# Characterizations for the generalized fractional integral operators on Morrey spaces

ERIDANI, Hendra GUNAWAN, Eiichi NAKAI and Yoshihiro SAWANO

**Abstract.** We present some characterizations for the boundedness of the generalized fractional integral operators on Morrey spaces. The characterizations follow from two key estimates, one for the norm of the characteristic functions of balls, and another for the values of the corresponding fractional integrals on smaller balls. Our results extends those obtained in our earlier paper [5]. We prove three theorems about necessary and sufficient conditions. We show that these theorems are independent by giving some examples.

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## 1. Introduction

In this paper, for a measurable function $\rho : (0,\infty) \to (0,\infty)$, we are interested in the generalized fractional integral operator $I_\rho$ given by the formula

$$I_\rho f(x) := \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} f(y)\,dy, \quad x \in \mathbb{R}^d,$$

for any suitable function $f$ on $\mathbb{R}^d$. This generalized fractional integral operator was initilally investigated in [14]. Nowadays many authors have been culminating important observations about $I_\rho$ especially in connection with Morrey spaces. Morrey spaces cover Lebesgue spaces as special cases and seem to describe the behavior of $I_\rho$ well. Here to highlight what we shall prove in this paper, we take up the works [2, 4, 5, 7, 11, 12, 15, 22]. In these works we proposed several conditions on $\rho$ for $I_\rho$ to be bounded on Morrey spaces. We aim to show that these conditions are necessary as well.

Note that the integral operators such as $(1-\Delta)^{-\alpha}$ and $L^{-\alpha}$ with $\alpha > 0$ fall under this scope, where $L$ is a suitable elliptic differential operator. Also, if a measurable function $V : \mathbb{R}^d \to (0,\infty)$ satisfies the reverse Hölder inequality, that is, there exist some constants $C > 0$, $q \gg 1$ such that, for all balls $B$, $\left(\int_B V(x)^q \frac{dx}{|B|}\right)^{1/q} \leq C \int_B V(x) \frac{dx}{|B|}$, then the operators $V^\gamma(-\Delta + V)^{-\beta}$ with $0 \leq \gamma \leq \beta \leq 1$ and $V^\gamma \partial_j (-\Delta + V)^{-\beta}$ with $0 \leq \gamma \leq \frac{1}{2} \leq \beta \leq 1$, $\beta - \gamma \geq \frac{1}{2}$ and $j = 1, 2, \cdots, d$ also fall under this scope [10]. We refer to [6, Sections 3 and 4] for a detailed description of these facts. Below, we assume that $\int_0^1 \frac{\rho(s)}{s} ds < \infty$, so that the fractional integrals $I_\rho f$ are well-defined, at least for characteristic functions of balls. In addition, we shall also assume that $\rho$ satisfies the *growth condition*: there exist constants $C_1 > 0$ and $0 < 2k_1 < k_2 < \infty$ such that

$$\sup_{r/2 < s \leq r} \rho(s) \leq C_1 \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s} ds, \quad r > 0. \tag{1.1}$$

This condition is weaker than the usual *doubling condition*: there exists a constant $C_2 > 0$ such that $\frac{1}{C_2} \leq \frac{\rho(r)}{\rho(s)} \leq C_2$ whenever $r$ and $s$ satisfy $r, s > 0$ and $\frac{1}{2} \leq \frac{r}{s} \leq 2$. See [24] for some examples and more explanation about these two conditions.

We note that if $\rho(r) = r^\alpha$, with $0 < \alpha < d$, then $I_\rho = I_\alpha$ is the classical fractional integral operator, also known as the Riesz potential, which is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ **if and only if** $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d}$, where $1 < p$, $q < \infty$ [19]. The necessary part is usually proved by using the scaling arguments. See [4, 7, 8, 26] for some recent results on the boundedness properties of $I_\rho$.

Our first theorem below characterizes the function $\rho$ for which $I_\rho$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ for $1 < p < q < \infty$.

**Theorem 1.1.** *Let $1 < p < q < \infty$. The operator $I_\rho$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if $\rho(r) \leq C\, r^{d(1/p - 1/q)}$ for all $r > 0$.*

Theorem 1.1 is a corollary of Theorem 1.3 and we prove and deal with Theorem 1.1 as such. For $\rho(r) = r^\alpha$, Theorem 1.1 reads that the operator $I_\rho$ is bounded from $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$ if and only if $\frac{\alpha}{d} = \frac{1}{p} - \frac{1}{q}$, where $1 < p < q < \infty$.

Our primary aim here is to characterize the function $\rho$ for which $I_\rho$ is bounded from one Morrey space to another. We obtain the characterizations by estimating the norm of the characteristic functions of balls and the values of the corresponding fractional integrals on smaller balls. Theorem 1.1 extends to the next result on Morrey spaces. For $1 \leq p < \infty$ and $0 \leq \lambda < d$, recall that the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^d)$ consists of all locally integrable functions $f$ on $\mathbb{R}^d$ for which

$$\|f : L^{p,\lambda}\| := \sup_{a \in \mathbb{R}^d,\, r > 0} \frac{1}{r^{(\lambda - d)/p}} \left[\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx\right]^{\frac{1}{p}} < \infty.$$

Note that $L^{p,0}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$. See [18] for more information about these spaces. The following theorem generalizes the previous characterization in Theorem 1.1. Theorem 1.2 follows from Theorem 1.3 and we prove and deal with Theorem 1.2 as such again.

**Theorem 1.2.** *Let* $1 < p < q < \infty$ *and* $0 \le \lambda < d$. *Assume that* $\rho$ *satisfies* (1.1). *Then the operator* $I_\rho$ *is bounded from* $L^{p,\lambda}(\mathbb{R}^d)$ *to* $L^{q,\lambda}(\mathbb{R}^d)$ *precisely when one of the following equivalent conditions holds.*

(*a*) $\rho(r) \le C\, r^{(d-\lambda)(1/p-1/q)}$ *for all* $r > 0$.

(*b*) $\tilde{\rho}(r) = \int_0^r \frac{\rho(s)}{s}\, ds \le C\, r^{(d-\lambda)(1/p-1/q)}$ *for all* $r > 0$.

Now we shall state our main result in full generality. We state result for generalized Morrey spaces. For a certain function $\phi : (0,\infty) \to (0,\infty)$, we say that a function $f$ belongs to the generalized Morrey space $L_{p,\phi} = L_{p,\phi}(\mathbb{R}^d)$, where $1 \le p < \infty$, if

$$\|f \,:\, L_{p,\phi}\| := \sup_{a \in \mathbb{R}^d,\, r>0} \frac{1}{\phi(r)} \left[ \frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx \right]^{1/p} < \infty.$$

Note that if $\phi(r) := r^{(\lambda-d)/p}$ for some $1 \le p < \infty$ and $0 \le \lambda < d$, then $L_{p,\phi}(\mathbb{R}^d) = L^{p,\lambda}(\mathbb{R}^d)$. In [13, p.446] we justified that $\phi$ is nonincreasing function such that $t \mapsto \phi(t)^p t^d$ is a nondecreasing for $L_{p,\phi}(\mathbb{R}^d) \ne \{0\}$. We refer to [12, 16] and [24, Section 1] for more information about these spaces.

In this paper, we shall assume that $\phi : (0,\infty) \to (0,\infty)$ is *almost decreasing* [that is, if $r \le s$, then $\phi(r) \ge C_3 \phi(s)$], and that $r^d \phi^p(r)$ is *almost increasing*, [that is, if $r \le s$, then $r^d \phi(r)^p \le C_3 s^d \phi(s)^p$]. These two conditions implies that $\phi$ also satisfies the doubling condition. Denote by $\mathcal{G}_p$ the set of all functions $\phi : (0,\infty) \to (0,\infty)$ such that $\phi$ is almost decreasing and that $r \mapsto r^{d/p}\phi(r)$ is almost increasing. Now we present three different criteria for the boundedness of $I_\rho$. Recall that we defined $\tilde{\rho}(r) = \int_0^r \frac{\rho(t)}{t}\, dt$.

**Theorem 1.3.** *Let* $1 < p < q < \infty$ *and* $\phi \in \mathcal{G}_p$. *Assume*

$$\int_r^\infty \frac{\phi(s)}{s}\, ds \le C\, \phi(r), \qquad \int_r^\infty \frac{\phi(s)\rho(s)}{s}\, ds \le C\, \phi(r)\rho(r) \quad (r > 0) \tag{1.2}$$

*for some constant* $C > 0$. *Then* $I_\rho$ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q}}(\mathbb{R}^d)$ *if and only if*

$$\tilde{\rho}(r) \le C\, \phi(r)^{p/q-1} \quad (r > 0) \tag{1.3}$$

*for some constant* $C > 0$.

**Theorem 1.4.** *Let* $1 < p < q < \infty$ *and* $\phi \in \mathcal{G}_p$.

(i) *If there exists a positive constant* $C > 0$ *such that*

$$\phi(r) \int_0^r \frac{\rho(t)}{t}\, dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \le C\phi(r)^{p/q} \quad (r > 0), \tag{1.4}$$

*then* $I_\rho$ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q}}(\mathbb{R}^d)$.

(ii) *Assume in addition that* $\phi$ *satisfies*

$$\int_0^r \frac{\phi(s)s^{d/p}}{s}\, ds \le C\phi(r) r^{d/p} \quad (r > 0) \tag{1.5}$$

*for some constant* $C > 0$ *and* $I_\rho$ *is bounded from* $L_{p,\phi}(\mathbb{R}^d)$ *to* $L_{q,\phi^{p/q}}(\mathbb{R}^d)$ *then* (1.4) *holds.*

**Theorem 1.5.** *Suppose that $\phi, \psi \in \mathcal{G}_1$.*

(i) *Assume that there exists a constant $C > 0$ such that*

$$\phi(r) \int_0^r \frac{\rho(t)}{t}\, dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \le C\psi(r) \quad (r > 0). \tag{1.6}$$

*Then $I_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi}(\mathbb{R}^d)$.*

(ii) *If there exists $C > 0$ such that*

$$\int_0^r \frac{\phi(s)s^d}{s}\, ds \le C\phi(r)r^d \quad (r > 0) \tag{1.7}$$

*and $I_\rho$ is bounded from $L_{1,\phi}(\mathbb{R}^d)$ to $L_{1,\psi}(\mathbb{R}^d)$, then* (1.6) *holds.*

*Remark* 1.6.

(i) The condition (1.4) appeared in [7] originally and it later appeared in a bilinear estimate of the form $g \cdot I_\alpha f$ (see [23, Theorem 1.6]).

(ii) If $\phi$ is decreasing, $\phi(r) \int_0^r \frac{\rho(t)}{t}\, dt + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt = \int_0^\infty \frac{\phi(\max(r,t))\rho(t)}{t}\, dt.$

Hereafter, the letter $C$ denotes a positive constant whose value may differ from line to line, which may depend on $d$, $\rho$, $p$ and $q$, but not on the functions $f$ and the variables $x$.

This paper is organized as follows: In Section 2, we calculate roughly the image by $I_\rho$ of the characteristic function of balls. Based upon a preliminary result, Lemma 2.1, in Section 2, we consider the boundedness of $I_\rho$ in Section 3. Some examples are presented in Section 4.

## 2. Some norm and integral estimates

Let us first consider the characteristic functions of balls. For every $R > 0$, let $B_R := B(0, R)$ be the ball centered at 0 with radius $R$, and $\chi_{B_R}$ be the characteristic function of $B_R$. Next, for all $r > 0$, let $\tilde{\rho}(r) := \int_0^r \frac{\rho(s)}{s}\, ds$. Also we write $B(x, R) = \{y \in \mathbb{R}^d : |x - y| < R\}$.

The following lemmas will be used several times in this paper.

**Lemma 2.1.** *There exists $C > 0$ such that the inequality $\tilde{\rho}(R/2) \le C\, I_\rho \chi_{B_R}(x)$ holds whenever $x \in B_{R/2}$ and $R > 0$.*

*Proof.* Take $x \in B_{R/2}$. We write the integral out in full:

$$I_\rho \chi_{B_R}(x) = \int_{\mathbb{R}^d} \frac{\rho(|x-y|)}{|x-y|^d} \chi_{B_R}(y)\, dy = \int_{B_R} \frac{\rho(|x-y|)}{|x-y|^d}\, dy.$$

A geometric observation shows that $B(x, R/2) \subseteq B(0, R)$. Hence, we have

$$I_\rho \chi_{B_R}(x) \ge \int_{B(x,R/2)} \frac{\rho(|x-y|)}{|x-y|^d}\, dy = C \int_0^{R/2} \frac{\rho(s)}{s}\, ds.$$

Note that we only use the spherical coordinates to obtain the last integral. □

**Lemma 2.2.** *For every $R > 0$ and a measurable function $\phi : \mathbb{R}^d \to (0,\infty)$ satisfying the doubling condition*

$$\frac{1}{C} \le \frac{\phi(s)}{\phi(r)} \le C \quad (0 < r/2 \le s \le 2r), \tag{2.1}$$

*the inequality $C^{-1} \int_{2R}^{\infty} \frac{\phi(t)\rho(t)}{t}\, dt \le I_\rho g_R(x) \le C \int_{2R/3}^{\infty} \frac{\phi(t)\rho(t)}{t}\, dt$ holds whenever $x \in B_{R/3}$, where $g_R(x) = \phi(|x|)\chi_{B_R^c}(x)$.*

*Proof.* We prove the right-hand inequality, the left-hand inequality being similar. Writing $I_\rho g_R(x)$ out in full, we obtain $I_\rho g_R(x) = \int_{\mathbb{R}^d \setminus B_R} \frac{\phi(|y|)\rho(|x-y|)}{|x-y|^d}\, dy$. Since $\phi$ satisfies (2.1), it follows that

$$I_\rho g_R(x) \le \int_{\mathbb{R}^d \setminus B(x,2R/3)} \phi(|y|) \frac{\rho(|x-y|)}{|x-y|^d}\, dy \sim \int_{\mathbb{R}^d \setminus B_{2R/3}} \phi(|y|) \frac{\rho(|y|)}{|y|^d}\, dy.$$

Hence $I_\rho g_R(x) \le C \int_{2R/3}^{\infty} \frac{\phi(t)\rho(t)}{t}\, dt$. It remains to write the most right-hand side in terms of the spherical coordinates. □

The lemma below gives an estimate for the norm of $\chi_{B_R}$ in $L_{p,\phi}(\mathbb{R}^d)$.

**Lemma 2.3.** *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. There exists a constant $C > 0$ such that $C^{-1}\phi(R)^{-1} \le \|\chi_{B_R} : L_{p,\phi}\| \le C\,\phi(R)^{-1}$ for all $R > 0$.*

Lemma 2.3 is proven in [13, Lemma 3.3] and see [5] as well.

**Lemma 2.4.** *Let $1 \le p < \infty$ and $\phi \in \mathcal{G}_p$. Assume that there exists a positive constant $C > 0$ such that*

$$\int_0^R \phi(t) t^{d/p-1}\, dt \le C\phi(R)R^{d/p} \quad (R > 0). \tag{2.2}$$

*Then the function $x \mapsto \phi(|x|)$ belongs to $L_{p,\phi}(\mathbb{R}^d)$.*

*Proof.* Since there exists a non-increasing function $\phi_1$ such that $\phi(r) \le \phi_1(r)$ and that $L_{p,\phi}(\mathbb{R}^d)$ and $L_{p,\phi_1}(\mathbb{R}^d)$ are isomorphic [13, p.446], we can assume that $\phi$ itself is decreasing. In this case $x \mapsto \phi(|x|)$ is radial decreasing, so that

$$\left[\frac{1}{|B(a,r)|}\int_{B(a,r)} \phi(|x|)^p dx\right]^{1/p} \le \left[\frac{1}{|B_r|}\int_{B_r} \phi(|x|)^p dx\right]^{1/p} \quad (a \in \mathbb{R}^d).$$

We have $\frac{1}{|B_r|}\int_{B_r} \phi(|x|)^p dx = C\frac{1}{r^d}\int_0^r \phi(t)^p t^{d-1}\, dt \le C\phi(r)^p$, in view of [17, Lemma 7.1]. Combining these observations, we prove the lemma. □

## 3. Proof of Theorems 1.3, 1.4 and 1.5

A normalization allows us to assume $\|f : L_{p,\phi}\| = 1$ (resp. $\|f : L_{1,\phi}\| = 1$) in the proof of necessity of Theorems 1.3 and 1.4 (resp. Theorem 1.5).

**3.1. Proof of the sufficiency**

We remark that (1.4) includes (1.3). We prove the estimate (3.1). Once we prove (3.1), the estimate (3.1) gives us the boundedness of $I_\rho$ from $L_{p,\phi}(\mathbb{R}^d)$ to $L_{q,\phi^{p/q}}(\mathbb{R}^d)$. Here we use the fact that the Hardy-Littlewood maximal operator $M$ is bounded on $L_{p,\phi}(\mathbb{R}^d)$, if $\phi$ satisfies the integral condition [12]. See [12, 19, 20] for more information about $M$ on the space $L_{p,\phi}(\mathbb{R}^d)$.

**Lemma 3.1.** *If we normalize the norm of $f$ by $\|f : L_{p,\phi}\| = 1$, then we have*

$$|I_\rho f(x)| \le C \left( [Mf(x)]^{p/q} + \inf_{r>0} \phi(r)^{p/q} \right), \quad x \in \mathbb{R}^d. \tag{3.1}$$

*Proof.* We have $|I_\rho f(x)| \le C \left[ \sum_{j=-\infty}^{-1} + \sum_{j=0}^{\infty} \frac{\rho^*(2^j r)}{(2^j r)^d} \int_{|x-y|<2^j r} |f(y)|\, dy \right]$ for given $x \in \mathbb{R}^d$ and $r > 0$. Recall that $k_1$ and $k_2$ appeared in the condition (1.1) of $\rho$. Let $\rho^*(r) = \int_{k_1 r}^{k_2 r} \frac{\rho(s)}{s}\, ds$. Let $\Sigma_I$ and $\Sigma_{II}$ be the first and second summations above. Then, by the *overlapping property* [24], we have

$$\Sigma_I \le C \sum_{j=-\infty}^{-1} \rho^*(2^j r) Mf(x) \le C\, \tilde{\rho}(k_2 r) Mf(x) \le C\, \phi(r)^{p/q-1} Mf(x),$$

$$\Sigma_{II} \le C \sum_{j=0}^{\infty} \rho^*(2^j r)\phi(2^j r) \|f : L_{p,\phi}\| \le C \int_{k_1 r}^{\infty} \frac{\rho(s)\phi(s)}{s}\, ds.$$

By (2.2) and the doubling property of $\phi$, we obtain $\Sigma_{II} \le C\, \phi(r)^{p/q}$. Hence,

$$|I_\rho f(x)| \le C\, \phi(r)^{p/q-1} \left[ Mf(x) + \phi(r) \right] \quad (r > 0). \tag{3.2}$$

Thus, we can assume $Mf(x) > \inf_{r>0} \phi(r)$. Otherwise (3.1) is immediate from (3.2). From the definition of $Mf(x)$ and $\|f : L_{p,\phi}\| = 1$, and Hölder's inequality, one may observe that $\inf_{r>0} \phi(r) \le Mf(x) \le \sup_{r>0} \phi(r)$. We can thus find $R > 0$ such that $\frac{1}{2C}\, \phi(R) \le Mf(x) \le 2\phi(R)$ and with this $R$ we can obtain (3.1). □

*Proof of Theorem* 1.5. For a ball $B(z,r)$, let $f_1 = f\chi_{B(z,2r)}$ and $f_2 = f - f_1$. Then a geometric observation shows $|I_\rho f_1(x)| \le \int_{B(z,2r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\, dy$ for all $x \in B(z,2r)$. Hence by the Fubini theorem and the normalization,

$$\int_{B(z,r)} I_\rho f_1(y)\, dy \le \int_{B(z,2r)} \int_{B(y,3r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\, dx\, dy \le C\tilde{\rho}(3r)\phi(3r) r^d.$$

Thus, the estimate for $f_1$ is valid. As for $f_2$, we let $x \in B(z,r)$. Then we have

$$|I_\rho f_2(x)| \le \int_{B(z,2r)^c} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\, dy \le \int_{B(x,r)^c} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\, dy$$

and decomposing the right-hand side dyadically as we did in the proof of Theorem 1.3 for $\sum_{II}$, we obtain

$$|I_\rho f_2(x)| \le \sum_{j=1}^{\infty} \int_{B(x,2^j r)\setminus B(x,2^{j-1} r)} |f(y)| \frac{\rho(|x-y|)}{|x-y|^d}\, dy \le C \int_{2k_1 r}^{\infty} \frac{\phi(t)\rho(t)}{t}\, dt.$$

If we use (1.6), then we obtain $|I_\rho f_2(x)| \le C\psi(r)$. Thus, the estimate for $f_2$ is valid as well. □

*Remark* 3.2. The proof of the sufficient part is similar to, but not the same as, that in [7, 15]. In this paper, we do not assume that $\rho$ satisfies the doubling condition and that $\phi$ is surjective, as we did in [7].

**3.2. Proof of necessity**

*Proof of Theorem* 1.3. Note that $\tilde\rho(R/2) \le C\left[\int_{B_{R/2}} |I_\rho \chi_{B_R}(x)|^q dx\right]^{1/q}$ by Lemma 2.1. Notice also that $\|I_\rho \chi_{B_R} : L_{q,\phi^{p/q}}\| \le C\|\chi_{B_R} : L_{p,\phi}\|$ since $I_\rho$ is assumed bounded. If we invoke Lemma 2.1, and the doubling property of $\phi$, we have $\tilde\rho(R/2) \le C\,\phi(R/2)^{p/q}\,\|I_\rho \chi_{B_R} : L_{q,\phi^{p/q}}\| \le C\,\phi(R/2)^{p/q-1}$, for all $R > 0$. This completes the proof. □

*Proof of Theorem* 1.4. If we integrate the conclusion of Lemma 2.1 over $B_{R/2}$ and use the boundedness of $I_\rho$ is bounded from $L_{p,\phi}(\mathbb{R}^d)$ to $L_{q,\phi^{p/q}}(\mathbb{R}^d)$, $\int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r)^{p/q-1}$ for $r > 0$. By virtue of Lemma 2.2, we obtain

$$\int_{2R}^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \left(\frac{1}{R^d}\int_{B_{R/3}} I_\rho g(x)^q\,dx\right)^{1/q} \le C\phi(R)^{p/q}\|I_\rho g_R : L_{q,\phi^{p/q}}\|.$$

Since $I_\rho$ is bounded, we obtain $\int_{2R}^\infty \frac{\phi(t)\rho(t)}{t}\,dt \le C\phi(R)^{p/q}\|g_R : L_{p,\phi}\|$. Now we invoke Lemma 2.4 to conclude $\int_{2R}^\infty \frac{\phi(t)\rho(t)}{t}\,dt \le C\phi(R)^{p/q} \le C\phi(2R)^{p/q}$. Thus, Theorem 1.4 is proven. □

*Proof of Theorem* 1.5. By Lemma 2.1 and $\|\chi_{B_r} : L_{1,\phi}\| \sim \phi(r)^{-1}$, we obtain

$$\tilde\rho(r) \sim r^{-d}\int_{B_{r/2}} I_\rho\chi_{B_r}(x)\,dx \le \psi\left(\frac{r}{2}\right)\|I_\rho\chi_{B_r} : L_{1,\psi}\| \le C\frac{\psi(r)}{\phi(r)}.$$

Meanwhile, by Lemma 2.2 and $\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \sim r^{-d}\int_{B_{2r/3}} I_\rho g_r(x)\,dx$, we have

$$\int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\psi\left(\frac{2r}{3}\right)\|I_\rho g_r : L_{1,\psi}\| \le C\psi(r)\|g_r : L_{1,\phi}\| \le C\psi(r).$$

Thus, Theorem 1.5 is proved. □

## 4. Examples

In this section, we show by examples that Theorems 1.3. 1.4 and 1.5 have independent interest. Here and below we write

$$\ell_{\beta_1,\beta_2}(r) := \begin{cases} \left(\log\frac{1}{r}\right)^{-\beta_1} & (0 < r < e^{-1}), \\ 1 & (e^{-1} \le r \le e), \\ (\log r)^{\beta_2} & (e < r). \end{cases}$$

This function is used to describe the "log"-growth and "log"-decay properties. Also, we fix $p$ and $q$ so that $1 < p < q < \infty$. As Proposition 4.1 below shows, generalized Morrey spaces occur natuarally.

**Proposition 4.1.** [25, Theorem 5.1] *Let* $1 < p < \infty$ *and* $s \in (0, d/p)$*. There exists* $C_{p,s} > 0$ *such that* $\int_{B_r} |f(y)|dy \le C_{p,s} \frac{r^d \ell_{-1,0}(r)}{(1+r)^s} \|(1-\Delta)^{s/2} f : L^{p,d-ps}\|$ *holds for all* $r > 0$ *and* $f \in L^{p,d-ps}(\mathbb{R}^d)$ *with* $(1-\Delta)^{s/2} f \in L^{p,d-ps}(\mathbb{R}^d)$*.*

We can improve Proposition 4.1 to $p = 1$ while we cannot delete $\ell_{-1,0}$ because it is necessary condition for this estimate. See Example 5 below. The following example deals more deeply with Proposition 4.1.

*Example* 1. Let $\mu_1, \mu_2$ satisfy $\mu_1, \mu_2 \ge 0$. Set $\alpha = \frac{d}{p} - \frac{d}{q}$ and $\beta_i = \left(\frac{p}{q} - 1\right) \mu_i$ for $i = 1, 2$. Define $\rho(r) = r^\alpha \ell_{\beta_1,\beta_2}(r)$, $\quad \phi(r) = r^{-\frac{d}{p}} \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair $(\rho, \phi)$ fulfills the assumption (1.2) and (1.3) in Theorem 1.3 but fails (1.5) in Theorem 1.4. More precisely, since $\alpha > 0$, we have $\tilde{\rho}(r) \sim \rho(r)$ and $\displaystyle\int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \sim \phi(r)\rho(r)$.

Example 1 is an endpoint case of the next example.

*Example* 2. Let $\lambda$ satisfy $0 < \left(\frac{p}{q} - 1\right) \lambda < d$ and $-\frac{d}{p} < \lambda < 0$. Take $\mu_1, \mu_2$ arbitrarily. Set $\alpha = \left(\frac{p}{q} - 1\right) \lambda$ and $\beta_i = \left(\frac{p}{q} - 1\right) \mu_i$ for $i = 1, 2$. Define $\rho(r) = r^\alpha \ell_{\beta_1,\beta_2}(r)$ and $\phi(r) = r^\lambda \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair $(\rho, \phi)$ fulfills the assumption (1.2)–(1.5) in Theorems 1.3 and 1.4. Indeed, $\tilde{\rho}(r) \sim \rho(r)$, $\quad \int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \sim \phi(r)\rho(r)$.

The next example concerns the case when the spaces are close to $L^\infty(\mathbb{R}^d)$ and the smoothing order of $I_\rho$ is "almost 0".

*Example* 3. Let $\mu_1, \mu_2 < 0$. Set $\beta_1 = \left(\frac{p}{q} - 1\right) \mu_1 + 1 \in (1, \infty)$ and $\beta_2 = \left(\frac{p}{q} - 1\right) \mu_2 - 1 \in (-1, \infty)$. Define $\rho(r) = \ell_{\beta_1,\beta_2}(r)$ and $\phi(r) = \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair $(\rho, \phi)$ fulfills the assumptions (1.4) and (1.5) in Theorem 1.4 but fails (1.2) in Theorem 1.3. More precisely, for all $r > 0$, we have $\tilde{\rho}(r) \sim \ell_{\beta_1-1,\beta_2+1}(r)$, and $\int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \sim \ell_{\mu_1+\beta_1-1,\mu_2+\beta_2+1}(r)$.

We consider a case when the target space is close to $L^\infty(\mathbb{R}^d)$.

*Example* 4. Let $\alpha, \beta_1, \mu_1, \mu_2$ satisfy $0 < \alpha < \frac{d}{p}$, $\mu_1 + \beta_1 < 1$, $\mu_2 < 0$. Set $\beta_2 = \left(\frac{p}{q} - 1\right) \mu_2 - 1 \in (-1, \infty)$. Define $\rho(r) = \min(1, r^\alpha) \ell_{\beta_1,\beta_2}(r)$ and $\phi(r) = \max(1, r^{-\alpha}) \ell_{\mu_1,\mu_2}(r)$ for $r > 0$. Then this pair $(\rho, \phi)$ fulfills the assumptions (1.4) and (1.5) in Theorem 1.4 but fails (1.2) in Theorem 1.3. More precisely, $\phi(r)\tilde{\rho}(r) \sim \ell_{\mu_1+\beta_1,\mu_2+\beta_2+1}(r)$ and $\int_r^\infty \frac{\phi(t)\rho(t)}{t}\, dt \sim \ell_{\mu_1+\beta_1-1,\mu_2+\beta_2+1}(r)$ for $r > 0$.

Finally, we explain how we improve Proposition 4.1 from our results.

*Example* 5. Let $0 < s < d$. Define $\phi(r) = r^{-s}$ and $\psi(r) = (1 + r)^{-s} \ell_{-1,0}(r)$ for $r > 0$. Let $\rho(r) = r^d G_s(r)$, where $G_s$ denotes the Bessel kernel, the kernel of $(1 - \Delta)^{s/2}$. Observe that $\tilde{\rho}(r) \sim \min(r^s, 1)$ and hence $\phi(r)\tilde{\rho}(r) \sim$

$\min(1, r^{-s})$. Note also that $\int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \begin{cases} \log(e/r) & (r < 1), \\ r^{d-s}G_s(r) & (r \geq 1). \end{cases}$ Then we have $\phi(r)\tilde{\rho}(r) + \int_r^\infty \frac{\phi(t)\rho(t)}{t}\,dt \sim \psi(r)$. Hence it follows from Theorem 1.5 that $\|I_\rho f : L_{1,\psi}\| \leq C\|f : L_{1,\phi}\|$, extending Proposition 4.1. This triple $(\rho, \phi, \psi)$ fulfills the assumptions (1.6) and (1.7) in Thoerem 1.5 but it fails (1.2) in Theorem 1.3 and (1.4) in Theorem 1.4.

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## References

[1] D. R. Adams, “A note on Riesz potentials”, *Duke Math. J.* **42** (1975), 765–778.

[2] V. Burenkov; and V. Guliyev, Necessary and sufficient conditions for the boundedness of the Riesz potential in local Morrey-type spaces, *Potential Anal.* **30** (2009), no. 3, 211–249.

[3] F. Chiarenza and M. Frasca, “Morrey spaces and Hardy-Littlewood maximal function”, *Rend. Mat.* **7** (1987), 273–279.

[4] Eridani; H. Gunawan; and E. Nakai, “On generalized fractional integral operators”, *Sci. Math. Jpn.* **60** (2004), 539–550.

[5] Eridani; H. Gunawan; and M.I. Utoyo, “A revisit on Nakai’s results on maximal functions and frational integrals”, submitted.

[6] V. Guliyev; J. Hasanov; and Y. Zeren, “Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces”, *J. Math. Inequal.* **5** (2011), no. 4, 491–506.

[7] H. Gunawan, “A note on the generalized fractional integral operators”, *J. Indones. Math. Soc.* **9** (2003), 39–43.

[8] H. Gunawan and Eridani, “Fractional integrals and generalized Olsen inequalities”, *Kyungpook Math. J.* **49** (2009), 31–39.

[9] V. Guliyev; and R. Mustafayev, Boundedness of the anisotropic maximal and anisotropic singular integral operators in generalized Morrey spaces, *Acta Math. Sin.* (Engl. Ser.) **27** (2011), no. 12, 2361–2370.

[10] K. Kurata and S. Sugano, “A remark on estimates for uniformly elliptic operators on weighted $L^p$ spaces and Morrey spaces”, *Math. Nachr.* **209** (2000), 137–150.

[11] Y. Mizuta; E. Nakai; T. Ohno; and T. Shimomura, Boundedness of fractional integral operators on Morrey spaces and Sobolev embeddings for generalized Riesz potentials, *J. Math. Soc. Japan* **62** (2010), no. 3, 707–744.

[12] E. Nakai, “Hardy-Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces”, *Math. Nachr.* **166** (1994), 95–103.

[13] E. Nakai, “A characterization of pointwise multipliers on the Morrey spaces”, *Sci. Math.* **3** (2000), 445-454.

[14] E. Nakai, “On generalized fractional integrals”, *Taiwanese J. Math.* **5** (2001), 587–602.

[15] E. Nakai, “ On generalized fractional integrals on the weak Orlicz spaces”, $BMO_\varphi$, the Morrey spaces and the Campanato spaces, *Function spaces, interpolation theory and related topics (Lund, 2000)*, 389–401, de Gruyter, Berlin, 2002.

[16] E. Nakai, “Orlicz-Morrey spaces and the Hardy-Littlewood maximal function”, *Studia Math.* **188** (2008), no. 3, 193–221.

[17] E. Nakai, “A generalization of Hardy spaces $H^p$ by using atoms”, *Acta Math. Sin.* (Engl. Ser.) **24** (2008), no. 8, 1243–1268.

[18] J. Peetre, “On the theory of $\mathcal{L}_{p,\lambda}$ spaces”, *J. Funct. Anal.* **4** (1969), 71–87.

[19] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1970.

[20] E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, New Jersey, 1993.

[21] Y. Sawano; S. Sugano; and H. Tanaka, “A note on generalized fractional integral operators on generalized Morrey spaces”, *Bound. Value Probl.* 2009, Art. ID 835865, 18 pp.

[22] Y. Sawano, Generalized Morrey Spaces for Non-doubling Measures, *Nonlinear differ. equ. appl.* **15** (2008), 413–425.

[23] Y. Sawano; S. Sugano; and H. Tanaka, “Generalized fractional integral operators and fractional maximal operators in the framework of Morrey spaces”, *Trans. Amer. Math. Soc.* **363** (2011), no. 12, 6481–6503.

[24] Y. Sawano; S. Sugano and H. Tanaka, “Orlicz-Morrey spaces and fractional operators”, *Potential Anal.* **36**, No. 4 (2012), 517-556.

[25] Y. Sawano; and H. Wadade, *On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space*, *J. Fourier Anal. Appl.*, online.

[26] S. Sugano; and H. Tanaka, “Boundedness of fractional integral operators on generalized Morrey spaces”, *Sci. Math. Jpn. Online* **8** (2003), 233–242.

ERIDANI
Department of Mathematics, Airlangga University, Surabaya 60115, Indonesia.
e-mail: `eridani.dinadewi@gmail.com`

Hendra GUNAWAN
Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia.
e-mail: `hgunawan@math.itb.ac.id`

Eiichi NAKAI
Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan
e-mail: `enakai@mx.ibaraki.ac.jp`

Yoshihiro SAWANO
Department of Mathematics and Information Scinence, Tokyo Metropolitan University, 1-1 Minami-Ohsawa, Hachioji, 192-0397, Japan.
e-mail: `yoshihiro-sawano@celery.ocn.ne.jp`