ON ANGLES BETWEEN SUBSPACES OF INNER PRODUCT SPACES

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ABSTRACT. We discuss the notion of angles between two subspaces of an inner product space, as introduced by Risteski and Trenčevski [11] and Gunawan, Neswan and Setya-Budhi [7], and study its connection with the so-called canonical angles. Our approach uses only elementary calculus and linear algebra.

1. INTRODUCTION

The notion of angles between two subspaces of a Euclidean space $\mathbb{R}^n$ has attracted many researchers since the 1950's (see, for instance, [1, 2, 3]). These angles are known to statisticians and numerical analysts as canonical or principal angles. A few results on canonical angles can be found in, for example, [4, 8, 10, 12]. Recently, Risteski and Trenčevski [11] introduced an alternative definition of angles between two subspaces of $\mathbb{R}^n$ and explained their connection with canonical angles.

Let $U = \text{span}\{u_1, \dots, u_p\}$ and $V = \text{span}\{v_1, \dots, v_q\}$ be p- and q-dimensional subspaces of $\mathbb{R}^n$, where $\{u_1, \dots, u_p\}$ and $\{v_1, \dots, v_q\}$ are orthonormal, with $1 \le p \le q$. Then, as is suggested by Risteski and Trenčevski [11], one can define the angle θ between the two subspaces U and V by

$$\cos^2 \theta := \det(M^{\mathrm{T}} M)$$

where $M := [\langle u_i, v_k \rangle]^{\mathrm{T}}$ is a $q \times p$ matrix, M^{T} is its transpose, and $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^n$. (Actually, Risteski and Trenčevski gave a more general formula for $\cos \theta$, but their formula is erroneous and is corrected by Gunawan, Neswan and Setya-Budhi in [7].)

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Now let $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_p$ be the canonical angles between U and V, which are defined recursively by

$$\cos\theta_1 := \max_{u\in U, \|u\|=1} \max_{v\in V, \|v\|=1} \langle u, v\rangle = \langle u_1, v_1\rangle$$
$$\cos\theta_{i+1} := \max_{u\in U_i, \|u\|=1} \max_{v\in V_i, \|v\|=1} \langle u, v\rangle = \langle u_{i+1}, v_{i+1}\rangle$$

where U_i is the orthogonal complement of u_i relative to U_{i-1} and V_i is the orthogonal complement of v_i relative to V_{i-1} (with $U_0 = U$ and $V_0 = V$). Then we have

Theorem 1 (Risteski & Trenčevski). $\cos^2\theta = \prod_{i=1}^{p} \cos^2\theta_i$.

The proof of Theorem 1 uses matrix theoretic arguments (involving the notions of orthogonal, positive definite, symmetric, and diagonalizable matrices). In this note, we present an alternative approach towards Theorem 1. For the case $p = 2$, we reprove Theorem 1 by using only elementary calculus. For the general case, we prove a statement similar to Theorem 1 by using elementary calculus and linear algebra. As in [7], we replace $\mathbb{R}^n$ by an inner product space of arbitrary dimension. Related results may be found in [9].

2. Main results

In this section, let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space of dimension 2 or higher (may be infinite), and $U = \mathrm{span}\{u_1, \ldots, u_p\}$ and $V = \mathrm{span}\{v_1, \ldots, v_q\}$ be two subspaces of X with $1 \leq p \leq q < \infty$. Assuming that $\{u_1, \ldots, u_p\}$ and $\{v_1, \ldots, v_q\}$ are orthonormal, let θ be the angle between U and V, given by

$$\cos^2\theta := \det(M^{\mathrm{T}}M)$$

with $M := [\langle u_i, v_k\rangle]^{\mathrm{T}}$ being a $q \times p$ matrix.

2.1. **The case $p = 2$.** For $p = 2$, we have the following reformulation of Theorem 1.

Fact 2. $\cos\theta = \max_{u\in U, \|u\|=1} \max_{v\in V, \|v\|=1} \langle u, v\rangle \cdot \min_{u\in U, \|u\|=1} \max_{v\in V, \|v\|=1} \langle u, v\rangle.$

Proof. First note that, for each $u \in X$, $\langle u, v\rangle$ is maximized on $\{v \in V : \|v\| = 1\}$ at $v = \frac{\mathrm{proj}_V u}{\|\mathrm{proj}_V u\|}$ and $\max_{v\in V, \|v\|=1} \langle u, v\rangle = \left\langle u, \frac{\mathrm{proj}_V u}{\|\mathrm{proj}_V u\|}\right\rangle = \|\mathrm{proj}_V u\|$. Here of course $\mathrm{proj}_V u = \sum_{k=1}^{q} \langle u, v_k\rangle v_k$, denoting the orthogonal projection of u on V. Now consider the function

$f(u) := \|\mathrm{proj}_V u\|$, $u \in U$, $\|u\| = 1$. To find its extreme values, we examine the function $g := f^2$, given by

$$g(u) = \|\mathrm{proj}_V u\|^2 = \sum_{k=1}^{q} \langle u, v_k \rangle^2, \quad u \in U,\ \|u\| = 1.$$

Writing $u = (\cos t)u_1 + (\sin t)u_2$, we can view g as a function of $t \in [0, \pi]$, given by

$$g(t) = \sum_{k=1}^{q} \big[\cos t \langle u_1, v_k \rangle + \sin t \langle u_2, v_k \rangle\big]^2.$$

Expanding the series and using trigonometric identities, we can rewrite g as

$$g(t) = \frac{1}{2}\big[C + \sqrt{A^2 + B^2}\cos(2t - \alpha)\big]$$

where

$$A = \sum_{k=1}^{q} \big[\langle u_1, v_k \rangle^2 - \langle u_2, v_k \rangle^2\big], \quad B = 2\sum_{k=1}^{q} \langle u_1, v_k \rangle \langle u_2, v_k \rangle,$$

$$C = \sum_{k=1}^{q} \big[\langle u_1, v_k \rangle^2 + \langle u_2, v_k \rangle^2\big],$$

and $\tan\alpha = B/A$. From this expression, we see that g has the maximum value $m := \frac{C + \sqrt{A^2+B^2}}{2}$ and the minimum value $n := \frac{C - \sqrt{A^2+B^2}}{2}$. (Note particularly that n is nonnegative.) Accordingly the extreme values of f are

$$\max_{u \in U, \|u\|=1} \max_{v \in V, \|v\|=1} \langle u, v \rangle = \sqrt{m} \quad \text{and} \quad \min_{u \in U, \|u\|=1} \max_{v \in V, \|v\|=1} \langle u, v \rangle = \sqrt{n}.$$

Now we observe that

$$\begin{aligned} mn &= \frac{C^2 - (A^2 + B^2)}{4} \\ &= \sum_{k=1}^{q} \langle u_1, v_k \rangle^2 \sum_{k=1}^{q} \langle u_2, v_k \rangle^2 - \Big[\sum_{k=1}^{q} \langle u_1, v_k \rangle \langle u_2, v_k \rangle\Big]^2 \\ &= \det(M^{\mathrm{T}} M) \\ &= \cos^2\theta. \end{aligned}$$

This completes the proof. □

Geometrically, the value of $\cos\theta$, where θ is the angle between $U = \mathrm{span}\{u_1, u_2\}$ and $V = \mathrm{span}\{v_1, \ldots, v_q\}$ with $2 \le q < \infty$, represents the ratio between the area of the parallelogram spanned by $\mathrm{proj}_V u_1$ and $\mathrm{proj}_V u_2$ and the area of the parallelogram spanned by u_1 and u_2. The area of the parallelogram spanned by two vectors x and y in X is given by $\big[\|x\|^2\|y\|^2 - \langle x, y \rangle^2\big]^{1/2}$ (see [6]). In particular, when $\{u_1, u_2\}$ and

$\{v_1, \ldots, v_q\}$ are orthonormal, the area of the parallelogram spanned by $\mathrm{proj}_V u_1$ and $\mathrm{proj}_V u_2$ is $\Big[\sum_{k=1}^q \langle u_1, v_k\rangle^2 \sum_{k=1}^q \langle u_2, v_k\rangle^2 - \big[\sum_{k=1}^q \langle u_1, v_k\rangle\langle u_2, v_k\rangle\big]^2\Big]^{1/2}$, while the area of the parallelogram spanned by u_1 and u_2 is 1.

Our result here says that, in statistical terminology, the value of $\cos\theta$ is the product of the maximum and the minimum correlations between $u \in U$ and its best predictor $\mathrm{proj}_V u \in V$. The maximum correlation is of course the cosine of the first canonical angle, while the minimum correlation is the cosine of the second canonical angle. The fact that the cosine of the last canonical angle is in general the minimum correlation between $u \in U$ and $\mathrm{proj}_V u \in V$ is justified in [10].

2.2. **The general case.** The main ideas of the proof of Theorem 1 are that the matrix $M^{\mathrm{T}}M$ is diagonalizable so that its determinant is equal to the product of its eigenvalues, and that the eigenvalues are the square of the cosines of the canonical angles. The following statement is slightly less sharp than Theorem 1, but much easier to prove (in the sense that there are no sophisticated arguments needed).

Theorem 3. $\cos\theta$ *is equal to the product of all critical values of* $f(u) := \max_{v\in V, \|v\|=1} \langle u, v\rangle$, $u \in U,\ \|u\| = 1$.

Proof. For each $u \in X$, $\langle u, v\rangle$ is maximized on $\{v \in V : \|v\| = 1\}$ at $v = \frac{\mathrm{proj}_V u}{\|\mathrm{proj}_V u\|}$ and $\max_{v\in V,\|v\|=1} \langle u, v\rangle = \|\mathrm{proj}_V u\|$. Now consider the function $g := f^2$, given by

$$g(u) = \|\mathrm{proj}_V u\|^2, \quad u \in U,\ \|u\| = 1.$$

Writing $u = \sum_{i=1}^p t_i u_i$, we have

$$\mathrm{proj}_V u = \sum_{k=1}^q \langle u, v_k\rangle v_k = \sum_{k=1}^q \sum_{i=1}^p t_i \langle u_i, v_k\rangle v_k,$$

and so we can view g as $g(t_1, \ldots, t_p)$, given by

$$g(t_1, \ldots, t_p) := \Big\|\sum_{k=1}^q \sum_{i=1}^p t_i\langle u_i, v_k\rangle v_k\Big\|^2 = \sum_{k=1}^q \Big(\sum_{i=1}^p t_i\langle u_i, v_k\rangle\Big)^2,$$

where $\sum_{i=1}^p t_i^2 = 1$. In terms of the matrix $M = [\langle u_i, v_k\rangle]^{\mathrm{T}}$, one may write

$$g(\mathbf{t}) = \mathbf{t}^{\mathrm{T}} M^{\mathrm{T}} M \mathbf{t} = |M\mathbf{t}|_q^2, \quad |\mathbf{t}|_p = 1,$$

where $\mathbf{t} = [t_1 \ \cdots \ t_p]^{\mathrm{T}} \in \mathbb{R}^p$ and $|\cdot|_p$ denotes the usual norm in $\mathbb{R}^p$.

To find its critical values, we use the Lagrange method, working with the function

$$h_\lambda(\mathbf{t}) := |M\mathbf{t}|_q^2 + \lambda\big(1 - |\mathbf{t}|_p^2\big), \quad \mathbf{t} \in \mathbb{R}^p.$$

For the critical values, we must have

$$\nabla h_\lambda(\mathbf{t}) = 2M^{\mathrm{T}}M\mathbf{t} - 2\lambda\mathbf{t} = \mathbf{0},$$

whence

$$M^{\mathrm{T}}M\mathbf{t} = \lambda\mathbf{t}. \tag{2.1}$$

Multilpying both sides by $\mathbf{t}^{\mathrm{T}}$ on the left, we obtain

$$\lambda = \mathbf{t}^{\mathrm{T}}M^{\mathrm{T}}M\mathbf{t} = g(\mathbf{t}),$$

because $\mathbf{t}^{\mathrm{T}}\mathbf{t} = |\mathbf{t}|^2 = 1$. Hence any Lagrange multiplier λ is a critical value of g.

But (2.1) also tells us that λ must be an eigenvalue of $M^{\mathrm{T}}M$. From elementary linear algebra, we know that λ must satisfy

$$\det(M^{\mathrm{T}}M - \lambda I) = 0,$$

which has in general p roots, $\lambda_1 \geq \cdots \geq \lambda_p \geq 0$, say. Since $M^{\mathrm{T}}M$ is diagonalizable, we conclude that

$$\det(M^{\mathrm{T}}M) = \prod_{i=1}^{p} \lambda_i.$$

This proves the assertion. □

A geometric interpretation of the value of $\cos\theta$ is provided below.

Fact 4. *$\cos\theta$ is equal to the volume of the p-dimensional parallelepiped spanned by $\mathrm{proj}_V u_i,\ i = 1, \ldots, p$.*

Proof. Since we are assuming that $\{v_1, \ldots, v_q\}$ is orthonormal, we have $\mathrm{proj}_V u_i = \sum_{k=1}^{q} \langle u_i, v_k\rangle v_k$, for each $i = 1, \ldots, p$. Thus, for $i, j = 1, \ldots, p$, we obtain

$$\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j\rangle = \sum_{k=1}^{q} \langle u_i, v_k\rangle\langle u_j, v_k\rangle,$$

whence

$$\det[\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j\rangle] = \det(M^{\mathrm{T}}M) = \cos^2\theta.$$

But $\det[\langle \mathrm{proj}_V u_i, \mathrm{proj}_V u_j\rangle]$ is the square of the volume of the p-dimensional parallelepiped spanned by $\mathrm{proj}_V u_i,\ i = 1, \ldots, p$ (see [6]). □

3. Concluding remarks

Statisticians use canonical angles as measures of dependency of one set of random variables on another. The drawback is that finding these angles between two given subspaces is rather involved (see, for example, [2, 4]). Many researchers often use only the first canonical angle for estimation purpose (see, for instance, [5]). Geometrically, however, the first canonical angle is not a good measurement for approximation (in $\mathbf{R}^3$, for instance, the first canonical angle between two arbitrary subspaces is 0 even though the two subspaces do not coincide).

The work of Risteski and Trenčevski [11] suggests that if we multiply the cosines of the canonical angles (instead of computing each of them separately), we get the cosine of some value θ that can be considered as the 'geometrical' angle between the two given subspaces. In general, as is shown in [7], the value of $\cos\theta$ represents the ratio between the volume of the parallelepiped spanned by the projection of the basis vectors of the lower dimension subspace on the higher dimension subspace and the volume of the parallelepiped spanned by the basis vectors of the lower dimension subspace. (Thus, the notion of angles between two subspaces developed here can be thought of as a generalization of the notion of angles in trigonometry taught at schools.) What is particularly nice here is that we have an explicit formula for $\cos\theta$ in terms of the basis vectors of the two subspaces.

References

[1] T.W. Anderson, *An Introduction to Multivariate Statistical Analysis*, John Wiley & Sons, Inc., New York, 1958.

[2] Å. Björck and G.H. Golub, "Numerical methods for computing angles between linear subspaces", *Math. Comp.* **27** (1973), 579–594.

[3] C. Davies and W. Kahan, "The rotation of eigenvectors by a perturbation. III", *SIAM J. Numer. Anal.* **7** (1970), 1–46.

[4] Z. Drmač, "On principal angles between subspaces of Euclidean space", *SIAM J. Matrix Anal. Appl.* **22** (2000), 173–194 (electronic).

[5] S. Fedorov, "Angle between subspaces of analytic and antianalytic functions in weighted L_2 space on a boundary of a multiply connected domain" in *Operator Theory, System Theory and Related Topics*, Beer-Sheva/Rehovot, 1997, 229–256.

[6] F.R. Gantmacher, *The Theory of Matrices*, Vol. I, Chelsea Publ. Co., New York (1960), 247–256.

[7] H. Gunawan, O. Neswan and W. Setya-Budhi, "A formula for angles between two subspaces of inner product spaces", submitted.

[8] A.V. Knyazev and M.E. Argentati, "Principal angles between subspaces in an A-based scalar product: algorithms and perturbation estimates", *SIAM J. Sci. Comput.* **23** (2002), 2008–2040.

[9] J. Miao and A. Ben-Israel, "Product cosines of angles between subspaces", *Linear Algebra Appl.* **237/238** (1996), 71–81.
[10] V. Rakočević and H.K. Wimmer, "A variational characterization of canonical angles between subspaces", *J. Geom.* **78** (2003), 122–124.
[11] I.B. Risteski and K.G. Trenčevski, "Principal values and principal subspaces of two subspaces of vector spaces with inner product", *Beiträge Algebra Geom.* **42** (2001), 289–300.
[12] H.K. Wimmer, "Canonical angles of unitary spaces and perturbations of direct complements", *Linear Algebra Appl.* **287** (1999), 373–379.

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