Inclusion Property of Morrey Spaces

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Harmonic Analysis Seminar
Ibaraki University, 7 August 2017

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What we shall do

In this talk, we shall discuss the inclusion relation between Morrey spaces and between weak Morrey spaces, as well as between generalized Morrey spaces and between generalized weak Morrey spaces.

In particular, we are interested in necessary and sufficient consitions for the inclusion property of these spaces.

Morrey spaces

Let $L^p_{\text{loc}}(\mathbb{R}^d)$ denote the space of all p-locally integrable functions on $\mathbb{R}^d$.

For $1 \le p \le q < \infty$, we define the *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ by

$$\mathcal{M}^p_q := \{f \in L^p_{\text{loc}}(\mathbb{R}^d) \,:\, \|f\|_{\mathcal{M}^p_q} < \infty\},$$

where $\|\cdot\|_{\mathcal{M}^p_q}$ is given by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\ r>0} |B(a,r)|^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Here, $B(a,r)$ denotes the open ball in $\mathbb{R}^d$ centered at a with radius r, and $|B(a,r)| = c_d r^d$ is its Lebesgue measure.

Fractional integral operators on Morrey spaces

Note: $\|\cdot\|_{\mathcal{M}^p_q}$ defines a norm on $\mathcal{M}^p_q$ and $(\mathcal{M}^p_q, \|\cdot\|_{\mathcal{M}^p_q})$ is a Banach space.

If $p = q$, then $\mathcal{M}^p_q = L^p$. Thus, $\mathcal{M}^p_q$ can be viewed as a generalization of the Lebesgue space L^p.

Morrey spaces were first introduced by C.B. Morrey in 1938. [1][1]

[1] [1] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938)

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One of the important results on Morrey spaces is the boundedness of the *fractional integral operator* I_α, which is defined for $0 < \alpha < d$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy,$$

for any locally integrable function f on $\mathbb{R}^d$.

[1] [1] C.B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. **43** (1938)

Theorem 1.1

Let $1 < p \le q < \dfrac{d}{\alpha}$ *and* $1 < s \le t < \infty$. *If*

$$\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{d} \text{ and } \frac{p}{q} = \frac{s}{t},$$

then there exists a constant $C > 0$ *such that*

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q},$$

for every $f \in \mathcal{M}^p_q$.

See [2][2] and [3][3] for the proof.

[2] [2] D.R. Adams, A note on Riesz potentials, Duke Math. J. **42** (1975)

[3] [3] F. Chiarenza and M. Frasca, Morrey spaces and Hardy-Littlewood maximal function, Rend. Mat. **7** (1987)

Hardy-Littlewood maximal operator

Theorem 1.1 may be proved by using Morrey norm estimates for the *Hardy-Littlewood maximal operator* M, which is defined by

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

for any locally integrable function f on $\mathbb{R}^d$.

Theorem 1.2

[3] Let $1 < p \le q < \infty$. Then, there exists a constant $C > 0$ such that

$$\|Mf\|_{\mathcal{M}^p_q} \le C\|f\|_{\mathcal{M}^p_q}$$

for every $f \in \mathcal{M}^p_q$.

Inclusion property of Morrey spaces

Note: The inclusion $\mathcal{M}_q^p \subseteq \mathcal{M}_q^1$ is used in the proof of Theorem 1.2.

This inclusion is a special case of the inclusion property of Morrey spaces which is stated in the following theorem [4][4].

Theorem 1.3

For $1 \le p_1 \le p_2 \le q < \infty$, the following inclusion holds:

$$L^q = \mathcal{M}_q^q \subseteq \mathcal{M}_q^{p_2} \subseteq \mathcal{M}_q^{p_1}.$$

[4] [4] Y. Sawano and H. Tanaka, Morrey spaces for non-doubling measures, Acta Math. Sin. (Engl. Ser.) **21**(6) (2005)

Notes

For $d \geq 2$ and $1 \leq p < q$, the inclusion $\mathcal{M}^q_q \subseteq \mathcal{M}^p_q$ is proper.

To see this, take $f_{p,q}(x) := |x|^{-\frac{d}{q}}$. Then $f_{p,q} \in \mathcal{M}^p_q \setminus \mathcal{M}^q_q$.

[5][5] Y. Sawano, A non-dense subspace in $\mathcal{M}^q_p$ with $1 < q < p < \infty$, Trans. A. Razmadze Math. Inst. (2017)

Notes

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One may also find an example to show that for $p_1 < p_2$ the inclusion $\mathcal{M}^{p_2}_q \subseteq \mathcal{M}^{p_1}_q$ is also proper.

Very recently, Sawano proved that $\mathcal{M}^{p_2}_q$ is not dense in $\mathcal{M}^{p_1}_q$ for $p_1 < p_2$.[5]

[5][5] Y. Sawano, A non-dense subspace in $\mathcal{M}^q_p$ with $1 < q < p < \infty$, Trans. A. Razmadze Math. Inst. (2017)

What next

In connection with Theorem 1.3, we shall prove the inclusion properties of weak Morrey spaces, generalized Morrey spaces, and generalized weak Morrey spaces.

We do not only give the sufficient condition for the inclusion properties of the generalized (weak) Morrey spaces, but also the necessary condition.

The inclusion of generalized Orlicz-Morrey spaces can be found, for instance, in Theorem 4.4 of [6][6] and Remark 1 of [7][7].

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[6] [6] E. Nakai, Orlicz-Morrey spaces and the Hardy-Littlewood maximal function, Studia Math. **188**(3) (2008)

[7] [7] D.I. Hakim, E. Nakai, and Y. Sawano, Generalized fractional maximal operators and vector-valued inequalities on generalized Orlicz-Morrey spaces, Rev. Mat. Complut. **29**(1) (2016)

The boundedness of fractional integral operators I_α from Morrey spaces $\mathcal{M}_q^p$ to $\mathcal{M}_t^s$ only holds for $1 < p \leq q < \frac{d}{\alpha}$ (and suitable s and t). The same is true on Lebesgue spaces, the function $I_\alpha f$ fails to be in $L^{\frac{d}{d-\alpha}}$ for $f \in L^1$. However, weaker results are available for $p = 1$: I_α is bounded from L^1 to a weaker space than $L^{\frac{d}{d-\alpha}}$.

[8] [8] J. García-Cuerva and A.E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math. **162** (2004)

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In general, by weakening its membership condition, one can enlarge L^p to obtain weak Lebesgue spaces wL^p, which is also known as Lorentz spaces. Boundedness of operators on weak Lebesgue spaces may lead to boundedness of operators on 'strong' Lebesgue spaces through Marcinkiewicz's interpolation theorem.[8] This motivates us to study weak Morrey spaces and, later on, generalized weak Morrey spaces as well.

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[8] [8] J. García-Cuerva and A.E. Gatto, Boundedness properties of fractional integral operators associated to non-doubling measures, Studia Math. **162** (2004)

Weak Morrey spaces

Definition 2.1

Let $1 \le p \le q < \infty$. The *weak Morrey space* $w\mathcal{M}^p_q = w\mathcal{M}^p_q(\mathbb{R}^d)$ is the set of all measurable functions f for which $\|f\|_{w\mathcal{M}^p_q} < \infty$, where

$$\|f\|_{w\mathcal{M}^p_q} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \gamma \left|\{x \in B(a,r) \colon |f(x)| > \gamma\}\right|^{\frac{1}{p}}$$

$$= \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \left\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\right\|_{L^p(B(a,r))}.$$

Note: $\|\cdot\|_{w\mathcal{M}^p_q}$ defines a quasi-norm in $w\mathcal{M}^p_q$ and $(w\mathcal{M}^p_q, \|\cdot\|_{w\mathcal{M}^p_q})$ forms a quasi-Banach space.

If $p = q$, then $\|\cdot\|_{w\mathcal{M}^p_q} = \|\cdot\|_{wL^p}$, where $\|f\|_{wL^p} := \sup_{a\in\mathbb{R}^d,\, r>0} \|f\|_{wL^p(B(a,r))} = \sup_{a\in\mathbb{R}^d,\, r>0} \sup_{\gamma>0} \left\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\right\|_{L^p(B(a,r))}$.

The weak contains the strong one

Theorem 2.2

Let $1 \le p \le q < \infty$. Then $\mathcal{M}_q^p \subseteq w\mathcal{M}_q^p$, with

$$\|f\|_{w\mathcal{M}_q^p} \le \|f\|_{\mathcal{M}_q^p}$$

for every $f \in \mathcal{M}_q^p$.

Proof. For every $x \in B = B(a,r)$, we have $\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}(x) \le |f(x)|$. Hence

$$|B|^{\frac{1}{q}-\frac{1}{p}} \|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B)} \le |B|^{\frac{1}{q}-\frac{1}{p}} \|f\|_{L^p(B)} \le \|f\|_{\mathcal{M}_q^p}.$$

By taking the supremum over all $B = B(a,r)$ and γ, we conclude that $f \in w\mathcal{M}_q^p$ with $\|f\|_{w\mathcal{M}_q^p} \le \|f\|_{\mathcal{M}_q^p}$. □

Notes

Note that $f(x) := |x|^{-d/q} \in w\mathcal{M}^q_q \setminus \mathcal{M}^q_q$.

In general, the inclusion between Morrey spaces and weak Morrey spaces is proper [9][9]. In other words, weak Morrey spaces are strictly larger than their corresponding Morrey spaces.

[9] [9] H. Gunawan, D.I. Hakim, Y. Sawano, and I. Sihwaningrum, Weak type inequalities for some singular integral operators on generalized non-homogeneous Morrey spaces, J. Function Spaces Appl., Article ID 809704 (2013)

Inclusion property of weak Morrey spaces

The inclusion property of weak Morrey spaces is presented in the following theorem.

Theorem 2.3

If $1 \leq p_1 \leq p_2 \leq q < \infty$, *then* $w\mathcal{M}_q^{p_2} \subseteq w\mathcal{M}_q^{p_1}$ *with*

$$\|f\|_{w\mathcal{M}_q^{p_1}} \leq \|f\|_{w\mathcal{M}_q^{p_2}}$$

for every $f \in w\mathcal{M}_q^{p_2}$.

Proof of Theorem 2.3

Let $f \in w\mathcal{M}_q^{p_2}$, $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$. By using Hölder's inequality, we get

$$\begin{aligned}
&|B(a,r)|^{\frac{1}{q}-\frac{1}{p_1}} \|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^{p_1}(B(a,r))} \\
&\quad \leq |B(a,r)|^{\frac{1}{q}-\frac{1}{p_1}} \left[\left(\int_{B(a,r)} \gamma^{p_2} \chi_{\{x\,:\,|f(x)|>\gamma\}}(x)dx \right)^{\frac{p_1}{p_2}} |B(a,r)|^{1-\frac{p_1}{p_2}} \right]^{\frac{1}{p_1}} \\
&\quad = |B(a,r)|^{\frac{1}{q}-\frac{1}{p_2}} \|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^{p_2}(B(a,r))} \\
&\quad \leq \|f\|_{w\mathcal{M}_q^{p_2}}.
\end{aligned}$$

Hence $f \in w\mathcal{M}_q^{p_1}$ with $\|f\|_{w\mathcal{M}_q^{p_1}} \leq \|f\|_{w\mathcal{M}_q^{p_2}}$. □

One might ask whether we can have a relation between weak Morrey spaces $w\mathcal{M}_{q_2}^{p_2}$ and $w\mathcal{M}_{q_1}^{p_1}$ for distinct values of q_1 and q_2.

The answer is negative, as we shall find out in a more general setting, in the next section.

Let $\phi : (0,\infty) \to (0,\infty)$ be of class $\mathcal{G}_p$, that is, ϕ is *almost decreasing* $[r \le s \Rightarrow \phi(r) \ge C\phi(s)]$ and $t \mapsto t^{d/p}\phi(t)$ is *almost increasing* $[r \le s \Rightarrow r^{d/p}\phi(r) \le Cs^{d/p}\phi(s)]$.

Note that if $\phi \in \mathcal{G}_p$, then ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C$$

for every $r, s > 0$ with $\frac{1}{2} \le \frac{r}{s} \le 2$.

See [4], [10][10] and [11][11] for related results.

[10] [10] Eridani and H. Gunawan, Stummel class and Morrey spaces, Southeast Asian Bull. Math. **29**(6) (2005)

[11] [11] H. Gunawan, E. Nakai, Y. Sawano, and H. Tanaka, Generalized Stummel class and Morrey spaces, Publ. Inst. Math. (Beograd) (N.S.) **92**(106) (2012)

Generalized Morrey spaces

Definition 3.1

For each $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$, the *generalized Morrey space* $\mathcal{M}_\phi^p = \mathcal{M}_\phi^p(\mathbb{R}^d)$ is defined as the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}_\phi^p} := \sup_{a \in \mathbb{R}^d,\ r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Note: $\|\cdot\|_{\mathcal{M}_\phi^p}$ defines a norm on $\mathcal{M}_\phi^p$ and $(\mathcal{M}_\phi^p, \|\cdot\|_{\mathcal{M}_\phi^p})$ is a Banach space.

Observe that, if $1 \leq p \leq q < \infty$ and $\phi(r) := r^{-\frac{d}{q}}$, then $\mathcal{M}_\phi^p = \mathcal{M}_q^p$ is the classical Morrey space that we already know.

The characteristic functions of balls

As shown in [12][12] and [13][13], the characteristic functions of balls are contained in the generalized Morrey spaces.

Lemma 3.2

Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Then there exists $C > 1$ such that

$$\frac{1}{\phi(r_0)} \le \|\chi_{B_0}\|_{\mathcal{M}^p_\phi} \le \frac{C}{\phi(r_0)}, \qquad (1)$$

for every ball $B_0 := B(0, r_0)$.

[12] [12] Eridani, H. Gunawan, E. Nakai, and Y. Sawano, Characterizations for the generalized fractional integral operators on Morrey spaces, Math. Ineq. Appl. **17** (2014)

[13] [13] Eridani, H. Gunawan, and M.I. Utoyo, A characterization for fractional integral operators on generalized Morrey spaces, Anal. Theory Appl. **28**(3) (2012)

Proof of Lemma 3.2

Let $r_0 > 0$. By the definition of $\|\cdot\|_{\mathcal{M}^p_\phi}$, we have

$$\|\chi_{B_0}\|_{\mathcal{M}^p_\phi} = \sup_{a\in\mathbb{R}^d,\ r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |\chi_{B_0}(x)|^p\ dx\right)^{1/p}$$
$$\geq \frac{1}{\phi(r_0)} \left(\frac{|B_0 \cap B_0|}{|B_0|}\right)^{1/p} = \frac{1}{\phi(r_0)}.$$

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$$\geq \frac{1}{\phi(r_0)}\left(\frac{|B_0\cap B_0|}{|B_0|}\right)^{1/p} = \frac{1}{\phi(r_0)}.$$

To prove the other inequality, we consider two cases.

First, if $r \leq r_0$, then we have $\phi(r) \geq C\phi(r_0)$. Thus,

$$\frac{1}{\phi(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |\chi_{B_0}(x)|^p\ dx\right)^{1/p} \leq \frac{C}{\phi(r_0)}\left(\frac{|B(a,r)\cap B_0|}{|B(a,r)|}\right)^{1/p}$$
$$\leq \frac{C}{\phi(r_0)}.$$

Next, suppose that $r \geq r_0$. Since $r_0^{\frac{d}{p}}\phi(r_0) \leq Cr^{\frac{d}{p}}\phi(r)$, we have

$$\frac{1}{\phi(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}|\chi_{B_0}(x)|^p\,dx\right)^{1/p} \leq C\frac{r^{\frac{d}{p}}r_0^{-\frac{d}{p}}}{\phi(r_0)}\left(\frac{|B(a,r)\cap B_0|}{|B(a,r)|}\right)^{1/q} \leq C\frac{r^{\frac{d}{p}}r_0^{-\frac{d}{p}}}{\phi(r_0)}\left(\frac{|B_0|}{|B(a,r)|}\right)^{1/p} = \frac{C}{\phi(r_0)}.$$

From these two cases, we can conclude that $\|\chi_{B_0}\|_{\mathcal{M}^p_\phi} \leq \frac{C}{\phi(r_0)}$. □

Inclusion property of generalized Morrey spaces

Theorem 3.3

Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, *and* $\phi_2 \in \mathcal{G}_{p_2}$. *Then, the following statements are equivalent:*

(a) $\phi_2 \leq C\phi_1$.

(b) $\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$.

(c) There exists a constant $C > 0$ *such that* $\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}$ *for every* $f \in \mathcal{M}^{p_2}_{\phi_2}$.

Proof of Theorem 3.3: (a) implies (b)

Suppose that (a) holds and let $f \in \mathcal{M}^{p_2}_{\phi_2}$. For every $a \in \mathbb{R}^d$ and $r > 0$, we have

$$\frac{1}{\phi_1(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^{p_1} \, dx \right)^{\frac{1}{p_1}}$$

$$\leq \frac{C}{\phi_2(r)} \left[\frac{1}{|B(a,r)|} \left(\int_{B(a,r)} (|f(x)|^{p_1})^{\frac{p_2}{p_1}} \, dx \right)^{\frac{p_1}{p_2}} \left(\int_{B(a,r)} dx \right)^{1-\frac{p_1}{p_2}} \right]^{\frac{1}{p_1}}$$

$$\leq \frac{C}{\phi_2(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^{p_2} \, dx \right)^{\frac{1}{p_2}} \leq C \|f\|_{\mathcal{M}^{p_2}_{\phi_2}}.$$

Hence $f \in \mathcal{M}^{p_1}_{\phi_1}$, so that $\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$ [14][14].

[14] [14] I. Sihwaningrum, Fractional Integral Operators and Generalized Morrey Spaces of Nonhomogeneous Type), Dissertation, Bandung Institute of Technology (2010)

Next, since $(\mathcal{M}^{p_2}_{\phi_2}, \mathcal{M}^{p_1}_{\phi_1})$ is a Banach pair, it follows from Chapter I, Lemma 3.3 of [15][15] that (b) and (c) are equivalent.

It thus remains to prove that (b) implies (a) **OR** (c) implies (a). By using Lemma 3.2, we shall prove that (c) implies (a).

[15] [15] S.G. Kreĭn, Yu.Ī. Petunīn, and E.M. Semënov, Interpolation of Linear Operators, Translations of Mathematical Monographs **54**, American Mathematical Society, Providence, R.I. (1982)

Proof that (c) implies (a)

Suppose that (c) holds. Let $B_0 := B(0, r_0)$, where $r_0 > 0$. Then there exists a constant $C > 1$ such that

$$\|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}} \le C\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}} \tag{2}$$

for every ball $B_0 = B(0, r_0)$. By using Lemma 3.2, we obtain

$$\frac{1}{\phi_1(r_0)} \le \|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}} \tag{3}$$

and

$$\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}} \le \frac{C}{\phi_2(r_0)}. \tag{4}$$

The inequalities (2), (3), and (4) imply that $\phi_2(r_0) \le C\phi_1(r_0)$. Since r_0 is an arbitrary positive real number, we obtain $\phi_2 \le C\phi_1$. □

Remarks

(1) As a consequence of Theorem 3.3, we see that we cannot have $\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$ for distinct values of q_1 and q_2 since we do not have the inequality $r^{-d/q_2} \leq C\, r^{-d/q_1}$ for every $r > 0$.

Remarks

(1) As a consequence of Theorem 3.3, we see that we cannot have $\mathcal{M}^{p_2}_{q_2} \subseteq \mathcal{M}^{p_1}_{q_1}$ for distinct values of q_1 and q_2 since we do not have the inequality $r^{-d/q_2} \le C\, r^{-d/q_1}$ for every $r > 0$.

(2) We can also say something about the inequality for the fractional integral operator I_α on the classical Morrey spaces that is presented in Theorem 1.1, namely

$$\|I_\alpha f\|_{\mathcal{M}^s_t} \le C\|f\|_{\mathcal{M}^p_q},$$

for $p < s$ and $q < t$. According to Theorem 3.3, with $\phi_1(r) := r^{-\frac{d}{q}}$ and $\phi_2(r) := r^{-\frac{d}{t}}$, there is no inclusion relation between the range $\mathcal{M}^s_t$ and the domain $\mathcal{M}^p_q$.

Generalized weak Morrey spaces

We now move on to the generalized weak Morrey spaces, which we define as follows. See [16][16] for related works.

Definition 4.1

Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. The *generalized weak Morrey space* $w\mathcal{M}^p_\phi = w\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{w\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} \frac{\left\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\right\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{1/p}} < \infty.$$

[16] [16] A. Akbulut, V. Guliyev, and R. Mustafayev, On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces, Math. Bohem. **137** (2012)

The weak contains the strong one

Theorem 4.2

Let $1 \le p < \infty$ *and* $\phi \in \mathcal{G}_p$. *Then* $\mathcal{M}^p_\phi \subseteq w\mathcal{M}^p_\phi$ *with*

$$\|f\|_{w\mathcal{M}^p_\phi} \le \|f\|_{\mathcal{M}^p_\phi}$$

for every $f \in \mathcal{M}^p_\phi$.

Proof of Theorem 4.2

Let $f \in \mathcal{M}^p_\phi$, $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$. We observe that

$$\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B(a,r))} \le \left(\int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}}.$$

Dividing both side by $\phi(r)|B(a,r)|^{\frac{1}{p}}$, we get

$$\begin{aligned} \frac{\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{\frac{1}{p}}} &\le \frac{1}{\phi(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(x)|^p \, dx\right)^{\frac{1}{p}} \\ &\le \|f\|_{\mathcal{M}^p_\phi}. \end{aligned}$$

Therefore, $f \in w\mathcal{M}^p_\phi$ with $\|f\|_{w\mathcal{M}^p_\phi} \le \|f\|_{\mathcal{M}^p_\phi}$. □

An analog of Lemma 3.2

Lemma 4.3

Let $1 \le p < \infty$ *and* $\phi \in \mathcal{G}_p$. *Then there exists* $C > 1$ *such that*

$$\frac{1}{\phi(r_0)} \le \|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \le \frac{C}{\phi(r_0)},$$

for every ball $B_0 := B(0, r_0)$.

Proof of Lemma 4.3

Let $r_0 > 0$. By using Lemma 3.2 and Theorem 4.2, we get

$$\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \le \|\chi_{B_0}\|_{\mathcal{M}_\phi^p} \le \frac{C}{\phi(r_0)}.$$

Next, by using the definition of $w\mathcal{M}_\phi^p$, we have

$$\begin{aligned}\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} &\ge \frac{\gamma}{\phi(r_0)} \left(\frac{|\{x : |\chi_{B_0}(x)| > \gamma\}|}{|B_0|} \right)^{\frac{1}{p}} \\ &= \frac{\gamma}{\phi(r_0)} \left(\frac{|B_0|}{|B_0|} \right)^{\frac{1}{p}} = \frac{\gamma}{\phi(r_0)}\end{aligned}$$

for every $\gamma \in (0, 1)$. Therefore $\|\chi_{B_0}\|_{w\mathcal{M}_\phi^p} \ge \frac{1}{\phi(r_0)}$, and the lemma is proved. □

Inclusion property of generalized weak Morrey spaces

Theorem 4.4

Let $1 \leq p_1 \leq p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, *and* $\phi_2 \in \mathcal{G}_{p_2}$. *Then, the following statements are equivalent:*

(a) $\phi_2 \leq C\phi_1$.

(b) $w\mathcal{M}^{p_2}_{\phi_2} \subseteq w\mathcal{M}^{p_1}_{\phi_1}$.

(c) There exists a constant $C > 0$ *such that* $\|f\|_{w\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$ *for every* $f \in w\mathcal{M}^{p_2}_{\phi_2}$.

Proof of Theorem 4.4

As a corollary of the Open Mapping Theorem (see Appendix G of [17][17]), one can check that Lemma 3.3 in [15] still holds for quasi-Banach spaces, and so (b) and (c) are equivalent.

For convenience, however, we shall prove that (b) implies (c), in companion with the proof that (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (a).

[17] L. Grafakos, Classical Fourier Analysis, Graduate Texts in Mathematics **249**, Springer, New York (2009)

Proof that (a) implies (b)

Suppose that (a) holds. Let $f \in w\mathcal{M}^{p_2}_{\phi_2}$, $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$. By using Hölder's inequality, we get

$$\frac{\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^{p_1}(B(a,r))}}{\phi_1(r)|B(a,r)|^{1/p_1}} \le C\, \frac{\|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^{p_2}(B(a,r))}}{\phi_2(r)|B(a,r)|^{1/p_2}} \le C\, \|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}.$$

Since this holds for every $a \in \mathbb{R}^d$, $r > 0$, and $\gamma > 0$, we conclude that $f \in w\mathcal{M}^{p_1}_{\phi_1}$.

Proof that (b) implies (c)

Let T be the identity mapping from $w\mathcal{M}^{p_2}_{\phi_2}$ into $w\mathcal{M}^{p_1}_{\phi_1}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subseteq w\mathcal{M}^{p_2}_{\phi_2}$ converges to f_0 in $w\mathcal{M}^{p_2}_{\phi_2}$ and $\{Tf_n\}_{n=1}^{\infty}$ converges to g_0 in $w\mathcal{M}^{p_1}_{\phi_1}$. We want to show that $Tf_0 = g_0$.

Proof that (b) implies (c)

Let T be the identity mapping from $w\mathcal{M}_{\phi_2}^{p_2}$ into $w\mathcal{M}_{\phi_1}^{p_1}$. Suppose that $\{f_n\}_{n=1}^{\infty} \subseteq w\mathcal{M}_{\phi_2}^{p_2}$ converges to f_0 in $w\mathcal{M}_{\phi_2}^{p_2}$ and $\{Tf_n\}_{n=1}^{\infty}$ converges to g_0 in $w\mathcal{M}_{\phi_1}^{p_1}$. We want to show that $Tf_0 = g_0$.

Fix $B(a,r)$. Observe that f_n converges to f_0 in measure in $B(a,r)$. Indeed, for any $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that for $n > n_0$,

$$\frac{\gamma\,|\{x \in B(a,r)\colon |f_n(x) - f_0(x)| > \gamma\}|^{\frac{1}{p_2}}}{\phi_2(r)|B(a,r)|^{1/p_2}} < \varepsilon^{\frac{1}{p_2}+1}$$

for all $\gamma > 0$. By taking $\gamma = \varepsilon$, we have

$$|\{x \in B(a,r)\colon |f_n(x) - f_0(x)| > \varepsilon\}| < \phi_2^{p_2}(r)|B(a,r)|\,\varepsilon$$

for $n > n_0$.

Consequently, there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} f_{n_j}(x) = f_0(x) = Tf_0(x),$$

for almost every $x \in B(a, r)$.

Consequently, there exists a subsequence $\{f_{n_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} f_{n_j}(x) = f_0(x) = Tf_0(x),$$

for almost every $x \in B(a,r)$.

By Fatou's lemma for weak Lebesgue spaces, we have

$$\begin{aligned}
&\frac{\gamma\,|\{x \in B(a,r) : |Tf_0(x) - g_0(x)| > \gamma\}|^{\frac{1}{p_1}}}{\phi_1(r)|B(a,r)|^{1/p_1}} \\
&\quad\le \frac{\left\|\liminf_{j\to\infty} f_{n_j} - g_0\right\|_{wL^{p_1}(B(a,r))}}{\phi_1(r)|B(a,r)|^{1/p_1}} \\
&\quad\le \liminf_{j\to\infty} \frac{\|f_{n_j} - g_0\|_{wL^{p_1}(B(a,r))}}{\phi_1(r)|B(a,r)|^{1/p_1}} \\
&\quad\le \liminf_{j\to\infty} \|f_{n_j} - g_0\|_{w\mathcal{M}^{p_1}_{\phi_1}} = 0.
\end{aligned}$$

Hence, we obtain $\|Tf_0 - g_0\|_{w\mathcal{M}^{p_1}_{\phi_1}} = 0$, whence $Tf_0 = g_0$.

It follows from the Closed Graph Theorem that T is bounded, that is, there exists $C > 0$ such that

$$\|Tf\|_{w\mathcal{M}^{p_1}_{\phi_1}} \leq C\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}},$$

for every $f \in w\mathcal{M}^{p_2}_{\phi_2}$. Therefore, $\|f\|_{w\mathcal{M}^{p_1}_{\phi_1}} \leq C\,\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$ for every $f \in w\mathcal{M}^{p_2}_{\phi_2}$.

Proof that (c) implies (a)

Finally, suppose that (c) holds, and let $r_0 > 0$. Then, there exists a constant $C > 0$ such that

$$\|\chi_{B_0}\|_{w\mathcal{M}^{p_1}_{\phi_1}} \leq C\|\chi_{B_0}\|_{w\mathcal{M}^{p_2}_{\phi_2}}, \tag{5}$$

for every $B_0 := B(0, r_0)$. By using Lemma 4.3, we get

$$\frac{1}{\phi_1(r_0)} \leq \|\chi_{B_0}\|_{w\mathcal{M}^{p_1}_{\phi_1}} \tag{6}$$

and

$$\|\chi_{B_0}\|_{w\mathcal{M}^{p_2}_{\phi_2}} \leq \frac{C}{\phi_2(r_0)}. \tag{7}$$

It now follows from the inequalities (5), (6), and (7) that $\phi_2(r_0) \leq C\phi_1(r_0)$. Since $r_0 > 0$ is arbitrary, we conclude that $\phi_2 \leq C\phi_1$. This completes the proof. □

Remarks

It follows from Theorem 4.4 that there cannot be an inclusion relation between $w\mathcal{M}^{p_2}_{q_2}$ and $w\mathcal{M}^{p_1}_{q_1}$ for distinct values of q_1 and q_2.

When the parameters are the same, we know that the generalized Morrey spaces are contained in generalized weak Morrey spaces. Together with Theorems 3.3 and 4.4, we have the following inclusion relations

$$\begin{array}{ccc} \mathcal{M}^{p_2}_{\phi_2} & \rightarrow & \mathcal{M}^{p_1}_{\phi_1} \\ \downarrow & \searrow & \downarrow \\ w\mathcal{M}^{p_2}_{\phi_2} & \rightarrow & w\mathcal{M}^{p_1}_{\phi_1} \end{array}$$

for $1 \le p_1 \le p_2 < \infty$ and $\phi_2 \le C\phi_1$, where the arrows mean 'contained in' or 'embedded into'.

When the parameters are the same, we know that the generalized Morrey spaces are contained in generalized weak Morrey spaces. Together with Theorems 3.3 and 4.4, we have the following inclusion relations

$$\begin{array}{ccc} \mathcal{M}^{p_2}_{\phi_2} & \rightarrow & \mathcal{M}^{p_1}_{\phi_1} \\ \downarrow & \searrow & \downarrow \\ w\mathcal{M}^{p_2}_{\phi_2} & \rightarrow & w\mathcal{M}^{p_1}_{\phi_1} \end{array}$$

for $1 \le p_1 \le p_2 < \infty$ and $\phi_2 \le C\phi_1$, where the arrows mean 'contained in' or 'embedded into'.

One question remains: what is the relation between $w\mathcal{M}^{p_2}_{\phi_2}$ and $\mathcal{M}^{p_1}_{\phi_1}$ for $1 \le p_1 < p_2 < \infty$ and $\phi_2 \le C\phi_1$? The answer is given in following theorem.

Theorem 5.1

Let $1 \leq p_1 < p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, *and* $\phi_2 \in \mathcal{G}_{p_2}$. *If* $\phi_2 \leq C\phi_1$, *then* $w\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$ *with*

$$\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\left(\frac{p_1}{p_2 - p_1}\right)^{1/p_2} \|f\|_{w\mathcal{M}^{p_2}_{\phi_2}} \tag{8}$$

for all $f \in w\mathcal{M}^{p_2}_{\phi_2}$. *Conversely, if* $w\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$, *then it is necessary that* $\phi_2 \leq C\phi_1$.

Proof of Theorem 5.1

The second part of the theorem follows directly from Proposition 4.2 and Theorems 3.3 or 4.4.

To prove the first part, we use the idea from Exercise 1.1.11 in [17].

Let $f \in w\mathcal{M}^{p_2}_{\phi_2}$ and $B = B(a, r)$ be any ball in $\mathbb{R}^d$.

Proof of Theorem 5.1

By the distribution formula, the definition of $\| \cdot \|_{w\mathcal{M}^{p_2}_{\phi_2}}$, and $\phi_2 \le C\phi_1$, we have for any $R > 0$

$$\begin{aligned}
\int_{B(a,r)} |f(y)|^{p_1}\, dy &= p_1 \int_0^\infty \gamma^{p_1-1} |\{x \in B : |f(x)| > \gamma\}|\, d\gamma \\
&\le p_1 |B(a,r)| \int_0^R \gamma^{p_1-1}\, d\gamma \\
&\quad + p_1 \phi_2(r)^{p_2} |B(a,r)|\, \|f\|^{p_2}_{w\mathcal{M}^{p_2}_{\phi_2}} \int_R^\infty \gamma^{p_1-p_2-1}\, d\gamma \\
&= |B(a,r)| R^{p_1} \\
&\quad + C \frac{p_1}{p_2 - p_1} \phi_1(r)^{p_2} |B(a,r)|\, \|f\|^{p_2}_{w\mathcal{M}^{p_2}_{\phi_2}} R^{p_1-p_2}.
\end{aligned}$$

Proof of Theorem 5.1 (continued)

Therefore,

$$\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(y)|^{p_1}\,dy \le R^{p_1} + \frac{p_1}{p_2-p_1}\phi_1(r)^{p_2}\|f\|^{p_2}_{w\mathcal{M}^{p_2}_{\phi_2}}R^{p_1-p_2}. \tag{9}$$

Proof of Theorem 5.1 (continued)

Therefore,

$$\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(y)|^{p_1}\,dy \le R^{p_1} + \frac{p_1}{p_2-p_1}\phi_1(r)^{p_2}\|f\|^{p_2}_{w\mathcal{M}^{p_2}_{\phi_2}} R^{p_1-p_2}. \tag{9}$$

Now we see that the number

$$R = \left(\frac{p_1}{p_2-p_1}\right)^{1/p_2}\phi_1(r)\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$$

will minimize the right-hand side of the inequality (9). For this choice of R, we have

$$\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(y)|^{p_1}\,dy \le 2\left(\frac{p_1}{p_2-p_1}\right)^{p_1/p_2}\phi_1(r)^{p_1}\|f\|^{p_1}_{w\mathcal{M}^{p_2}_{\phi_2}}.$$

Hence,

$$\frac{1}{\phi_1(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)}|f(y)|^{p_1}\,dy\right)^{1/p_1}\leq 2^{1/p_1}\left[\frac{p_1}{p_2-p_1}\right]^{1/p_2}\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$$

By taking supremum over all balls B, we obtain the desired inclusion and the inequality (8). □

Remarks

In total, if we assume $1 \leq p_1 < p_2 < \infty$ and $\phi_2 \leq C\phi_1$, then we get the following chain of inclusions:

$$\mathcal{M}^{p_2}_{\phi_2} \subseteq w\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1} \subseteq w\mathcal{M}^{p_1}_{\phi_1}.$$

In these inclusions, the condition $\phi_2 \leq C\phi_1$ is also necessary.

Most results presented here are extracted from the paper by G., Hakim, Limanta, and Masta, "Inclusion Properties of Generalized Morrey Spaces", Math. Nachr. **290** (2017).

The present work on necessary and sufficient conditions for proper inclusion properties of generalized Morrey spaces are supported by P3MI-ITB 2017.

I personally would like to thank Prof. Nakai for inviting me to Ibaraki University and giving me the chance to present this talk.