THE n-DUAL SPACE OF THE SPACE OF p-SUMMABLE SEQUENCES

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Abstract. In the theory of normed spaces, we have the concept of bounded linear functionals and dual spaces. Now, given an n-normed space, we are interested in bounded multilinear n-functionals and n-dual spaces. The concept of bounded multilinear n-functionals on an n-normed space was initially intoduced by White [10], and studied further by Gozali *et al.* [4] and Batkunde *et al.* [1]. In this paper, we shall refine the definition of bounded multilinear n-functionals, introduce the concept of n-dual spaces, and then determine the n-dual spaces of ℓ^p spaces.

1. Introduction

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$. A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties,

(1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent,
(2) $\|x_1, \dots, x_n\|$ is invariant under permutation,
(3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$ for all $\alpha \in \mathbb{R}$,
(4) $\|x_1 + x_1', \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an *n-normed space* [2, 3]. Note that on an n-normed space $(X, \|\cdot, \dots, \cdot\|)$, we have $\|x_1, x_2, \dots, x_n\| = \|x_1 + y, x_2, \dots, x_n\|$ for any linear combination y of $x_2, \dots, x_2 \in X$.

To give an example, let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then we can equip the space ℓ^p of p-summable sequences with an n-norm $\|\cdot, \dots, \cdot\|_p^G$ which is given by

$$\|x_1, \dots, x_n\|_p^G := \sup_{y_j \in \ell^q, \, \|y_j\|_q \leq 1} \left| \det \left[\sum_{k=1}^{\infty} x_{ik} y_{jk} \right]_{i,j} \right|, \quad x_1, \dots, x_n \in \ell^p.$$

Here ℓ^q is the dual space of ℓ^p, and $\|\cdot\|_q$ denotes the usual norm on ℓ^q (see, for instance, [7]). The above n-norm is due to Gähler [1,2]. Another n-norm can be defined on ℓ^p, namely

$$\|x_1, \dots, x_n\|_p^H := \left(\frac{1}{n!} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left| \det \left[x_{ik_j} \right]_{i,j} \right|^p \right)^{\frac{1}{p}}, \quad x_1, \dots, x_n \in \ell^p.$$

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This n-norm is introduced by Gunawan [5]. As shown in [11], these two n-norms on ℓ^p are equivalent, that is,

$$(n!)^{\frac{1}{p}-1}\,\|x_1,\dots,x_n\|_p^H \le \|x_1,\dots,x_n\|_p^G \le (n!)^{\frac{1}{p}}\,\|x_1,\dots,x_n\|_p^H \tag{1.1}$$

for all $x_1,\dots,x_n \in \ell^p$.

Any real-valued function f on X^n, where X is a real vector space of dimension $d \ge n$, is called an *n-functional* on X. Furthermore, an n-functional f satisfying the following two properties:

(1) $f(x_1+y_1,\dots,x_n+y_n) = \displaystyle\sum_{h_i\in\{x_i,y_i\},\,1\le i\le n} f(h_1,\dots,h_n)$

(2) $f(\alpha_1 x_1,\dots,\alpha_n x_n) = \alpha_1\cdots\alpha_n f(x_1,\dots,x_n)$

is called a *multilinear n-functional* on X.

Next, suppose that f is an n-functional on a normed space $(X,\|\cdot\|)$ [respectively an n-normed space $(X,\|\cdot,\dots,\cdot\|)$]. If there exists a constant $K>0$ such that

$$|f(x_1,\dots,x_n)| \le K\|x_1\|\cdots\|x_n\| \quad [\text{resp. } |f(x_1,\dots,x_n)| \le K\|x_1,\dots,x_n\|\,]$$

for all $x_1,\dots,x_n \in X$, then f is said to be *bounded* on $(X,\|\cdot\|)$ [resp. *bounded* on $(X,\|\cdot,\dots,\cdot\|)$].

It is easy to check that every bounded multilinear n-functional f on an n-normed space $(X,\|\cdot,\dots,\cdot\|)$ satisfies

$$f(x_1,\dots,x_n) = 0$$

whenever $x_1,\dots,x_n$ are linearly dependent. Further, it is antisymmetric, that is,

$$f(x_1,\dots,x_n) = \operatorname{sgn}(\sigma) f(x_{\sigma(1)},\dots,x_{\sigma(n)})$$

for any $x_1,\dots,x_n \in X$ and any permutation σ of $(1,\dots,n)$. Here $\operatorname{sgn}(\sigma)=1$ if σ is an even permutation and $\operatorname{sgn}(\sigma)=-1$ if σ is an odd permutation. These properties do not hold for bounded multilinear n-functionals on a normed space $(X,\|\cdot\|)$.

Inspired by the concept of the dual space of a normed space, the space of bounded multilinear n-functionals on $(X,\|\cdot\|)$ [resp. on $(X,\|\cdot,\dots,\cdot\|)$] is called the *$n$-dual space* of $(X,\|\cdot\|)$ [resp. the *n-dual space* of $(X,\|\cdot,\dots,\cdot\|)$]. This space can be equipped with the following norm

$$\|f\|_{n,1} := \sup_{\|x_1\|,\dots,\|x_n\|\neq 0} \frac{|f(x_1,\dots,x_n)|}{\|x_1\|\dots\|x_n\|} \quad \left[\text{resp. } \|f\|_{n,n} := \sup_{\|x_1,\dots,x_n\|\neq 0} \frac{|f(x_1,\dots,x_n)|}{\|x_1,\dots,x_n\|}\right].$$

In the following sections, we shall focus on $X=\ell^p$, where $1\le p<\infty$. For convenient, we shall first discuss the 2-dual spaces of ℓ^p, and then generalize the result for all $n\ge 2$. This work is part of the first author thesis [9].

2. The 2-Dual Spaces of ℓ^p

We shall here identify the 2-dual space of ℓ^p as a normed space, and then use the result to determine the 2-dual space of ℓ^p as a 2-normed space, equipped with Gähler's 2-norm as well as Gunawan's 2-norm. From now on, we shall always assume that $1\le p<\infty$ and q is the dual exponent of p, that is, $\frac{1}{p}+\frac{1}{q}=1$, unless otherwise stated.

To achieve our goals, we need to introduce the following normed space. We say that a double index sequence $\theta := (\theta_{kj})$ (of real numbers) belongs to the space $Y^q_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_p=1} \left[\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right]^{\frac{1}{q}} < \infty.$$

Here $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ defines a norm on $Y^q_{\mathbb{N}\times\mathbb{N}}$. For $q=\infty$, a double index sequence $\theta := (\theta_{kj})$ is in $Y^\infty_{\mathbb{N}\times\mathbb{N}}$ if

$$\|\theta\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x\|_1=1}\sup_{j\in\mathbb{N}}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right| < \infty.$$

Our first result is the following.

Theorem 1. *If $1 < p < \infty$, then the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$. Moreover, the mapping $f \mapsto \theta := (f(e_k, e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p, \|\cdot\|_p)$ to $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$.*

Proof. For $\theta := (\theta_{kj}) \in Y^q_{\mathbb{N}\times\mathbb{N}}$, we define a 2-functional f on ℓ^p by

$$f(x,y) := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} x_k y_j \theta_{kj},$$

where $x := (x_i) = \sum\limits_{i=1}^{\infty} x_i e_i$ and $y := (y_i) = \sum\limits_{i=1}^{\infty} y_i e_i$. Note that $f(e_k, e_j) = \theta_{kj}$ for $k, j \in \mathbb{N}$. Further, f is a bilinear 2-functional on $(l^p, \|\cdot\|_p)$, and for $x, y \in \ell^p$ with $\|x\|_p = \|y\|_p = 1$, we have

$$\begin{aligned}
|f(x,y)| &= \left|\sum_{j=1}^{\infty}\left(y_j\sum_{k=1}^{\infty} x_k\theta_{kj}\right)\right| \\
&\le \left(\sum_{j=1}^{\infty}|y_j|^p\right)^{\frac{1}{p}}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} x_k\theta_{kj}\right|^q\right)^{\frac{1}{q}} \\
&\le \sup_{\|z\|_p=1}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} z_k\theta_{kj}\right|^q\right)^{\frac{1}{q}} \\
&= \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}.
\end{aligned}$$

Hence, for $x, y \neq 0$, we have

$$\frac{|f(x,y)|}{\|x\|_p\|y\|_p} \le \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}.$$

This means that f is a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$ with

$$\|f\|_{2,1} \le \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}. \tag{2.1}$$

Conversely, let f be a bounded bilinear 2-functional on $(\ell^p, \|\cdot\|_p)$. We claim that $\theta := (f(e_k, e_j)) \in Y^q_{\mathbb{N}\times\mathbb{N}}$. For each $x \in \ell^p$ with $\|x\|_p = 1$, let f_x be the functional on $(\ell^p, \|\cdot\|_p)$ given by

$$f_x(y) := f(x, y), \quad y \in \ell^p.$$

It is clear that f_x is a linear functional on $(\ell^p, \|\cdot\|_p)$. Moroever, if $y \neq 0$, then

$$\frac{|f_x(y)|}{\|y\|_p} = \frac{|f(x,y)|}{\|x\|_p\|y\|_p} \le \|f\|_{2,1}.$$

Hence f_x is bounded with $\|f_x\| \le \|f\|_{2,1}$. Since the dual space of $(\ell^p, \|\cdot\|_p)$ is $(\ell^q, \|\cdot\|_q)$, the bounded linear functional f_x is identified by $(f_x(e_j)) = (f(x, e_j))$ with

$$\left(\sum_{j=1}^{\infty} |f(x, e_j)|^q\right)^{\frac{1}{q}} = \|f_x\| \le \|f\|_{2,1}.$$

Therefore, we obtain

$$\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} = \sup_{\|x\|_p=1} \left(\sum_{j=1}^{\infty} \left|\sum_{k=1}^{\infty} x_k f(e_k, e_j)\right|^q\right)^{\frac{1}{q}} \le \|f\|_{2,1}, \tag{2.2}$$

and this proves our claim.

It follows from (2.1) and (2.2) that the mapping $f \mapsto \theta := (f(e_k, e_j))$ is an isometric bijection from the 2-dual space of $(\ell^p, \|\cdot\|_p)$ to $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$. □

For $p = 1$, we can also prove easily that the 2-dual space of $(\ell^1, \|\cdot\|_1)$ is identified by $\left(Y^\infty_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^\infty_{\mathbb{N}\times\mathbb{N}}}\right)$. Hence we have the following corollary.

Corollary 2. *For $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$.*

Now we shall discuss the 2-dual space of $(\ell^p, \|\cdot,\cdot\|^G_p)$. For this purpose, we need to invoke the concept of g-orthogonality on ℓ^p, where g is the semi-inner product on ℓ^p given by the formula

$$g(x, y) := \|x\|_p^{2-p} \sum_{j=1}^{\infty} |x_j|^{p-1}\mathrm{sgn}(x_j) y_j, \quad x := (x_j), y := (y_j).$$

If $g(x, y) = 0$, then we say that x and y are *g-orthogonal*, and we write $x \perp_g y$. (See [8] for some properties of g-orthogonality.)

As in [6], we may define the "volume" of the parallelpiped spanned by linearly independent $x_1, \dots, x_n \in \ell^p$ by the formula

$$V(x_1, \dots, x_n) := \|x_1^\circ\|_p \cdots \|x_n^\circ\|_p,$$

where $\{x_1^\circ, \dots, x_n^\circ\}$ is the left g-orthogonal sequence obtained from $\{x_1, \dots, x_n\}$ through a Gram-Schmidt process. If $x_1, \dots, x_n$ are linearly dependent, then we simply define $V(x_1, \dots, x_n) = 0$.

In [11], it is shown that

$$V(x_{i_1}, \dots, x_{i_n}) \leq \|x_1, \dots, x_n\|_p^G \tag{2.3}$$

for all $x_1, \dots, x_n \in \ell^p$ and any permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$. Using this fact (for the case where $n = 2$), we get the following theorem.

Theorem 3. *A bilinear 2-functional f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$ if and only if f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have*

$$\frac{1}{2}\|f\|_{2,1} \leq \|f\|_{2,2}^G \leq \|f\|_{2,1},$$

where $\|\cdot\|_{2,2}^G$ is the norm on the 2-dual space of $(\ell^p, \|\cdot,\cdot\|_p^G)$.

Proof. Suppose that f is a bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$. It is clear that f is antisymmetric, that is, $f(x, y) = -f(y, x)$ for all $x, y \in \ell^p$. Next, for $x, y \in \ell^p$, we have $\|x, y\|_p^G \leq 2^{\frac{1}{p}}\|x, y\|_p^H$ (by (1.1) for $n = 2$) and $\|x, y\|_p^H \leq 2^{1-\frac{1}{p}}\|x\|_p\|y\|_p$ (see [5]), so that $\|x, y\|_p^G \leq 2\,\|x\|_p\|y\|_p$. Thus, for any linearly independent $x, y \in \ell^p$, we obtain

$$\frac{1}{2}\frac{|f(x,y)|}{\|x\|_p\|y\|_p} \leq \frac{|f(x,y)|}{\|x,y\|_p^G} \leq \|f\|_{2,2}^G.$$

Hence f is bounded on $(\ell^p, \|\cdot\|_p)$ with

$$\frac{1}{2}\|f\|_{2,1} \leq \|f\|_{2,2}^G. \tag{2.4}$$

Conversely, suppose that f is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Given linearly independent $x, y \in \ell^p$, we observe that $f(x, y) = f(x^\circ, y^\circ)$ where $\{x^\circ, y^\circ\}$ is the left g-orthogonal set obtained from $\{x, y\}$. Moreover, we have

$$\frac{|f(x,y)|}{\|x,y\|_p^G} \leq \frac{|f(x,y)|}{V(x,y)} = \frac{|f(x^\circ,y^\circ)|}{\|x^\circ\|_p\|y^\circ\|_p} \leq \|f\|_{2,1}.$$

Since f is also antisymmetric, we have

$$|f(a,b)| \leq \|f\|_{2,1}\|x,y\|_p^G$$

for all $x, y \in \ell^p$, that is, f is bounded on $(\ell^p, \|\cdot,\cdot\|_p^G)$ with

$$\|f\|_{2,2}^G \leq \|f\|_{2,1}. \tag{2.5}$$

Finally, from (2.4) and (2.5), we conclude that

$$\frac{1}{2}\|f\|_{2,1} \leq \|f\|_{2,2}^G \leq \|f\|_{2,1},$$

as desired. □

To identify the 2-dual space of of $(\ell^p, \|\cdot,\cdot\|_p^G)$, we consider some subspace of $Y_{\mathbb{N}\times\mathbb{N}}^q$. A double index sequence $\theta := (\theta_{kj})$ belongs to $Z_{\mathbb{N}\times\mathbb{N}}^q$ if $\theta \in Y_{\mathbb{N}\times\mathbb{N}}^q$ dan $\theta_{kj} = -\theta_{jk}$ for all

$k, j \in \mathbb{N}$. Note that $Z^q_{\mathbb{N}\times\mathbb{N}}$ can be viewed as a normed space equipped with the norm inherited from $Y^q_{\mathbb{N}\times\mathbb{N}}$.

Previously, we have shown that the 2-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$. Hence the space of all antisymmetric bounded bilinear 2-functionals on $(\ell^p, \|\cdot\|_p)$ can be identified by the space $\left(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}\right)$. From this and the previous theorem, we get the following corollaries.

Corollary 4. *The function* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^G_p \neq 0} \frac{\left|\sum_{j=1}^\infty \sum_{k=1}^\infty x_k y_j \theta_{kj}\right|}{\|x,y\|^G_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$. *Furthermore,* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$\frac{1}{2}\|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 5. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|^G_p)$ *is identified by* $\left(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

Using (1.1) for the case where $n = 2$, we obtain the following corollaries.

Corollary 6. *The function* $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *on* $Z^q_{\mathbb{N}\times\mathbb{N}}$ *defined by*

$$\|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} := \sup_{\|x,y\|^H_p \neq 0} \frac{\left|\sum_{j=1}^\infty \sum_{k=1}^\infty x_k y_j \theta_{kj}\right|}{\|x,y\|^H_p}$$

defines a norm on $Z^q_{\mathbb{N}\times\mathbb{N}}$. *Furthermore,* $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *and* $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$ *are equivalent norms on* $Z^q_{\mathbb{N}\times\mathbb{N}}$, *with*

$$2^{\frac{1}{p}-1}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq \|\theta\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}} \leq 2^{\frac{1}{p}}\|\theta\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$$

for all $\theta \in Z^q_{\mathbb{N}\times\mathbb{N}}$.

Corollary 7. *The* 2*-dual space of* $(\ell^p, \|\cdot,\cdot\|^H_p)$ *is identified by* $\left(Z^q_{\mathbb{N}\times\mathbb{N}}, \|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}\right)$.

`Remark`. Here $\|\cdot\|^H_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, $\|\cdot\|^G_{Z^q_{\mathbb{N}\times\mathbb{N}}}$, and $\|\cdot\|_{Y^q_{\mathbb{N}\times\mathbb{N}}}$ are three equivalent norms on $Z^q_{\mathbb{N}\times\mathbb{N}}$.

3. The n-Dual Spaces of ℓ^p

The results for the case $n = 2$ can be extended easily to the case $n \geq 2$. For $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, we define $Y^q_{\mathbb{N}^n}$ to be the space of all (real) n-index sequence $\theta := (\theta_{k_1\cdots k_n})$ where

$$\|\theta\|_{Y^q_{\mathbb{N}^n}} := \sup_{\|a_1\|_p = \cdots = \|a_{n-1}\|_p = 1} \left[\sum_{k_n=1}^\infty \left|\sum_{k_1,\ldots,k_{n-1}=1}^\infty a_{1k_1} \cdots a_{n-1,k_{n-1}} \theta_{k_1\cdots k_n}\right|^q\right]^{\frac{1}{q}} < \infty.$$

For $q=\infty$, an n-index sequence $\theta := (\theta_{k_1\cdots k_n})$ belongs to the space $Y^{\infty}_{\mathbb{N}^n}$ if

$$\|\theta\|_{Y^{\infty}_{\mathbb{N}^n}} := \sup_{\|a_1\|_1=\cdots=\|a_{n-1}\|_1=1} \sup_{k_n\in\mathbb{N}} \left| \sum_{k_1,\ldots,k_{n-1}=1}^{\infty} a_{1k_1}\cdots a_{n-1,k_{n-1}}\theta_{k_1\cdots k_n} \right| < \infty.$$

Here $\mathbb{N}^n := \mathbb{N}\times\cdots\times\mathbb{N}$ (n factors). Note also that the inner sum above is a multiple sum.

We also define the generalization of $Z^q_{\mathbb{N}\times\mathbb{N}}$ spaces as follows. An n-index sequence $\theta := (\theta_{k_1\cdots k_n})$ belongs to the space $Z^q_{\mathbb{N}^n}$ if $\theta \in Y^q_{\mathbb{N}^n}$ and $\theta_{k_1\cdots k_n} = \operatorname{sgn}(\sigma)\theta_{\sigma(k_1)\cdots\sigma(k_n)}$, for all $k_1,\ldots,k_n\in\mathbb{N}$ and any permutation σ of $(k_1,\ldots,k_n)$.

Analogous to the case $n=2$, we have the following result for $n\geq 2$. (We leave the proof to the reader.)

Theorem 8. *The n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$. Moreover, the mapping $f\mapsto \theta := (f(e_{k_1},\ldots,e_{k_n}))$ is an isometric bijection from the n-dual space of $(\ell^p, \|\cdot\|_p)$ to $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$.*

Using (2.3) and the following two inequalities from [5, 11]:

$$\|x_1,\ldots,x_n\|^G_p \leq (n!)^{\frac{1}{p}}\, \|x_1,\ldots,x_n\|^H_p$$

and

$$\|x_1,\ldots,x_n\|^H_p \leq (n!)^{1-\frac{1}{p}}\, \|x_1\|_p\cdots\|x_n\|_p,$$

we can prove the following theorem by using similar arguments as in the case where $n=2$.

Theorem 9. *A multilinear n-functional f is bounded on $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$ if and only if it is antisymmetric and bounded on $(\ell^p, \|\cdot\|_p)$. Furthermore, we have*

$$\frac{1}{n!}\|f\|_{n,1} \leq \|f\|^G_{n,n} \leq \|f\|_{n,1}$$

where $\|\cdot\|^G_{n,n}$ is the norm on the n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$.

From Theorems 9 and 10, we get the following result.

Corollary 10. *The n-dual space of $(\ell^p, \|\cdot,\ldots,\cdot\|^G_p)$ is identified by $\left(Z^q_{\mathbb{N}^n}, \|\cdot\|^G_{Z^q_{\mathbb{N}^n}}\right)$, where $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ is given by*

$$\|\theta\|^G_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\ldots,x_n\|^G_p\neq 0} \frac{\left|\sum_{k_1,\ldots,k_n=1}^{\infty} x_{1k_1}\cdots x_{nk_n}\theta_{k_1\cdots k_n}\right|}{\|x_1,\ldots,x_n\|^G_p}.$$

Using (1.1), we also get the following theorem.

Corollary 11. *The n-dual space of $(\ell^p, \|\cdot, \dots, \cdot\|_p^H)$ is identified by $\left(Z^q_{\mathbb{N}^n}, \|\cdot\|^H_{Z^q_{\mathbb{N}^n}}\right)$, where $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ is given by*

$$\|\theta\|^H_{Z^q_{\mathbb{N}^n}} := \sup_{\|x_1,\dots,x_n\|_p^H \neq 0} \frac{\left|\sum\limits_{k_1,\dots,k_n=1}^{\infty} x_{1k_1}\cdots x_{nk_n}\theta_{k_1\cdots k_n}\right|}{\|x_1,\dots,x_n\|_p^H}.$$

4. Concluding Remarks

In the theory of normed spaces, we know that the dual space of $(\ell^p, \|\cdot\|_p)$ is (identified by) $(\ell^q, \|\cdot\|_q)$, where $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Here we show that the n-dual space of $(\ell^p, \|\cdot\|_p)$ is identified by $\left(Y^q_{\mathbb{N}^n}, \|\cdot\|_{Y^q_{\mathbb{N}^n}}\right)$. We see similarities between the two results. Similar relations also occur for the n-dual space of ℓ^p when ℓ^p is viewed as an n-normed space with Gähler's n-norm or Gunawan's n-norm. All these results are identical in the case where $n = 1$. For $n \ge 2$, however, we still have a question whether the norm $\|\cdot\|_{Y^q_{\mathbb{N}^n}}$ on $Y^q_{\mathbb{N}^n}$, as well as the norms $\|\cdot\|^H_{Z^q_{\mathbb{N}^n}}$ and $\|\cdot\|^G_{Z^q_{\mathbb{N}^n}}$ on $Z^q_{\mathbb{N}^n}$, can be reduced to

$$\|\theta\|^*_{Y^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}}$$

and

$$\|\theta\|^*_{Z^q_{\mathbb{N}^n}} := \left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}}.$$

One may easily check that if $\theta := (\theta_{k_1\dots k_n})$ satisfies

$$\left(\sum_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}} < \infty,$$

then $\|\theta\|_{Y^q_{\mathbb{N}^n}}$, $\|\theta\|^H_{Z^q_{\mathbb{N}^n}}$, and $\|\theta\|^G_{Z^q_{\mathbb{N}^n}}$ are all dominated by $\left(\sum\limits_{k_1,\dots,k_n=1}^{\infty} |\theta_{k_1\cdots k_n}|^q\right)^{\frac{1}{q}}$. We just do not know whether the converse is also true. See [1] for related problems.

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