Inclusion relation between generalized Stummel classes and Morrey spaces

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Stummel classes

Stummel classes were introduced 2 decades ago in [DH][1] and [RZ][2].

For $0 < \alpha < n$, the *Stummel class* $S_\alpha = S_\alpha(\mathbb{R}^n)$ is defined by

$$S_\alpha := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \eta_\alpha f(r) \searrow 0 \quad \text{for} \quad r \searrow 0 \right\},$$

where

$$\eta_\alpha f(r) := \sup_{x \in \mathbb{R}^n} \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, dy, \quad r > 0.$$

For $\alpha = 2$, S_2 is known as the *Stummel-Kato class*.

[1] E. B. Davis and A. Hinz, “Kato class potentials for higer order elliptic operators”, *J. London Math. Soc.* **58** (1998), 669–678

[2] M. A. Ragusa and P. Zamboni, “A potential theoretic inequality”, *Czech. Mat. J.* **51** (2001), 55–56

Stummel classes

Knowledge of Stummel classes is important when one is studying the behavior of the solutions of some partial differential equations, see for instance [CRR][3], [CFG][4], [D][5], and [Moh][6].

[3] R. E. Castillo, J. C. Ramos-Fernandes, and E. M. Rojas, "Properties of scales of Kato classes, Bessel potentials, Morrey spaces, and weak Harnack inequality for nonnegative solution of elliptic equations", *J. Differential Equations* **92** (2017), 1–17

[4] F. Chiarenza, E. Fabes, and N. Garofalo, "Harnack's inequality for Schrödinger operators and the continuity of solutions", *Proc. Amer. Math. Soc.* **98** (1986), 415–425

[5] G. Di Fazio, "Hölder continuity of solutions for some Schrödinger equations", *Rend. Sem. Mat. Univ. Padova.* **79** (1988), 173–183

[6] A. Mohamed, "Weak Harnack's inequality for nonnegative solutions of elliptic equations with potential", *Proc. Amer. Math. Soc.* **129** (2001), 2617–2621

Morrey spaces

In the mean time, the study of Morrey spaces, which were introduced by C. B. Morrey in [Mor][7], has attracted many researchers.
For $1 \le p < \infty$ and $0 \le \lambda \le n$, the *Morrey space* $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the set of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{L^{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x,r))} < \infty,$$

where $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ and

$$\|f\|_{L^p(B(x,r))} := \left(\int_{|x-y|<r} |f(y)|^p \, dy\right)^{\frac{1}{p}}.$$

Note that for $\lambda = 0$, we have $L^{p,\lambda} = L^p$.

[7] C. B. Morrey, "On the solutions of quasi-linear elliptic partial differential equations", *Trans. Amer. Math. Soc.* **43** (1938), 126–166

Relation between Stummel Classes and Morrey spaces

As shown in [RZ], one may observe that $L^{1,\lambda} \subseteq S_\alpha$ provided that $n - \lambda < \alpha < n$. (For the case $\alpha = 2$, this fact was proved in [D].)

Conversely, if $V \in S_\alpha$ for $0 < \alpha < n$ and $\eta_\alpha f(r) \sim r^\sigma$ for some $\sigma > 0$, then $V \in L^{1,n-\alpha+\sigma}$.

In [EG][8], the definition of Stummel classes was generalized and the inclusion relation between these classes and generalized Morrey spaces was studied.

[8] Eridani and H. Gunawan, "Stummel classes and Morrey spaces", *Southeast Asian Bull. Math.* **29** (2005), 1053–1056

Generalized Morrey spaces

For $1 \leq p < \infty$ and a measurable function $\Psi : (0, \infty) \to (0, \infty)$, we define the *generalized Morrey space* $\mathcal{M}^{p,\Psi} = \mathcal{M}^{p,\Psi}(\mathbb{R}^n)$ to be the set of all $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^{p,\Psi}} := \sup_{x \in \mathbb{R}^n,\, r>0} \frac{|B(x,r)|^{-\frac{1}{p}}}{\Psi(r)} \|f\|_{L^p(B(x,r))} < \infty,$$

where $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$.

Observe that, for $\Psi(t) := t^{\frac{\lambda-n}{p}}$ $(0 \leq \lambda \leq n)$, we have $\mathcal{M}^{p,\Psi} = L^{p,\lambda}$.

Further works on the inclusion relation between generalized Stummel classes and Morrey spaces can be found in [GNST][9] and [S][10].

[9] H. Gunawan, E. Nakai, Y. Sawano, and H. Tanaka, "Generalized Stummel class and Morrey spaces", *Publ. Inst. Math.* (New Series) **92**(106) (2012), 127–138

[10] S. Samko, "Morrey spaces are closely embedded between vanishing Stummel class", *Math. Ineq. Appl.* **17** (2014), 627–639

Aim of Talk

The aim of this talk is to refine the definition of generalized Stummel classes and study the inclusion relation between these classes.

We also study the inclusion relation between Stummel classes and Morrey spaces using assumptions that are different from the assumptions used in [EG], [GNST], and [S].

We give an example of a function which belongs to the generalized Stummel class but not to the generalized Morrey space.

Furthermore, we also prove that the Stummel class contains weak Morrey spaces under certain conditions.

Weak Morrey spaces

For $1 \le p < \infty$ and $0 \le \lambda \le n$, the *weak Morrey space* $wL^{p,\lambda} = wL^{p,\lambda}(\mathbb{R}^n)$ is the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ for which

$$\|f\|_{wL^{p,\lambda}} := \sup_{x \in \mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{wL^p(B(x,r))} < \infty,$$

where

$$\|f\|_{wL^p(B(x,r))} := \sup_{t>0} t \left|\{y \in B(x,r) : |f(y)| > t\}\right|^{\frac{1}{p}} .$$

Observe that, by taking $\lambda = 0$, we obtain the weak Lebesgue space wL^p.

Assumptions

Throughout this talk we assume that $\Psi : (0, \infty) \to (0, \infty)$ is a measurable function.

Whenever required, we consider the following conditions on Ψ:

$$\int_0^1 \frac{\Psi(t)}{t} dt < \infty; \tag{1}$$

$$\frac{1}{A_1} \le \frac{\Psi(s)}{\Psi(r)} \le A_1 \quad \text{for} \quad 1 \le \frac{s}{r} \le 2; \tag{2}$$

$$\frac{\Psi(r)}{r^n} \le A_2 \frac{\Psi(s)}{s^n} \quad \text{for} \quad s \le r, \tag{3}$$

where $A_i > 0$, $i = 1, 2$, are independent of $r, s > 0$.

The condition (2) is known as the *doubling condition* on Ψ.

Assumptions

In some cases, we can weaken the doubling condition by the *right doubling condition*:

$$\frac{\Psi(s)}{\Psi(r)} \leq A_3 \quad \text{for} \quad 1 \leq \frac{s}{r} \leq 2, \tag{4}$$

where A_3 is independent of $r, s > 0$.

In this talk, the constant $c > 0$ that appears in the proof of all theorems may vary from line to line, and the notation $c = c(\alpha, \beta, \ldots, \zeta)$ indicates that c depends on $\alpha, \beta, \ldots, \zeta$.

Definition 3.1

For $1 \le p < \infty$, we define the **generalized Stummel p-class** $S_{\Psi,p} = S_{\Psi,p}(\mathbb{R}^n)$ by

$$S_{\Psi,p} := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : \eta_{\Psi,p} f(r) \searrow 0 \quad \text{for} \quad r \searrow 0\},$$

where

$$\eta_{\Psi,p} f(r) := \sup_{x\in\mathbb{R}^n} \left(\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n} \, dy \right)^{\frac{1}{p}}, \quad r > 0.$$

We call $\eta_{\Psi,p} f$ the **Stummel p-modulus** of f.

Observe that the Stummel p-modulus is nondecreasing on $(0, \infty)$.

For $p = 1$, we have $S_{\Psi,1} := S_\Psi$ — the generalized Stummel class introduced in [EG].

For $\Psi(t) := t^\alpha$ $(0 < \alpha < n)$, we write $S_{\alpha,p}$ instead of $S_{\Psi,p}$ and $\eta_{\alpha,p}$ instead of $\eta_{\Psi,p}$.

Observe that $S_{\alpha,1} := S_\alpha$ — the Stummel class introduced in [DH] and [RZ].

Proposition 3.2

If $f \in S_{\Psi,p}$, then $\eta_{\Psi,p} f$ is continuous on $(0, \infty)$.

Proposition 3.3

Let Ψ satisfy the condition (3). If $f \in S_{\Psi,p}$, then $\eta_{\Psi,p} f$ satisfies the doubling condition.

The previous facts will be used to prove the relationship between Stummel classes and the classes that we define below.

Definition 3.4

Let $1 \leq p < \infty$ and $\Phi : (0, \infty) \to (0, \infty)$ be a continuous and nondecreasing function such that $\lim_{t \to 0^+} \Phi(t) = 0$.

We say that a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to the class $S^p_{\Psi,\Phi} = S^p_{\Psi,\Phi}(\mathbb{R}^n)$ if there exists a nondecreasing function $\Theta : (0, \infty) \to (0, \infty)$ with $\lim_{r \to 0^+} \Theta(r) = 0$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{|y-x|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)} \, dy \leq \Theta(r), \quad r > 0.$$

Note that for $p = 1$ and $\Psi(t) := t^\alpha$ with $0 < \alpha < n$, we have $S^p_{\Psi,\Phi} := S_{\alpha,\Phi}$ which is the Stummel class introduced in [RZ]. This class is encountered in the nonalgebraic Adams' type inequality.

Theorem 3.5

For $1 \le p < \infty$, we have $S^p_{\Psi,\Phi} \subseteq S_{\Psi,p}$.

Proof.

Let $f \in S^p_{\Psi,\Phi}$, $x \in \mathbb{R}^n$, and $r > 0$. We observe that

$$\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n}\, dy \le \Phi(r) \int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)} dy$$
$$\le \Phi(r)\Theta(r),$$

where $\Theta : (0,\infty) \to (0,\infty)$ is nondecreasing and $\lim_{r\to 0^+} \Theta(r) = 0$.
Hence $0 \le \eta_{\Psi,p} f(r) \le (\Phi(r)\Theta(r))^{\frac{1}{p}}$. Since $\lim_{r\to 0^+} (\Phi(r)\Theta(r))^{\frac{1}{p}} = 0$, we conclude that f belongs to $S_{\Psi,p}$. □

The following theorem gives a sufficient condition for a function $f \in S_{\Psi,p}$ to be in $S^p_{\Psi,\Phi}$ for some function Φ.

Theorem 3.6

Let $1 \le p < \infty$ *and* Ψ *satisfy the condition (3). If* $f \in S_{\Psi,p}$ *and*

$$\int_0^1 \left[\eta_{\Psi,p} f(t)\right]^p \left[\Phi(t)\right]^{-1} t^{-1} dt < \infty,$$

then $f \in S^p_{\Psi,\Phi}$.

Proof.

Let $f \in S_{\Psi,p}$. By the hypotheses and the doubling condition, we have

$$\begin{aligned}
&\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)}\, dy \\
&= \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+1}} \le |x-y| < \frac{r}{2^k}} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n \Phi(|x-y|)}\, dy \\
&\le \sum_{k=0}^{\infty} \left[\Phi\left(\frac{r}{2^{k+1}}\right)\right]^{-1} \int_{|x-y|<\frac{r}{2^k}} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n} dy \\
&\le c(n,p) \sum_{k=0}^{\infty} \left[\Phi\left(\frac{r}{2^{k+2}}\right)\right]^{-1} \left[\eta_{\Psi,p} f\left(\frac{r}{2^{k+2}}\right)\right]^p \\
&\le c(n,p) \sum_{k=0}^{\infty} \int_{\frac{r}{2^{k+2}}}^{\frac{r}{2^{k+1}}} \left[\eta_{\Psi,p} f(t)\right]^p \left[\Phi(t)\right]^{-1} t^{-1} dt \\
&= c(n,p) \int_0^{\frac{r}{2}} \left[\eta_{\Psi,p} f(t)\right]^p \left[\Phi(t)\right]^{-1} t^{-1} dt =: \Theta(r).
\end{aligned}$$

For $f \in S_{\Psi,p}$, Proposition 3.2 tells us that the function $\eta_{\Psi,p}f$. Moreover, we also know that it is nondecreasing and $\lim_{t\to 0^+} \eta_{\Psi,p}f(t) = 0$. The next corollary generalizes the result in [RZ].

Corollary 3.7

Let $1 \le p < \infty$. If $f \in S_{\Psi,p}$ and

$$\int_0^1 [\eta_{\Psi,p}f(t)]^{p-\vartheta}\, t^{-1} dt < \infty,$$

for some $\vartheta \in (0,1)$, then $f \in S^p_{\Psi,\Phi}$ with $\Phi := [\eta_{\Psi,p}f]^{\vartheta}$. In particular, for each $x \in \mathbb{R}^n$ and $r > 0$, we have

$$\int_{|x-y|<r} \frac{|f(y)|^p \Psi(|x-y|)}{|x-y|^n [\eta_{\Psi,p}f(|x-y|)]^{\vartheta}}\, dy \le \Theta(r),$$

where $\Theta(r) := c(n,p) \int_0^{\frac{r}{2}} [\eta_{\Psi,p}f(t)]^{p-\vartheta}\, t^{-1} dt$.

In this section, we are going to investigate the inclusion between two Stummel classes. (Unless otherwise stated, we always assume that $1 \leq p < \infty$.)

The proposition below is obvious.

Proposition 4.1

Suppose that Ψ_2 *satisfies the condition (3) and that there exist* $c > 0$ *and* $\delta > 0$ *such that* $\Psi_2(t) \leq c\,\Psi_1(t)$ *for every* $t \in (0, \delta)$. *Then* $S_{\Psi_1,p} \subseteq S_{\Psi_2,p}$.

As an immediate consequence of Proposition 4.1, we have the following corollary.

Corollary 4.2

If $0 < \alpha \leq \beta < n$, *then* $S_{\alpha,p} \subseteq S_{\beta,p}$.

Remark 4.3

For $0 < \alpha < \beta < n$, *the above inclusion is proper. Indeed, for* $0 < \beta < n$, *define* $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ *by the formula*

$$f(y) := \left(\frac{\chi_B(y)}{|y|^\beta \, |\ln|y||^2} \right)^{\frac{1}{p}}, \quad y \in \mathbb{R}^n,$$

where $B := B(0, e^{-\frac{2}{\beta}})$. *Then* $f \in S_{\beta,p} \backslash S_{\alpha,p}$ *whenever* $0 < \alpha < \beta$.

The next proposition shows the relationship between two Stummel classes with different parameters p.

Proposition 4.4

If $1 \leq p_2 \leq p_1 < \infty$ *and* Ψ *satisfies (1), then* $S_{\Psi,p_1} \subseteq S_{\Psi,p_2}$.

Proof of Proposition 4.4

Let $f \in S_{\Psi,p_1}$, $x \in \mathbb{R}^n$, and $0 < r \leq 1$. Then by Hölder's inequality

$$\begin{aligned}
\int_{|y-x|<r} \frac{|f(y)|^{p_2}\Psi(|y-x|)}{|y-x|^n}\,dy &\leq \left(\int_{|y-x|<r} \frac{|f(y)|^{p_1}\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{\frac{p_2}{p_1}} \\
&\quad \times \left(\int_{|y-x|<r} \frac{\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{1-\frac{p_2}{p_1}} \\
&= \left(\int_{|y-x|<r} \frac{|f(y)|^{p_1}\Psi(|y-x|)}{|y-x|^n}\,dy\right)^{\frac{p_2}{p_1}} \\
&\quad \times \left(c(n)\int_0^r \frac{\Psi(t)}{t}\,dt\right)^{1-\frac{p_2}{p_1}}.
\end{aligned}$$

Therefore ...

Proof (continued)

Therefore

$$\eta_{\Psi,p_2} f(r) \leq c(n,p_1,p_2)\, \eta_{\Psi,p_1} f(r) \left(\int_0^r \frac{\Psi(t)}{t}\, dt \right)^{\frac{1}{p_2} - \frac{1}{p_1}} \searrow 0,$$

for $r \searrow 0$, which tells us that $f \in S_{\Psi,p_2}$.

We conclude that $S_{\Psi,p_1} \subseteq S_{\Psi,p_2}$. □

According to Proposition 4.4, we have the following corollary.

Corollary 4.5

If $1 \leq p_2 \leq p_1 < \infty$*, then* $S_{\alpha,p_1} \subseteq S_{\alpha,p_2}$.

Remark 4.6

For $1 \leq p_2 < p_1 < \infty$*, the above inclusion is proper. Indeed, for* $\frac{\alpha}{p_1} < \gamma < \min\{\frac{\alpha}{p_2}, \frac{n}{p_1}\}$*, we have* $f(y) := |y|^{-\gamma} \in S_{\alpha,p_2} \backslash S_{\alpha,p_1}$.

Using Proposition 4.1 and Proposition 4.4, we get the following corollary.

Corollary 4.7

Suppose that $1 \leq p_2 \leq p_1 < \infty$, Ψ_2 *satisfies the conditions (1) and (3), and there exist* $c > 0$ *and* $\delta > 0$ *such that* $\Psi_2(t) \leq c\,\Psi_1(t)$ *for every* $t \in (0, \delta)$. *Then* $S_{\Psi_1, p_1} \subseteq S_{\Psi_2, p_2}$.

Our next theorem gives an inclusion relation between generalized Morrey spaces and generalized Stummel classes.

Theorem 5.1

Let $1 \leq p_2 \leq p_1 < \infty$. Assume that Ψ_1 satisfies (2) and that Ψ_2 satisfies the right-doubling condition (4). If

$$\int_0^1 \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\, dt < \infty, \tag{5}$$

then $\mathcal{M}^{p_1,\Psi_1} \subseteq S_{\Psi_2,p_2}$.

Remark 5.2

Let $p_1 = p_2 = 1$, $\Psi_1(t) := t^{\lambda-n}$ where $0 \leq \lambda \leq n$, and $\Psi_2(t) := t^{\alpha}$ where $n - \lambda < \alpha < n$. Then, the above theorem reduces to the result in [RZ].

Proof of Theorem 5.1

Let $f \in \mathcal{M}^{p_1,\Psi_1}$, $x \in \mathbb{R}^n$, and $r > 0$. Since Ψ_2 satisfies (4), we have

$$\begin{aligned}
&\int_{|x-y|<r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\, dy \\
&\quad = \sum_{k=-\infty}^{-1} \int_{2^k r \le |x-y| < 2^{k+1} r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\, dy \\
&\quad \le c \sum_{k=-\infty}^{-1} \frac{\Psi_2(2^k r)}{|B(x, 2^{k+1} r)|} \int_{B(x,2^{k+1}r)} |f(y)|^{p_2}\, dy.
\end{aligned}$$

Proof (continued)

Combining the last inequality and Hölder's inequality, we get

$$\begin{aligned}\int_{|x-y|<r} \frac{|f(y)|^{p_2}\Psi_2(|x-y|)}{|x-y|^n}\, dy \\ &\leq c \sum_{k=-\infty}^{-1} \frac{\Psi_2(2^k r)}{|B(x,2^{k+1}r)|^{p_2/p_1}} \|f\|^{p_2}_{L^{p_1}(B(x,2^{k+1}r))} \\ &\leq c\, \|f\|^{p_2}_{\mathcal{M}^{p_1,\Psi_1}} \sum_{k=-\infty}^{-1} \Psi_2(2^k r)\Psi_1(2^{k+1}r)^{p_2}. \qquad (6)\end{aligned}$$

Proof (continued)

Using (4) and the monotonicity of Ψ_1, we get

$$\sum_{k=-\infty}^{-1} \Psi_2(2^k r)\Psi_1(2^{k+1} r)^{p_2} \leq c \sum_{k=-\infty}^{-1} \int_{2^{k-1} r}^{2^k r} \frac{\Psi_2(t)\Psi_1(t)^{p_1}}{t} \, dt$$

$$= c \int_0^{r/2} \frac{\Psi_2(t)\Psi_1(t)^{p_2}}{t} \, dt. \qquad (7)$$

We combine (6) and (7) to obtain

$$\eta_{\Psi_2,p_2} f(r) \leq c \left(\int_0^{r/2} \frac{\Psi_2(t)\Psi_1(t)^{p_2}}{t} \, dt \right)^{\frac{1}{p_2}}. \qquad (8)$$

Proof (continued)

Since $\int_0^{r/2} \frac{\Psi_2(t)\Psi_1(t)^{p_2}}{t}\, dt < \infty$ for $0 < r < 1$, we see that $\lim_{r\to 0^+} \int_0^{r/2} \frac{\Psi_2(t)\Psi_1(t)^{p_2}}{t}\, dt = 0$. This fact and (8) imply $\lim_{r\to 0^+} \eta_{\Psi_2,p_2} f(r) = 0$. Hence, $f \in S_{\Psi_2,p_2}$.

This shows that $\mathcal{M}^{p_1,\Psi_1} \subseteq S_{\Psi_2,p_2}$. □

The following example shows that the inclusion in Theorem 5.1 is proper.

Example 5.3

Let $1 \le p_2 \le p_1 < \infty$, Ψ_2 satisfy the condition (4), $\Psi_2(t)\,|\ln(t)|^2$ be nondecreasing on $(0,\delta)$ for some $\delta > 0$, and $\Psi_1(r)^{p_2}\,\Psi_2(r)\,|\ln(r)|^2 \searrow 0$ as $r \searrow 0$. Define $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ by the formula

$$f(y) := \left(\frac{\chi_B(y)}{\Psi_2(|y|)\,|\ln|y||^2}\right)^{\frac{1}{p_2}}, \quad y \in \mathbb{R}^n,$$

where $B := B(0,\delta)$. Then $f \in S_{\Psi_2,p_2} \backslash \mathcal{M}^{p_1,\Psi_1}$.

Remark 5.4

Let $1 \le p_2 \le p_1 < \infty$, $\Psi_1(t) := t^{\frac{\lambda-n}{p_1}}$ *where* $0 \le \lambda \le n$, *and* $\Psi_2(t) := t^{\alpha}$ *where* $(n-\lambda)\frac{p_2}{p_1} < \alpha < n$. *It can be shown that* Ψ_1 *and* Ψ_2 *satisfy all conditions in Theorem 5.1 and Example 5.3.*

Next, we are going to investigate the relation between generalized Stummel classes and generalized weak Morrey spaces. The generalized weak Morrey spaces are defined as follows.

Definition 5.5

Let $1 \leq p < \infty$ and $\Psi : (0, \infty) \to (0, \infty)$. The **generalized weak Morrey space** $w\mathcal{M}^{p,\Psi} = w\mathcal{M}^{p,\Psi}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f for which

$$\|f\|_{w\mathcal{M}^{p,\Psi}} := \sup_{a\in\mathbb{R}^n,\, r>0,\, t>0} \frac{t|\{x \in B(a,r) : |f(x)| > t\}|^{1/p}}{\Psi(r)|B(a,r)|^{1/p}} < \infty$$

The inclusion between generalized Stummel Classes and generalized weak Morrey spaces is given in the following theorem.

Theorem 5.6

Let $1 \leq p_2 < p_1 < \infty$. *Assume that* Ψ_1 *satisfies* (2) *and that* Ψ_2 *satisfies* (4). *If*

$$\int_0^1 \frac{\Psi_1(t)^{p_2}\Psi_2(t)}{t}\, dt < \infty,$$

then $w\mathcal{M}^{p_1,\Psi_1} \subseteq S_{\Psi_2,p_2}$.

Proof of Theorem 5.6

Since $p_2 < p_1$, by virtue of Theorem 5.1 of [GHLM][11], we have $w\mathcal{M}^{p_1,\Psi_1} \subseteq \mathcal{M}^{p_2,\Psi_1}$.
By Theorem 5.1, we have $\mathcal{M}^{p_2,\Psi_1} \subseteq S_{\Psi_2,p_2}$.
It thus follows that $w\mathcal{M}^{p_1,\Psi_1} \subseteq S_{\Psi_2,p_2}$. □

[11] H. Gunawan, D. I. Hakim, K. M. Limanta, and A. A. Masta, "Inclusion properties of generalized Morrey spaces", *Math. Nachr.* **290** (2017), 332–340

For the classical weak Morrey spaces and Stummel classes, we have the following result.

Theorem 5.7

For $1 < p < \infty$, *if* $0 \le \lambda < n$ *and* $\frac{n-\lambda}{p} < \alpha < n$, *then* $wL^{p,\lambda} \subseteq S_\alpha$. *Conversely, for* $1 \le p < \infty$, *if* $f \in S_{\alpha,p}$ *for* $0 < \alpha < n$ *and* $\eta_{\alpha,p} f(r) \le c r^{\frac{\sigma}{p}}$ *for some* $\sigma > 0$, *then* $f \in wL^{p,n-\alpha+\sigma}$.

Proof of Theorem 5.7

The first assertion follows from Theorem 5.6 by taking $p_1 = p$, $p_2 = 1$, $\Psi_1(t) := t^{\frac{\lambda - n}{p}}$ where $0 \leq \lambda < n$, and $\Psi_2(t) := t^{\alpha}$ where $n - \lambda < \alpha < n$.

To prove the converse, let $f \in S_{\alpha,p}$ ($1 \leq p < \infty$ and $0 < \alpha < n$), and take $x \in \mathbb{R}^n$ and $r > 0$. Using Tchebyshev's inequality, we obtain

$$\begin{aligned}\|f\|_{wL^p(B(x,r))} &\leq c(p)\left(r^{n-\alpha}\right)^{\frac{1}{p}} \eta_{\alpha,p} f(r) \\ &\leq c(p)\left(r^{n-\alpha}\right)^{\frac{1}{p}} r^{\frac{\sigma}{p}} = c(p)\left(r^{n-\alpha+\sigma}\right)^{\frac{1}{p}}.\end{aligned}$$

Therefore

$$\|f\|_{wL^{p,n-\alpha+\sigma}} = \sup_{x\in\mathbb{R}^n,\, r>0} \left(r^{n-\alpha+\sigma}\right)^{-\frac{1}{p}} \|f\|_{wL^p(B(x,r))} \leq c(p) < \infty.$$

This completes the proof.

Remark 5.8

The second part of Theorem 5.7 generalizes the result in [RZ].

For the case $p = 1$, the first part of Theorem 5.7 does not generally hold.

To see this, consider the function $f(y) := |y|^{-n},\ y \in \mathbb{R}^n$.

Then $f \in wL^{1,\lambda}$ for $0 \leq \lambda < n$, but $f \notin S_\alpha$ for $n - \lambda < \alpha < n$.

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