Fractional Integral Operators (FIO's) on Generalized Morrey Spaces

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Abstract

We shall discuss the boundedness of a generalized fractional integral operators (FIO) on generalized Morrey spaces.

Through the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces and a Hedberg type inequality for the fractional integral (pointwise), we prove a result which improves those previously obtained by **[E. Nakai: 2001]**, **[G: 2003]**, and **[Eridani; G; and Nakai: 2004]**.

The result is obtained in a joint work with **Eridani; E. Nakai; and Y. Sawano**. [The paper, which contains much more results, will appear soon in *Mathematical Inequalities & Applications*.]

A new results on weak-type inequalities for the generalized FIO (joint w/**D. Hakim and I. Sihwaningrum**) will also be presented.

Previous Works, Among Others . . . I

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1938: S.L. Sobolev, “On a theorem in functional analysis” (Russian), *Mat. Sob.* **46**
1940: C.B. Morrey, “Functions of Several Variables and Absolute Continuity”, *Duke Math. J.* **6**
1972: L.I. Hedberg, “On certain convolution inequalities”, *Proc. Amer. Math. Soc.* **36**
1987: F. Chiarenza and M. Frasca, “Morrey spaces and Hardy-Littlewood maximal function”, *Rend. Mat.* **7**
1994: E. Nakai, “Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces”, *Math. Nachr.* **166**

Previous Works, Among Others . . . II

2001: V.S. Guliyev; S.S. Aliyev; and R.Ch. Mustafayev, "On generalized fractional integrals", *Trans. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **21**
2001: E. Nakai, "On generalized fractional integrals", *Taiwanese J. Math.* **5**
2003: H. Gunawan, "A note on the generalized fractional integral operators", *J. Indones. Math. Soc.* **9**
2003: S. Sugano and H. Tanaka, "Boundedness of fractional integral operators on generalized Morrey spaces", *Sci. Math. Jpn.* **58** (2003)
2004: Eridani; H. Gunawan; and E. Nakai, "On the generalized fractional integral operators", *Sci. Math. Jpn.* **60**

Previous Works, Among Others . . . III

2004: J. García-Cuerva and A.E. Gatto, "Boundedness properties of fractional integral operators associated to non-doubling measures", *Studia Math.* **162**

2008: Y. Sawano, "Generalized Morrey spaces for non-doubling measures", *Nonlinear Differential Equations Appl.* **15**

2009: H. Gunawan; Y. Sawano; and I. Sihwaningrum, "Fractional integral operators in nonhomogeneous spaces", *Bull. Austral. Math. Soc.* **80**

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2011: Y. Sawano; S. Sugano; and H. Tanaka, "Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces", *Trans. Amer. Math. Soc.* **363**

Generalized FIO's

For a measurable function $\rho : (0, \infty) \to (0, \infty)$, we define the *generalized FIO* I_ρ by the formula

$$I_\rho f(x) := \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y)\, dy, \quad x \in \mathbb{R}^d,$$

for any suitable function f on $\mathbb{R}^d$.

This operator was initially investigated in **[Nakai: 2001]**.

Note that for $\rho(t) = t^\alpha,\ 0 < \alpha < d$, we get the classical FIO

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-\alpha}}\, dy, \quad x \in \mathbb{R}^d,$$

which is known to be *bounded* from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ whenever $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{d}$ [**H-L-S Inequality**].

To make sure that the fractional integrals $I_\rho f$ are well-defined, at least for characteristic functions of balls, we assume that

$$\int_0^1 \frac{\rho(s)}{s}\,ds < \infty.$$

To prove the boundedness of FIO, we usually assume that ρ satisfies the *doubling condition*: there exists a constant $C > 0$ such that

$$\frac{1}{C} \le \frac{\rho(r)}{\rho(s)} \le C$$

whenever $\frac{1}{2} \le \frac{r}{s} \le 2$.

Note that if ρ satisfies the doubling condition, then for every integer k and $r > 0$ we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t}\, dt \sim \rho(2^k r).$$

Further, it follows from the doubling condition that

$$\rho(r) \le C \int_0^r \frac{\rho(t)}{t}\, dt,$$

for every $r > 0$.

Note: Later on, we shall assume a condition which is weaker than the doubling condition, which gives us basic properties that we need only.

Generalized Morrey Spaces

Many authors have been culminating important observations about I_ρ especially in connection with Morrey spaces.

For a certain function $\phi : (0, \infty) \to (0, \infty)$, a function f belongs to the *generalized Morrey space* $L^p_\phi = L^p_\phi(\mathbb{R}^d)$, where $1 \le p < \infty$, if

$$\|f \,:\, L^p_\phi\| := \sup_{a \in \mathbb{R}^d,\, r>0} \frac{1}{\phi(r)} \left[\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx \right]^{1/p} < \infty.$$

Here ϕ is nonincreasing and $t \mapsto \phi(t)^p t^d$ is nondecreasing on $(0, \infty)$. Consequently, ϕ satisfies the doubling condition!

Note that if $\phi(r) := r^{(\lambda-d)/p}$ for some $1 \le p < \infty$ and $0 \le \lambda < d$, then $L^p_\phi(\mathbb{R}^d) = L^p_\lambda(\mathbb{R}^d)$, the classical Morrey spaces.

An Old Theorem on I_ρ

Theorem [G: 2003][EGN: 2004] Suppose that ϕ is surjective and satisfies $\int_r^\infty \frac{\phi(t)^p}{t} dt \le C\phi(r)^p$ (p-*integral condition*) and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q},$$

for all $r > 0$. Then there exists $C_{p,q} > 0$ such that

$$\|I_\rho f\|_{L^q_{\phi^{p/q}}} \le C_{p,q}\|f\|_{L^p_\phi}, \qquad f \in L^p_\phi;$$

that is, I_ρ is bounded from L^p_ϕ to $L^q_{\phi^{p/q}}$, for $1 < p < q < \infty$.

An Improvement of the Result

As we have indicated earlier, we shall now assume a condition on ρ which is weaker than the doubling condition. The condition is known as the *growth condition* (introduced by **[C. Perez: 1994]**):

There exist constants $C_1 > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r/2<s\le r} \rho(s) \le C_1 \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s}\, ds, \quad r > 0. \qquad (1)$$

Example: $\rho(t) = e^{-t}$ satisfies the growth condition, but not the doubling condition.

We also define a more *generalized Morrey space* $L^p_{\phi,\mu} = L^p_{\phi,\mu}(\mathbb{R}^d)$, where $1 \le p < \infty$, to be the set of all functions f such that

$$\|f \,:\, L^p_{\phi,\mu}\| := \sup_{a\in\mathbb{R}^d,\, r>0} \frac{1}{\phi(r)} \left[\frac{1}{r^n}\int_{B(a,r)} |f(x)|^p d\mu(x)\right]^{1/p} < \infty,$$

where μ is a Borel measure satisfying the *growth property*:

$$\mu(B(a,r)) \le C\, r^n.$$

This condition is weaker than the usual *doubling property* (used in the homogeneous setting):

$$\mu(2B) \le C\, \mu(B).$$

A New Theorem on I_ρ

Theorem [Eridani; G; Nakai; Sawano. 2013]: Suppose that ϕ satisfies $\int_r^\infty \frac{\phi(t)^p}{t}dt \le C\phi(r)^p$ (p-*integral condition*) and

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q},$$

for all $r > 0$. Then there exists $C_{p,q} > 0$ such that

$$\|I_\rho f\|_{L^q_{\phi^{p/q},\mu}} \le C_{p,q}\|f\|_{L^p_{\phi,\mu}}, \qquad f \in L^p_{\phi,\mu};$$

that is, I_ρ is bounded from $L^p_{\phi,\mu}$ to $L^q_{\phi^{p/q},\mu}$, for $1 < p < q < \infty$.

A Herberg's Type Inequality

The proof of the theorem invokes the following lemma:

Lemma: Let $\|f\|_{L^p_{\phi,\mu}} = 1$. Then, we have

$$|I_\rho f(x)| \le C\left(\left[M^n f(x)\right]^{p/q} + \inf_{r>0} \phi(r)^{p/q}\right),$$

for every $x \in \mathbb{R}^d$.

Here M^n is the *Hardy-Littlewood maximal operator*, given by

$$M^n f(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, d\mu(y), \quad x \in \mathbb{R}^d.$$

This operator is bounded on $L^p_{\phi,\mu}$ **[G; Sawano; Sihwaningrum: 2009]**

Once the lemma is proved, the theorem follows easily. Indeed, let $B = B(a, s)$ be an arbitrary ball. If we integrate the pointwise estimate, then we have

$$\frac{1}{|B|}\int_B |I_\rho f(x)|^q \, d\mu(x) \le C \left(\frac{1}{|B|}\int_B [M^n f(x)]^p \, d\mu(x) + \inf_{r>0} \phi(r)^p \right).$$

If we devide both sides by $\phi(s)^p$, then we have

$$\frac{1}{\phi(s)^p|B|}\int_B |I_\rho f(x)|^q \, d\mu(x) \le C \left(\frac{1}{\phi(s)^p|B|}\int_B [M^n f(x)]^p \, d\mu(x) + 1 \right)$$

$\le C_p$. Hence $\|I_\rho f\|_{L^p_{\phi^{p/q}}} \le C_{p,q}$.

The Idea of the Proof of the Lemma

The idea of the proof of the lemma is to split the integral (for every $x \in \mathbb{R}^d$) into two parts, using dyadic decomposition:

$$I_\rho f(x) \le C \left[\sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\rho^*(2^j r)}{(2^j r)^d} \int_{|x-y|<2^j r} |f(y)|\, d\mu(y) \right],$$

where $\rho^*(r) = \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s}\, ds$. (Here r is temporarily arbitrary.)

The first sum is then dominated by $C\,\phi(r)^{p/q-1} M^n f(x)$, while the second one is dominated by $C\,\phi(r)^{p/q}$. Then we choose $R > 0$ so that $\phi(R) \sim M^n f(x)$.
Here we do not require ϕ to be surjective on $(0, \infty)$; what is important is the doubling property of ϕ and the fact that $M^n f(x) \le \sup_{r>0} \phi(r)$ for every $x \in \mathbb{R}^d$.

Further Results

Further results include:

- Boundedness of I_ρ from $L^p_{\phi,\mu}$ to $L^q_{\psi,\mu}$ for certain ϕ and ψ;
- Necessary conditions for the boundedness of I_ρ on generalized Morrey spaces;
- Weak-type inequalities for I_ρ on generalized Morrey spaces (including in the non-homogeneous setting);
- ... and many other aspects.

A Weak-Type Inequality for I_ρ

Theorem [D.I. Hakim; G; Sihwaningrum: 2013]: Let $1 \le p < q < \infty$. Suppose that ϕ satisfies

$$\phi(r)\int_0^r \frac{\rho(t)}{t}dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\,\phi(r)^{p/q}$$

for every $r > 0$. Then, for any function $f \in L^p_{\phi,\mu}$ and any ball $B(a,r) \subseteq \mathbb{R}^d$, we have

$$\mu\left(\{x \in B(a,r) : |I_\rho f(x)| > \gamma\}\right) \le C\, r^n \phi(r)^p \left(\frac{\|f\|_{L^p_{\phi,\mu}}}{\gamma}\right)^q,$$

for every $\gamma > 0$.

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