On the Structure of Generalized Morrey Spaces

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Aim of Talk

We shall discuss Morrey spaces, weak Morrey spaces, generalized Morrey spaces, generalized weak Morrey spaces, and the relation between them.

We shall present necessary and sufficient conditions for the inclusion property of these spaces through a norm estimate for the characteristic functions of balls.

Previous Works, Among Others . . . I

1987: F. Chiarenza and M. Frasca, "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7**
2005: Y. Sawano and H. Tanaka, "Morrey spaces for non-doubling measures", *Acta Math. Sinica* **21**
2008: Y. Sawano, "Generalized Morrey spaces for non-doubling measures", *Nonlinear Differential Equations Appl.* **15**
2010: I. Sihwaningrum, "Operator integral fraksional dan ruang Morrey tak homogen yang diperumum" (Indonesia), Doctoral Thesis (ITB)

Morrey Spaces

Let $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ denote the space of all p-locally integrable functions on $\mathbb{R}^d$.
For $1 \le p \le q < \infty$, we define the *Morrey space* $\mathcal{M}^p_q = \mathcal{M}^p_q(\mathbb{R}^d)$ by

$$\mathcal{M}^p_q(\mathbb{R}^d) := \{f \in L^p_{\mathrm{loc}}(\mathbb{R}^d) \,:\, \|f\|_{\mathcal{M}^p_q} < \infty\},$$

where $\|\cdot\|_{\mathcal{M}^p_q}$ is given by

$$\|f\|_{\mathcal{M}^p_q} := \sup_{a \in \mathbb{R}^d,\ r>0} |B(a,r)|^{\frac{1}{q}} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(y)|^p \, dy \right)^{\frac{1}{p}}.$$

Here, $B(a,r)$ denotes the open ball in $\mathbb{R}^d$ centered at a with radius r, and $|B(a,r)|$ denotes its Lebesgue measure.
Note that if $p = q$, then $M^p_q = L^q$.

Boundedness of Fractional Integral Operators on Morrey Spaces

Let I_α be the Fractional Integral Operator, defined for $0 < \alpha < d$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy,$$

for any locally integrable function f on $\mathbb{R}^d$.

Let $1 < p \le q < \dfrac{d}{\alpha}$ and $1 < s \le t < \infty$. If

$$\frac{1}{t} = \frac{1}{q} - \frac{\alpha}{d} \text{ and } \frac{p}{q} = \frac{s}{t},$$

then there exists a constant $C > 0$ such that

$$\|I_\alpha f\|_{\mathcal{M}_t^s} \le C \|f\|_{\mathcal{M}_q^p},$$

for every $f \in \mathcal{M}_q^p$.

Inclusion Property of Strong Morrey Spaces

The inclusion $M_q^p \subseteq M_q^1$ is used in the proof of the boundedness of I_α. This inclusion property is a special case of the following.

Theorem [Sawano]. For $1 \le p_1 \le p_2 \le q < \infty$, the following inclusion holds:

$$L^q(\mathbb{R}^d) = \mathcal{M}_q^q(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_2}(\mathbb{R}^d) \subseteq \mathcal{M}_q^{p_1}(\mathbb{R}^d).$$

For $d \ge 2$ and $1 \le p < q$, the inclusion $\mathcal{M}_q^q(\mathbb{R}^d) \subseteq \mathcal{M}_q^p(\mathbb{R}^d)$ is proper. If $f_{p,q}(x) := |x|^{-\frac{d}{q}}$, then $f_{p,q} \in \mathcal{M}_q^p(\mathbb{R}^d) \setminus \mathcal{M}_q^q(\mathbb{R}^d)$.

The boundedness of fractional integral operators I_α from Morrey spaces $\mathcal{M}_q^p(\mathbb{R}^d)$ to $\mathcal{M}_t^s(\mathbb{R}^d)$ only holds for $1 < p \leq q < \frac{d}{\alpha}$ (and suitable s and t).

The same is true on Lebesgue spaces, the function $I_\alpha f$ fails to be in $L^{\frac{d}{\alpha}}(\mathbb{R}^d)$ for $f \in L^1(\mathbb{R}^d)$. However, weaker results are available for $p = 1$.

In general, by weakening its membership condition, one can define weak Lebesgue spaces $wL^p(\mathbb{R}^d)$ which is larger than $L^p(\mathbb{R}^d)$.

Weak Morrey Spaces

Let $1 \le p \le q < \infty$. The *weak Morrey space* $w\mathcal{M}_q^p(\mathbb{R}^d)$ is the set of all measurable functions f for which $\|f\|_{w\mathcal{M}_q^p} < \infty$, where

$$\begin{aligned} \|f\|_{w\mathcal{M}_q^p} &:= \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}}\, \gamma\, |\{x \in B(a,r) \colon |f(x)| > \gamma\}|^{\frac{1}{p}} \\ &= \sup_{a\in\mathbb{R}^d,\, r,\gamma>0} |B(a,r)|^{\frac{1}{q}-\frac{1}{p}} \|\gamma\, \chi_{\{x\,:\,|f(x)|>\gamma\}}\|_{L^p(B(a,r))}. \end{aligned}$$

Note that $\|\cdot\|_{w\mathcal{M}_q^p}$ forms a quasi-norm in $w\mathcal{M}_q^p(\mathbb{R}^d)$. If $p = q$, then $\|\cdot\|_{w\mathcal{M}_q^p} = \|\cdot\|_{wL^p}$.

The Weak Contains the Strong

For each $p \le q$, the weak Morrey space $w\mathcal{M}_q^p(\mathbb{R}^d)$ contains the Morrey space $\mathcal{M}_q^p(\mathbb{R}^d)$, as stated in the following proposition.

Proposition. Let $1 \le p \le q < \infty$. Then,

$$\mathcal{M}_q^p(\mathbb{R}^d) \subset w\mathcal{M}_q^p(\mathbb{R}^d),$$

with

$$\|f\|_{w\mathcal{M}_q^p} \le \|f\|_{\mathcal{M}_q^p}$$

for every $f \in \mathcal{M}_q^p(\mathbb{R}^d)$.

Inclusion Property of Weak Morrey Spaces

The inclusion property of weak Morrey spaces is presented in the following theorem.

Theorem. If $1 \leq p_1 \leq p_2 \leq q < \infty$, then

$$w\mathcal{M}_q^{p_2}(\mathbb{R}^d) \subset w\mathcal{M}_q^{p_1}(\mathbb{R}^d)$$

with

$$\|f\|_{w\mathcal{M}_q^{p_1}} \leq \|f\|_{w\mathcal{M}_q^{p_2}}$$

for every $f \in w\mathcal{M}_q^{p_2}(\mathbb{R}^d)$.

The generalized Morrey spaces $\mathcal{M}^p_\phi(\mathbb{R}^d)$ which we are going to define next is associated with two parameters, namely $1 \le p < \infty$ and a function $\phi : (0, \infty) \to (0, \infty)$.
We assume that ϕ is in the class $\mathcal{G}_p$, that is, ϕ is *almost decreasing* $[r \le s \Rightarrow \phi(r) \ge C\phi(s)]$ and $t \mapsto t^{d/p}\phi(t)$ is *almost increasing* $[r \le s \Rightarrow r^{d/p}\phi(r) \le Cs^{d/p}\phi(s)]$.
Note that $\phi \in \mathcal{G}_p$ implies that ϕ satisfies the *doubling condition*, that is, there exists $C > 0$ such that

$$\frac{1}{C} \le \frac{\phi(r)}{\phi(s)} \le C$$

for every $r, s > 0$ with $\frac{1}{2} \le \frac{r}{s} \le 2$.

For each $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$, the *generalized Morrey space* $\mathcal{M}^p_\phi = \mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined as the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{\mathcal{M}^p_\phi} := \sup_{a\in\mathbb{R}^d,\ r>0} \frac{1}{\phi(r)} \left(\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p \, dx \right)^{1/p} < \infty.$$

Inclusion Property, A Sufficient Condition

Theorem [Sihwaningrum]. Let $1 \leq p_1 \leq p_2 < \infty$. If $\phi_2 \leq C\,\phi_1$, then

$$\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$$

with

$$\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\,\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}.$$

Proof. Suppose that $\phi_2 \le C\,\phi_1$. Let $f \in \mathcal{M}^{p_2}_{\phi_2}$. For every $a \in \mathbb{R}^d,\ r > 0$, we have

$$
\begin{aligned}
&\frac{1}{\phi_1(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(x)|^{p_1}\,dx\right)^{\frac{1}{p_1}} \\
&\quad\le \frac{C}{\phi_2(r)}\left(\frac{1}{|B(a,r)|}\left(\int_{B(a,r)} (|f(x)|^{p_1})^{\frac{p_2}{p_1}}\,dx\right)^{\frac{p_1}{p_2}}\left(\int_{B(a,r)} dx\right)^{1-\frac{p_1}{p_2}} \right. \\
&\quad\le \frac{C}{\phi_2(r)}\left(\frac{1}{|B(a,r)|}\int_{B(a,r)} |f(x)|^{p_2}\,dx\right)^{\frac{1}{p_2}} \le C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}.
\end{aligned}
$$

Hence $f \in \mathcal{M}^{p_1}_{\phi_1}(\mathbb{R}^d)$ with $\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \le C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}$.

Inclusion Property, A Necessary Condition

Our first result is the following:

Theorem. Let $1 \leq p_1 \leq p_2 < \infty$. If

$$\mathcal{M}^{p_2}_{\phi_2} \subseteq \mathcal{M}^{p_1}_{\phi_1}$$

with

$$\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\,\|f\|_{\mathcal{M}^{p_2}_{\phi_2}},$$

then $\phi_2 \leq C\,\phi_1$.

The proof of the necessary condition uses the estimate

$$\frac{1}{\phi(r)} \leq \|\chi_{B(0,r)}\|_{\mathcal{M}^p_\phi} \leq \frac{C}{\phi(r)}$$

which holds for every $r > 0$.

Proof. Suppose that the inclusion holds. Let $B_0 := B(0, r_0)$, where $r_0 > 0$. Then

$$\|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}} \leq C\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}}. \qquad (1)$$

By using the estimate for $\|\chi_{B_0}\|$, we obtain

$$\frac{1}{\phi_1(r_0)} \leq \|\chi_{B_0}\|_{\mathcal{M}^{p_1}_{\phi_1}} \qquad (2)$$

and

$$\|\chi_{B_0}\|_{\mathcal{M}^{p_2}_{\phi_2}} \leq \frac{C}{\phi_2(r_0)}. \qquad (3)$$

The inequalities (1), (2), and (3) imply that $\phi_2(r_0) \leq C\phi_1(r_0)$.

Putting the two results together, we obtain:

Theorem Let $1 \le p_1 \le p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:
(a) $\phi_2 \le C\phi_1$.
(b) $\mathcal{M}^{p_2}_{\phi_2} \subset \mathcal{M}^{p_1}_{\phi_1}$ with

$$\|f\|_{\mathcal{M}^{p_1}_{\phi_1}} \le C\|f\|_{\mathcal{M}^{p_2}_{\phi_2}}$$

for every $f \in \mathcal{M}^{p_2}_{\phi_2}$.

Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. The *generalized weak Morrey space* $w\mathcal{M}^p_\phi = w\mathcal{M}^p_\phi(\mathbb{R}^d)$ is defined to be the set of all measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{w\mathcal{M}^p_\phi} := \sup_{a \in \mathbb{R}^d,\, r,\gamma > 0} \frac{\|\gamma\, \chi_{\{x : |f(x)| > \gamma\}}\|_{L^p(B(a,r))}}{\phi(r)|B(a,r)|^{1/p}} < \infty.$$

The relation between the generalized Morrey spaces and their weak type is given in the following proposition.

Theorem. Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. Then $\mathcal{M}^p_\phi \subseteq w\mathcal{M}^p_\phi$ with

$$\|f\|_{w\mathcal{M}^p_\phi} \leq \|f\|_{\mathcal{M}^p_\phi}$$

for every $f \in \mathcal{M}^p_\phi$.

Let $1 \leq p < \infty$ and $\phi \in \mathcal{G}_p$. Then there exists $C > 1$ such that

$$\frac{1}{\phi(r_0)} \leq \|\chi_{B_0}\|_{w\mathcal{M}^p_\phi} \leq \frac{C}{\phi(r_0)},$$

for every ball $B_0 := B(0, r_0)$.

Finally, we come to the inclusion property of generalized weak Morrey spaces.

Theorem. Let $1 \le p_1 \le p_2 < \infty$, $\phi_1 \in \mathcal{G}_{p_1}$, and $\phi_2 \in \mathcal{G}_{p_2}$. Then, the following statements are equivalent:
(a) $\phi_2 \le C\phi_1$.
(b) $w\mathcal{M}^{p_2}_{\phi_2} \subset w\mathcal{M}^{p_1}_{\phi_1}$ with

$$\|f\|_{w\mathcal{M}^{p_1}_{\phi_1}} \le C\|f\|_{w\mathcal{M}^{p_2}_{\phi_2}}$$

for every $f \in w\mathcal{M}^{p_2}_{\phi_2}$.

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