p)$

Our first result is the following.

Theorem 5.1

If $1 < p < \infty$, then the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

Moreover, the mapping $f \mapsto \theta := (f(e_k, e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p, \|\cdot\|_p)$ to $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

For $p = 1$, we can also prove easily that the 2-dual space of $(\ell^1, \|\cdot\|_1)$ is identified by $\left(Y^{\infty}_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^{\infty}_{\mathbb{N}\times\mathbb{N}}}\right)$.

Hence we have the following corollary.

Corollary 5.2

For $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^{q}_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^{q}_{\mathbb{N}\times\mathbb{N}}}\right)$.

Bilinear 2-functional on $(\ell^p, \|\cdot,\cdot\|_p^G)$

Using the previous fact (for the case where $n = 2$), we get the following theorem.

Theorem 5.3

A bilinear 2-functional f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$ if and only if f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have

$$\frac{1}{2}\|f\|_{2,1} \leq \|f\|_{2,2}^G \leq \|f\|_{2,1},$$

where $\|\cdot\|_{2,2}^G$ is the norm on the 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^G)$.

The space $Z^q_{\mathbb{N}\times\mathbb{N}}$

To identify the 2-dual space of of $(\ell^p, \|\cdot,\cdot\|^G_p)$, we consider some subspace of $Y^q_{\mathbb{N}\times\mathbb{N}}$.

A double index sequence $\theta := (\theta_{kj})$ belongs to $Z^q_{\mathbb{N}\times\mathbb{N}}$ if $\theta \in Y^q_{\mathbb{N}\times\mathbb{N}}$ and $\theta_{kj} = -\theta_{jk}$ for all $k, j \in \mathbb{N}$.

Note that $Z^q_{\mathbb{N}\times\mathbb{N}}$ can be viewed as a normed space equipped with the norm inherited from $Y^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 5.4

The function $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^G_p\neq 0} \frac{\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj}\right|}{\|x,y\|^G_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$.
Furthermore, $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$*, with*

$$\frac{1}{2}\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

The 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^G)$

Corollary 5.5

The 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|_p^G)$ *is identified by* $\left(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

Corollary 5.6

The function $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^H_p \neq 0} \frac{\left|\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj}\right|}{\|x,y\|^H_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$.
Furthermore, $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$2^{\frac{1}{p}-1}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq 2^{\frac{1}{p}}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

The 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^H)$

Corollary 5.7

The 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^H)$ *is identified by* $\left(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

Note: $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, and $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ are three equivalent norms on $Z^q_{\mathbb{N}\times\mathbb{N}}$.

The results for the case $n = 2$ can be extended easily to the case $n \geq 2$. For $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we define $Y^q_{\mathbb{N}^n}$ to be the space of all (real) n-index sequence $\theta := (\theta_{k_1 \cdots k_n})$ where

$$\|\theta\|_{Y^q_{\mathbb{N}^n}} := \sup_{\|a_i\|_p = 1} \left[\sum_{k_n=1}^{\infty} \left| \sum_{k_1, \ldots, k_{n-1}=1}^{\infty} a_{1k_1} \cdots a_{n-1,k_{n-1}} \theta_{k_1 \cdots k_n} \right|^q \right]^{\frac{1}{q}} < \infty.$$

For $q = \infty$, an n-index sequence $\theta := (\theta_{k_1 \cdots k_n})$ belongs to the space $Y^\infty_{\mathbb{N}^n}$ if

$$\|\theta\|_{Y^\infty_{\mathbb{N}^n}} := \sup_{\|a_i\|_1 = 1} \sup_{k_n \in \mathbb{N}} \left| \sum_{k_1, \ldots, k_{n-1}=1}^{\infty} a_{1k_1} \cdots a_{n-1,k_{n-1}} \theta_{k_1 \cdots k_n} \right| < \infty.$$

Here $\mathbb{N}^n := \mathbb{N} \times \cdots \times \mathbb{N}$ (n factors). Note also that the inner sum above is a multiple sum.

The n-dual space of $(\ell^p, \|\cdot\|_p)$

We also define the generalization of $Z^q_{\mathbb{N}\times\mathbb{N}}$ spaces as follows. An n-index sequence $\theta := (\theta_{k_1\cdots k_n})$ belongs to the space $Z^q_{\mathbb{N}^n}$ if $\theta \in Y^q_{\mathbb{N}^n}$ and $\theta_{k_1\cdots k_n} = \mathrm{sgn}(\sigma)\theta_{\sigma(k_1)\cdots\sigma(k_n)}$, for all $k_1, \ldots, k_n \in \mathbb{N}$ and any permutation σ of $(k_1, \ldots, k_n)$.

Analogous to the case $n = 2$, we have the following result for $n \geq 2$.

Theorem 6.1

The n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$. Moreover, the mapping $f \mapsto \theta := (f(e_{k_1}, \ldots, e_{k_n}))$ is an isometric bijection from the n-dual space of $(\ell^p, \|\cdot\|_p)$ to $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$.

Theorem 6.2

A multilinear n-functional f is bounded on $(\ell^p, \|\cdot, \ldots, \cdot\|_p^G)$ if and only if it is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have

$$\frac{1}{n!}\|f\|_{n,1} \leq \|f\|_{n,n}^G \leq \|f\|_{n,1}$$

where $\|\cdot\|_{n,n}^G$ is the norm on the n-dual space of $(\ell^p, \|\cdot, \ldots, \cdot\|_p^G)$.

The n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|_p^G)$

Corollary 6.3

The n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|_p^G)$ is identified by $\left(Z^q_{\mathbb{N}^n}, \|\cdot\|^G_{Z^q_{\mathbb{N}^n}}\right)$, where $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ is given by

$$\|\theta\|^G_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\dots,x_n\|_p^G \neq 0} \frac{\left|\sum_{k_1,\dots,k_n=1}^{\infty} x_{1k_1} \cdots x_{nk_n} \theta_{k_1 \cdots k_n}\right|}{\|x_1, \dots, x_n\|_p^G}.$$

The n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|_p^H)$

Corollary 6.4

The n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|_p^H)$ is identified by $\left(Z^q_{\mathbb{N}^n}, \|\cdot\|^H_{Z^q_{\mathbb{N}^n}}\right)$, where $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ is given by

$$\|\theta\|^H_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\ldots,x_n\|_p^H \neq 0} \frac{\left|\sum_{k_1,\ldots,k_n=1}^{\infty} x_{1k_1}\cdots x_{nk_n}\theta_{k_1\cdots k_n}\right|}{\|x_1,\ldots,x_n\|_p^H}.$$

In the theory of normed spaces, we know that the dual space of $(\ell^p, \|\cdot\|_p)$ is (identified by) $(\ell^q, \|\cdot\|_q)$, where $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Here we show that the n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$. We also obtain similar results for the n-dual space of ℓ^p when ℓ^p is viewed as an n-normed space.
All these results are identical in the case where $n = 1$. For $n \geq 2$, however, we still have a question whether the norm $\|\cdot\|_{Y^q_{\mathbb{N}^n}}$ on $Y^q_{\mathbb{N}^n}$, as well as the norms $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ and $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ on $Z^q_{\mathbb{N}^n}$, can be reduced to

$$\|\theta\|^*_{Y^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}}$$

and

$$\|\theta\|^*_{Z^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}}.$$

One may easily check that if $\theta := (\theta_{k_1 \dots k_n})$ satisfies

$$\|\theta\|_q^* := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1 \cdots k_n}|^q \right)^{\frac{1}{q}} < \infty,$$

then $\|\theta\|_{Y^q_{\mathbb{N}^n}}$, $\|\theta\|^H_{Z^q_{\mathbb{N}^n}}$, and $\|\theta\|^G_{Z^q_{\mathbb{N}^n}}$ are all dominated by $\|\theta\|_q^*$.

We just do not know whether the converse is also true.
See [1] for related problems.

Acknowledgement. This research is supported by ITB Research and Innovation Program 2012.

References I

[1] H. BATKUNDE, H. GUNAWAN, and Y.E.P. PANGALELA, *Bounded linear functionals on the n-normed space of p-summable sequences*, preprint.

[2] S. GÄHLER, *Untersuchungen uber verallgemeinerte m-metrische räume I*, Math. Nachr. **40** (1969), pp. 165–189.

[3] S. GÄHLER, *Untersuchungen uber verallgemeinerte m-metrische räume II*, Math. Nachr. **40** (1969), pp. 229–264.

[4] S.M. GOZALI, H. GUNAWAN, and O. NESWAN, *On n-norms and bounded n-linear functionals in a Hilbert space*, Ann. of Funct. Anal. **1** (2010), pp. 72–79.

References II

[5] H. GUNAWAN, *The space of p-summable sequences and its natural n-norm*, Bull. Austral. Math. Soc. **64** (2001), pp. 137–147.

[6] H. GUNAWAN, W. SETYA-BUDHI, MASHADI, and S. GEMAWATI, *On volumes of n-dimensional parallelepipeds in ℓ^p spaces*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **16** (2005), pp. 48–54.

[7] P.M. MILIČIĆ, *On the Gram-Schmidt projection in normed spaces*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **4** (1993), pp. 85–92.

[8] Y.E.P. PANGALELA, *Representasi Fungsional-2 Linear pada ℓ^p* (in Indonesian), Master Thesis, Institut Teknologi Bandung, 2012.

References III

[9] Y.E.P. PANGALELA and H. GUNAWAN, *The n-dual space of the space of p-summable sequences*, submitted.

[10] A.G. WHITE, *2-Banach spaces*, Math. Nachr. **42** (1969), pp. 43–60.

[11] R.A. WIBAWA-KUSUMAH and H. GUNAWAN, *Two equivalent n-norms on the space of p-summable sequences*, to appear in Period. Math. Hungarica.

THANK YOU FOR YOUR ATTENTION!

Note on Theorem 5.3

The proof of Theorem 5.3 involves the concept of g-**orthogonality** on ℓ^p, where g is the **semi-inner product** on ℓ^p given by the formula

$$g(x,y) := \|x\|_p^{2-p} \sum_{j=1}^{\infty} |x_j|^{p-1} \mathrm{sgn}(x_j) y_j, \quad x := (x_j), y := (y_j).$$

If $g(x,y) = 0$, then we say that x and y are g-**orthogonal**, and we write $x \perp_g y$ [7].

Using g-orthogonality, we may define the **volume** of the parallelpiped spanned by linearly independent $x_1, \ldots, x_n \in \ell^p$ by the formula

$$V(x_1, \ldots, x_n) := \|x_1^\circ\|_p \cdots \|x_n^\circ\|_p,$$

where $\{x_1^\circ, \ldots, x_n^\circ\}$ is the **left g-orthogonal sequence** obtained from $\{x_1, \ldots, x_n\}$ through a Gram-Schmidt process [6].

If $x_1, \ldots, x_n$ are linearly dependent, then we simply define $V(x_1, \ldots, x_n) = 0$.

In [11], it is shown that

$$V(x_{i_1}, \ldots, x_{i_n}) \le \|x_1, \ldots, x_n\|_p^G \tag{2}$$

for all $x_1, \ldots, x_n \in \ell^p$ and any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.